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UNIT FRACTIONS IN FIELDS

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In this note we transfer the notion of a unit fraction from the field of rational numbers to an arbitrary field.

Throughout the paper, the following standard notations will be used:

N — the set of all natural numbers,
Z — the set of all integers,
Q — the field of all rational numbers,
R — the field of all real numbers.

Let us recall first some elementary facts about valuation rings (see e.g. [1], [4], [5]).

A subring A of a field K is said to be a *valuation ring* of the field K if for every $x \in K$—$\{0\}$ either $x \in A$ or $x^{-1} \in A$.

A valuation ring has a unique maximal ideal which consists of all non-units of this ring.

Every valuation ring of the field K is integrally closed in K.

Every subring of the field K which is integrally closed in K is an intersection of valuation rings of the field K.

The main aim of this paper is to prove the following theorem.

Theorem. *Let K be a field, x an element of $K - \{0\}$. The following conditions are equivalent:*

(i) *There exist integers $c_1, \ldots, c_n$ such that the following identity*

$$c_1x^n + \ldots + c_nx - \bar{1} = 0 \tag{I}$$

holds (here $\bar{1}$ is the unit element of the field K).

(ii) *Let E be any subring of the field K such that $\bar{1} \notin E$. Then $x \notin E$.*

(iii) *The element x does not lie in any maximal ideal of valuation ring of the field K.*

(iv) *The element x^{-1} lies in all valuation rings of the field K.*

Definition. *A non zero element of the field K which satisfies one of the conditions* (i), (ii), (iii), (iv) *will be called a unit fraction of the field K.*

Proof of the theorem.

1. Proof of the implication (i) $\Rightarrow$ (ii)

Let $x \in K - \{0\}$ such that the identity (i) holds. Let F be any subring of the field K such that $x \in F$. Then also

$$c_1 x^n + \ldots + c_n x = \bar{1} \in F$$

and the implication is proved.

2. Proof of the implication (ii) $\Rightarrow$ (iii).

This implication follows from the fact that any maximal ideal of a valuation ring of the field K is a ring without the unit element $\bar{1}$ of the field K.

3. Proof of the implication (iii) $\Rightarrow$ (iv).

Do not let $x \in K$ belong to any maximal ideal of the valuation ring of the field K. Moreover, assume that there exists a valuation ring A of the field K such that $x^{\ 1} \notin A$. Then by the definition of the valuation ring $x \in A$. Since $x \in A$ and x is not a unit of the ring A, x lies in a proper ideal of the ring A. By Zorn's lemma, x lies in the maximal ideal of the valuation ring of the field K. This is a contradiction with our assumption and the implication is proved.

4. Proof of the implication (iv) $\Rightarrow$ (i).

Let us assume that x is a non-zero element of the field K such that $x^{\ 1}$ lies in all valuation rings of the field K. We claim that the intersection of all valuation rings of the field K is the integral closure of the ring

$$D = \{\bar{1} \cdot z / z \in Z\} \text{ in } K.$$

Indeed, every valuation ring contains the ring D. Since every valuation ring is integrally closed, it must contain also the integral closure of D in K. On the other hand, the integral closure of the ring D in K is integrally closed in K. Hence we have that the integral closure of D is K is the intersection of valuation rings of the field K. This proves our claim.

Since the element x^{-1} belongs to all valuation rings of the field K, the element x^{-1} is integral over the ring D. This means that there exist elements $d_1, \ldots, d_n \in Z$ such that

$$(x^{-1})^n + \mathrm{d}_1(x^{-1})^{n-1} + \ldots + d_n \bar{1} = 0 \quad \text{or} \quad -(d_n x^n + \ldots + d_1 x) = \bar{1}.$$

Hence the element x satisfies an identity of the form (i). This proves our implication.

The proof of the Theorem is completed.

Remark 1. The above definition of the unit fractions of the field K can be reformulated also in terms of Harrison's finite primes.

Let us recall Harrison's definitions of primes and finite primes of a commutative ring R with the unit element $\bar{\ }$ (see [2]).

By a prime of R we mean a subset P of R which is maximal among all the subsets A of R such that A is closed under addition and multiplication and $-\bar{1} \notin A$.

If the prime P of R does not contain $\bar{1}$, then P is called a finite prime of the ring R. If $\bar{1} \in P$, then P is called an infinite prime of the ring R.

It is easy to show that any finite prime P of the ring R is an additive subgroup of R, and hence it is a subring of R ([2], Proposition 1.2).

Statement 1. *Let G be the family of all subrings A of the ring R such that $\bar{1} \notin A$. The family G is naturally ordered by inclusion. Then the set of finite primes of the ring R is the set of maximal elements of the family G.*

Proof. Let us assume that P is a finite prime of the ring R. Then P is a subring of the ring R such that $\bar{1} \notin P$. Hence $P \in G$. Since any element B of the family G is a set closed under addition, multiplication, $-\bar{1} \notin B$ and because P is the maximal set among all such subsets of the ring R, P is the maximal element of the family G.

Now let us assume that the ring $A \in G$ is the maximal element of the family G. The ring A is a subset of the ring R closed under addition and multiplication. Furthermore, $-\bar{1} \notin A$. By Zorn's lemma, A is contained in some prime P of the ring R. There are two possibilities:

(1) P is a finite prime of the ring R;

(2) P is an infinite prime of the ring R.

(1) If P is a finite prime of the ring R, then $P \in G$. Since both A, P are maximal elements of the family G and $A \subseteqq P$, we have $A = P$. Hence A is a finite prime.

(2) We shall show that the second possibility cannot occur:

Let us assume that P is an infinite prime of the ring R. Then $\bar{1} \in P$ and char $R = 0$ (indeed, if there were char $R = n \neq 0$, then $-\bar{1} = (n-1)\bar{1} \in P$, which is a contradiction with the assumption that P is a prime of the ring R.).

Since A is a maximal element of the family G and the set $A + 2 \cdot \bar{1} \cdot Z$ is a subring of the ring R, we have either $2 \cdot \bar{1} \in A$ or there exist elements $a \in A$, $z \in Z$ such that $a + 2z\bar{1} = \bar{1}$ or equivalently $a = \bar{1}(1-2z) \neq 0$. In both cases there exists $m \in Z$, $m \neq 0$ such that $m \cdot 1 \in A$.

Since $\bar{1} \in P$, we have $-\bar{1} = -|m| \cdot \bar{1} + (|m|-1)\bar{1}$, where $|m|$ means an usual absolute value of the integer m and $-|m|\bar{1}A \in P$, $(|m|-1)\bar{1} \in P$.

Hence $-\bar{1} \in P$. This is a contradiction with our hypothesis that P is a prime of the ring R.

The proof of our statement is completed.

Now we can reformulate our definition of a unit fraction of the field as follows:

Statement 2. *The element $x \in K - \{0\}$* is a unit fraction of the field K if and only if x does not lie in any finite prime of the field K.

Proof. Let us assume that $x \in K - \{0\}$ is a unit fraction of the field K. Let P be any finite prime of the ring K. Then P is a subring of the field K such that $\bar{1} \notin P$.

Hence from the condition (ii) of the theorem we have $x \notin P$. This shows that a unit fraction of the field K does not lie in any finite prime of the field K.

Now let us assume that x does not belong to any finite prime of the field K. Contrary to our assertion let us assume that x belongs to some subring C of the field K such that $\bar{1} \notin C$. Then by Zorn's lemma there exists a maximal element A of the family G such that $C \subset A$. According to Statement 1, A is a Harrison finite prime of the field K. This is a contradiction with our hypothesis. This proves that the element x is a unit fraction of the field K.

The proof of Statement 2 is completed.

Example 1. Let $K = Q$. It is well known that the maximal ideals of the valuation rings of the field Q are in a one—to—one correspondence with the set of all prime numbers (see e.g. [1], 1.16 Theorem). This correspondence has the following form:

Every prime number p corresponds to the ring of rational numbers ab^{-1}, where a, b are relatively prime integers, $b \neq 0$ and p divides a.

From this and the condition (iii) of the theorem we get immediately that an element $c \in Q$ is a unit franction of the field Q if and only if it has the form $c = d^{-1}$, where $d \in Z - \{0\}$. Hence the notion of a unit fraction in our sense coincides with the old one in the field Q. (More precisely, this is true only up to the sign, because the unit fraction is usually understood as positive. See also Remark 2.)

Note that the equivalence of the properties (i), (iii) and (iii), (iv) can be immediately verified in the case $K = Q$. (The equivalence of the properties (i), (ii) is obvious.) Indeed, suppose that $c \in Q$ satisfies condition (iii). Then, as noted above, c has the form $c = d^{-1}$, where $d \in Z - \{0\}$. From the identity $d \cdot c - \bar{1} = 0$ we see that c satisfies an equation of the form (i).

On the other hand, suppose that there exist integers $c_1, c_2, \dots, c_n \in Z$ such that for the rational number $e = a \cdot b^{-1}$ (a, b are relatively prime integers, $b \neq 0$) the identity (i) holds. We then have

$$c_1 a^n + c_2 a^{n-1} b + \dots + c_n a b^{n-1} = b^n.$$

Hence a divides b^n. Since a, b are relatively prime, we have $a = \pm 1$. Hence e satisfies the condition (iii).

This proves the equivalence of the conditions (i), (iii).

Now if c satisfies the condition (iii), then $c^{-1} = d \in Z$ and hence c^{-1} belongs to each valuation ring of the field Q. On the other hand, if c^{-1}, $c \in Q - \{0\}$ belongs to all valuation rings of the field Q, c^{-1} is an integral element over the ring Z. Since Z is an integrally closed subring of the field Q, $c^{-1} = d \in Z$. Hence $c = d^{-1}$, $d \in Z - \{0\}$ and c does not belong to any maximal ideal of a valuation ring of the field Q. This proves the equivalence of the conditions (iii), (iv) in the case $K = Q$.

Remark 2. If we want to transfer the notion of the positive unit fraction from the field of rational numbers to an arbitrary field, we can propose the following definition.

Definition. *A non—zero element x of the field K will be called a positive unit fraction of the field K if and only if x is a unit fraction of the field K and x is positive in all orderings of the field K.*

(It is understood that if K has no ordering, then every element of K is positive in all orderings of the field K.)

The well—known theorem of Artin and Schreier asserts that a field K can be ordered if and only if K is formally real. (A field K is called formally real if -1 is not a sum of squares in K.)

Hence if the field K is not formally real, the notions of a unit fraction and a positive unit fraction are equivalent. If K is a formally real field, then the following well—known theorem of Artin and Schreier holds:

An element $x \neq 0$ in K is positive in each ordering of the field K if and only if x is a sum of squares of elements of K.

Hence we can reformulate our definition in the formally real case as follows:

Let K be a formally real field. Then the element $x \in K - \{0\}$ is a positive unit fraction if and only if x is a unit fraction of the field K and x is a sum of squares of the elements of K.

Example 1 is a special case of the following Example 2.

Example 2. Let K be a field of algebraic numbers. Then there is a one-to-one correspondence between the prime divisors of the field K and the maximal ideals of the valuation rings of the field K. To describe this correspondence we shall use the following notation.

Let x be any non-zero element of the field K. Let (x) be the principal divisor of the element x,

$$(x) = P_1^{e_1} \ldots P_n^{e_n}$$

the decomposition of (x) into the product of prime divisors of the field K. We shall say that (x) is a multiple of P_j if and only if $e_j > 0$.

The above correspondence has the following form:

$$\text{prime divisor } P \leftrightarrow \left\{ \begin{array}{l} \text{all non zero elements } x \text{ of the field } K, \\ \text{the principal divisors } (x) \text{ of which are} \\ \text{multiples of } P. \end{array} \right.$$

Hence an element $x \in K$ is a unit fraction of the field K if and only if (x) has the form

$$(x) = P_1^{e_1} \ldots P_n^{e_n}, \quad e_j \leqq 0, \ j = 1, \ldots, n,$$

where P_j are prime divisors and e_j are integral numbers.

Remark 3. Let us define the u nits of a given field in such a way as in an algebraic number field.

Definition. *Let K be a field. The element $x \in K - \{0\}$ is called a unit of the field K if both elements x, x^{-1} are contained in all valuation rings of the field K.*

Equivalently, x is a unit of the field K if and only if both elements x, x^{-1} are unit fractions of the field K.

Now we prove the following assertion, which is well known in algebraic number fields (see e.g. [3], Chapter 3, Proposition 3.3).

Proposition. *Let K be a field. Then the element $x \in K$ is a unit of the field K if and only if there exist*

$c_1, c_2, \ldots, c_{n-1} \in Z$ such that the following identity

$$x^n + c_1 x^{n-1} + \ldots + c_{n-1} x - \bar{1} = 0 \tag{U}$$

holds.

Proof. If $x \in K$ satisfies the equation (U), then the elements x, x^{-1} lie in all valuation rings of the field K, because they are integral over ring $D = \{z \cdot \bar{1}/z \in Z\}$. This means that x is a unit of the field K.

On the other hand, let x be a unit of the field K. Then because x is an integral element over the ring D, it satisfies an equation

$$x^m + d_1 x^{m-1} + \ldots + d_m = 0 \tag{1}$$

where $d_1, \ldots, d_m \in Z$, $m > 0$

Since x^{-1} is also an integral element over the ring D, the element x satisfies also an equation of the form

$$x(e_1 x^{r-1} + \ldots + e_r) = \bar{1}, \quad e_1 \neq 0. \tag{2}$$

Since we can take a power of the equality (2), we can assume that $r \geqq m + 1$. Now multiplying the equality (1) by $(e_1 - 1)x^{r-1-m}$ we get the equality of the form

$$(e_1 - 1)x^{r-1} + f_2 x^{r-2} + \ldots + f_{m+1} x^{e-1-m} = 0 \tag{3}$$

where $f_2, \ldots, f_{m+1}$ are integral numbers.

From (2) and (3) we get

$$\begin{gathered} \bar{1} = x(x^{r-1} + (e_1 - 1)x^{r-1} + e_2 x^{r-2} + \ldots + e_r) = \\ = x(x^{r-1} - f_2 x^{-2} - \ldots - f_{m+1} x^{r-1-m} + e_2 x^{r-2} + \ldots + e_r) = \\ = x(x^{r-1} + g_2 x^{r-2} + g_3 x^{r-3} + \ldots + g_r) \end{gathered}$$

or

$$x^r + g_2 x^{r-1} + g_3 x^{r-2} + \ldots + g_r x - \bar{1} = 0 \tag{4}$$

where $g_2, g_3, \ldots, g_r$ are integral numbers. Hence the equation (3) has the required form (U). The proof is completed.

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ФУНДАМЕНТАЛЬНЫЕ ДРОБЫ В ПОЛЯХ

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Резюме

Используя элементарную теорию нормирований, в статье приводятся три эквивалентных определения фундаментальной дроби в произвольных полях.