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TYPICAL CONTINUOUS FUNCTION WITHOUT CYCLES IS STABLE

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Let C be the metric space of all $I \to I$ continuous functions with the uniform metric where I is a real compact interval. For $f \in C$ put $\|f\| = \max\{|f(x)|, x \in I\}$. Denote by f^n the n-th iterate of f. If $f^n(x) = x$ for some $x \in I$, $n > 1$, and $f^m(x) \neq x$ whenever $m < n$, then f is said to have a cycle at x. The order of this cycle is n while its length is $d = \max\{|f^r(x) - f^s(x)|, 1 \leqq r, s \leqq n\}$. Let $\lambda(f)$ be the l.u.b. of the lengths of all cycles of f. The function f is said to be stable if $\lambda\colon C \to R$ is continuous at f (cf. [7]). It has been shown in [7] that the stable functions form a residual set in C. However, this result gives no information on the stability of functions without cycles, since the set A of these functions is nowhere dense in C (Theorem 1 below). The functions without cycles are very important in applications (see e.g. [5]). One of the reasons is that the sequences of their iterates are convergent [9] (see also [2]).

The main aim of the paper is to show that the unstable functions without cycles form a relatively small set. Namely, we show that the set A is a second Baire category set in itself (Theorem 2) while the unstable functions without cycles form a set of the first Baire category in A (Theorem 3). Thus the typical continuous function without cycles is stable.

The following notation will be used:

$$F = \{f \in A\,;\, f \text{ is unstable}\},$$
$$G = \{f \in A\,;\, f \text{ is stable}\}.$$

As it is known (see [6]) we have $F \neq \emptyset$. The property of "absence of the cycles" is not preserved in any neighbourhood of any function from C since the set of all functions having 3-cycles is dense in C (see [3]). However, it is possible to prove a stronger result.

Theorem 1. *The set of all continuous functions without a 3-cycle is nowhere dense in C.*

Proof. Take $f \in C$, $\varepsilon > 0$. Let $x^* \in I$, $f(x^*) = x^*$. The continuity of f at x^* implies the existence of a $\delta > 0$, $\delta < \varepsilon$ such that $|f(x) - x^*| < \varepsilon/2$, whenever $|x - x^*| < \delta$. Put $\eta = \delta/4$ and for $x \in [x^*, x^* + 2\eta]$ define

$$g^*(x)=\begin{cases}2x-x^* & \text{if } x^*\leqq x\leqq x^*+\eta\\ 3x^*+4\eta-2x & \text{otherwise.}\end{cases}$$

It is easy to see that for $x\in[x^*, x^*+2\eta]$ we have

$$|g^*(x)-f(x)|\leqq|g^*(x)-x^*|+|x^*-f(x)|<2\eta+\varepsilon/2<\varepsilon.$$

Now we define $g\colon I\to I$ such that $g(x)=g^*(x)$ for $x\in[x^*, x^*+2\eta]$, $g(x)=f(x)$ for $x\notin[x^*, x^*+3\eta]$, and let g be continuous in I and $\|g-f\|<\varepsilon$. It is easy to verify that for $x_0=x^*+\eta/2$ we have

$$g^3(x_0)<x_0<g(x_0)<g^2(x_0),$$

hence g has a 3-cycle (cf., e.g. [4]). By the continuity of g, for each continuous r from a sufficiently small neighbourhood $O(g)$ of g we have

$$r^3(x_0)<x_0<r(x_0)<r^2(x_0),$$

thus each $r\in O(g)$ has a 3-cycle, q.e.d.

As a direct consequence of Theorem 1 we obtain that the set A of all continuous functions without cycles is nowhere dense in C. We show it is a second Baire category set in itself.

Theorem 2. *The set A is a second Baire category set in itself.*

Proof. Let D be the set of all functions which have only 2-cycles. From Block's stability theorem [1] we have clos $A\subset A\cup D$, hence clos $A=A\cup D_0$ where D_0 is a suitable subset of D. According to Baire's theorem clos A is a second category set. To prove the theorem it suffices to show that D_0 is a first category set in clos A. We show that $D_0=\bigcup_{n=1}^{\infty} D_n$, where each $D_n=\{f\in D_0;\ \lambda(f)>1/n\}$ is nowhere dense in clos A. Let $f\in$ clos A. In any neighbourhood $O(f)$ of f there is a function $g\in A$. Let $X=\{x\in I;\ |x-g(x)|\geqq 1/2n\}$. Clearly X is a compact. Since g has no cycles and X contains no fixed points of g we have dist $(g_X, g_X^{-1})>0$, (see Lemma 1 in [6]), where g_X is the graph of g in X and g^{-1} is the inverse relation to g. From the continuity of g there is such a neighbourhood $O(g)\subset O(f)$ that for each $h\in O(g)$ dist $(h_X, h_X^{-1})>0$ and $\|h-g\|<1/2n$.

We show that h has no 2-cycle in X. Let $x_1, x_2\in X$, $x_1\to x_2\to x_1$, $x_1\neq x_2$. Let M be the point $M=(x_1, x_2)\in R^2$. Evidently $M\in h_X$. Since $h(x_2)=x_1$, we have $x_2\in h^{-1}(x_1)$, hence $M\in h_X^{-1}$ and dist $(h_X, h_X^{-1})=0$, which is impossible.

Hence, if h has a 2-cycle in I, $x_1\to x_2\to x_1$, then at least one of the points x_1, x_2, say x_1, belongs to $I\setminus X$. We have

$$|x_1-x_2|=|x_1-h(x_1)|\leqq|x_1-g(x_1)|+|g(x_1)-h(x_1)|<1/n.$$

Thus $h\notin D_n$ and the theorem is proved.

To show the main result it suffices to prove the next

Theorem 3. *The set F is a first category set in A.*

The proof is based on a result from [6] which can be restated as follows (Theorems 1 and 2 in [6]).

Theorem A. *The function λ: $C \to R$ is continuous at some $f \in A$ iff the set of fixed points of f contains no interval.*

Proof of Theorem 3. It suffices to show that F can be represented as $\bigcup_{n=1}^{\infty} A_n$ where each A_n is nowhere dense in A. Let $\{I_n\}$ be a sequence of all closed subintervals of I with rational endpoints. For each n let A_n be the set of all functions $f \in C$ with the property that the set of fixed points of f contains I_n. We show that $\bigcup_{n=1}^{\infty} A_n = F$. Evidently $\bigcup_{n=1}^{\infty} A_n \subset F$. If $f \in F$, then by Theorem A the set of fixed points of f contains an interval, and hence an interval I_k with rational endpoints. Thus $f \in A_k \subset \bigcup_{n=1}^{\infty} A_n$.

Further we show that each set A_n, $n = 1, 2, \ldots$, is nowhere dense in A. Let $f \in A$, $\varepsilon > 0$. If $f \in G$, then $f \notin A_n$ and there exists $\delta > 0$, $\delta < \varepsilon$ so that for $g \in C$ we have $g \notin A_n$ whenever $\|f - g\| < \delta$. If $f \in A_n \subset F$, we find $g \in C$, $\|f - g\| \leqq \varepsilon/2$ in the following way. Denote $I_n = [a_n, b_n]$, and choose $x_n \in (a_n, b_n)$. Let $\delta < \min(b - x_n, \varepsilon/2)$, $\delta > 0$. We shall define g^* so that

$$g^*(a_n) = a_n$$
$$g^*(b_n) = b_n$$
$$g^*(x_n) = x_n + \delta,$$

and g^* is continuous and linear in each of the intervals $[a_n, x_n]$, $[x_n, b_n]$. The function g is defined by

$$g(x) = \begin{cases} g^*(x) & \text{for } x \in I_n \\ f(x) & \text{for } x \in I \setminus I_n. \end{cases}$$

It is easy to verify that $g \in C$, $\|f - g\| < \varepsilon/2$, and $h \notin A_n$ whenever $\|h - g\| < \delta$. Thus an arbitrary ε-neighbourhood of f contains a δ-sphere disjoint with A_n and the theorem is proved.

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УСТОЙЧИВОСТЬ ТИПИЧНОЙ НЕПРЕРЫВНОЙ ФУНКЦИИ БЕЗ ЦИКЛОВ

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Резюме

Доказывается, что неустойчивые функции образуют относительно малое множество с топологической точки зрения. Именно показано, что множество A всех функций без циклов является множеством второй категории Бера в себе (Теорема 2), а неустойчивые функции образуют множество первой категории в A (Теорема 3).