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## LOCAL HOMEOMORPHISMS AND RELATED MAPPINGS ON GRAPHS

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Let a class  $\mathcal{S}$  of topological spaces and a class  $\mathcal{M}$  of mappings be given. The following problem is posed in [1], p. 26: characterize all spaces  $X \in \mathcal{S}$  having the property that if  $f: X \rightarrow f(X)$  is a mapping belonging to  $\mathcal{M}$  with non-degenerate  $f(X)$ , then there exists a homeomorphism of  $X$  onto  $f(X)$ . In other words, the question is to characterize all spaces in the class  $\mathcal{S}$  such that their images under mappings from  $\mathcal{M}$  are homeomorphic. In the present paper we discuss the above problem for the class  $\mathcal{S}$  of graphs taking as  $\mathcal{M}$  the classes of open mappings, of local homeomorphisms and of local homeomorphisms in the large sense. Using these and some other mappings we get a characterization of an arc and several characterizations of a simple closed curve. Some unsolved problems also are stated.

By a (linear) graph we mean a one-dimensional connected polyhedron. All mappings are assumed to be continuous. A mapping  $f: X \rightarrow Y$  is called open if images under  $f$  of open subsets of  $X$  are open in  $Y$ . We speak of a local homeomorphism in the large sense if every point  $x$  in  $X$  has an open neighbourhood  $U$  such that the partial mapping  $f|U: U \rightarrow f(U)$  is a homeomorphism (see [9], p. 51). If, moreover,  $f(U)$  is an open subset of  $Y$ , then  $f$  is said to be a local homeomorphism ([13], p. 199). Thus a mapping  $f$  is a local homeomorphism if and only if it is a local homeomorphism in the large sense and simultaneously an open mapping ([9], Theorem 1, p. 54). Further, it is known that a mapping  $f: X \rightarrow Y$  of  $X$  onto a continuum  $Y$  is a local homeomorphism if and only if  $f$  is open and there is a natural number  $n$  such that  $\text{card } f^{-1}(y) = n$  for each  $y \in Y$  ([10], Proposition, p. 64). If this last condition holds, we say that  $f$  is of the degree  $n$  (cf. [13], p. 199) or, equivalently, that it is an  $n$ -to-one mapping (see [5]).

**Proposition 1.** *The following are equivalent for a graph  $X$ :*

- (1) *every non-degenerate image of  $X$  under an open mapping is homeomorphic to  $X$ ,*
- (2)  *$X$  is an arc.*

Indeed, it is proved in Proposition 2 of [3] that an arbitrary arc  $A \subset X$  is the image of the graph  $X$  under an open retraction. Thus, if (1) holds, then  $A$  is

homeomorphic to  $X$ , i.e.,  $X$  is an arc. On the other hand, every open image of an arc is again an arc (see [13], (1.3), p. 184), thus (1) holds.

Note that the assumption of the considered space  $X$  being a graph is essential in Proposition 1. Namely it is proved in [12], Theorem 1.3, p. 260, that the pseudo-arc satisfies (1); and for the class of locally connected continua (even of continua which are regular in the sense of theory of order, see [8], p. 275 and Theorem 4, p. 301) the first author has found some (infinite) dendrites with the same property ([2], Theorem 1, p. 490, and Theorem 3, p. 493). These examples show that the class  $\mathcal{S}$  of spaces  $X$  for which the equivalence (1)  $\Leftrightarrow$  (2) holds cannot be reasonably extended, and therefore the result of Proposition 1 is (in this sense) the best possible.

Observe further that if we demand that the mapping under consideration is a local homeomorphism instead of being open only, we can find many graphs having the discussed property. For example, every local homeomorphism defined on a graph which does not contain any simple closed curve is a homeomorphism itself (see [13], Corollary, p. 199 and note that an open image of a dendrite is again a dendrite, [13], p. 185; cf. [10], Corollary, p. 67). It can be noted that the same property holds for every graph which is the union of a finite dendrite  $D$  and of finitely many simple closed curves  $C_1, C_2, \dots, C_m$  such that  $D \cap C_i$  is an end point of  $D$  for every  $i \in \{1, 2, \dots, m\}$  and if  $i \neq j$ , then either  $C_i \cap C_j = \emptyset$  or  $C_i \cap C_j = C_i \cap D = C_j \cap D$  (this means that some end points of  $D$  have been replaced by simply closed curves or by bundles of simple closed curves having exactly one point in common). But the simple closed curve itself does not have the considered property: the mapping  $z \rightarrow z^2$  is a local homeomorphism of the unit circumference  $S^1$  onto itself which is not a homeomorphism. Therefore the following two problems seem to be natural.

**Problem 1.** Characterize the graphs  $X$  having the property that every local homeomorphism on  $X$  is a homeomorphism.

**Problem 2.** Characterize the graphs  $X$  having the property that for every local homeomorphism  $f$  defined on  $X$  the image  $f(X)$  is homeomorphic to  $X$ .

However, there is no point in replacement open mappings by local homeomorphisms in the large sense in (1). It can be seen by.

**Proposition 2.** *There is no graph  $X$  such that every image of  $X$  under a local homeomorphism in the large sense is homeomorphic to  $X$ .*

In fact, let an arbitrary graph  $X$  be given, and consider a mapping  $f$  on  $X$  which identifies two different interior points  $p$  and  $q$  of an edge  $A$  in  $X$ . Thus  $f(X)$  is again a graph, the image  $f(A)$  of  $A$  is the union of a simple closed curve (namely  $f(pq)$ , where  $pq$  is a subarc of  $A$ ) and of two arcs emanating from the common point  $f(p) = f(q)$ . This point is a "new" ramification point of  $f(X)$  with respect to

those of  $X$ , hence  $f(X)$  is not homeomorphic to  $X$ . It is evident that  $f$  is a local homeomorphism in the large sense.

One can change the role of the domain and of the range space in Problem 1 investigating graphs  $Y$  for which

(3) every local homeomorphism from a continuum onto  $Y$  is a homeomorphism.

Recall ([4], Corollary 2) that a graph  $Y$  has the property (3) if and only if it is acyclic (i.e., it contains no simple closed curve).

We return now to the investigation of some properties of the domain space, proving

**Proposition 3.** *The following are equivalent for a graph  $X$ :*

- (4) *there exists a two-to-one mapping defined on  $X$ , and every image of  $X$  under such a mapping is homeomorphic to  $X$ ,*
- (5) *there exists a two-to-one mapping defined on  $X$  such that the image of  $X$  under this mapping is homeomorphic to  $X$ ,*
- (6) *there is a natural number  $n > 1$  such that there exists an  $n$ -to-one mapping defined on  $X$ , and every image of  $X$  under such a mapping is homeomorphic to  $X$ ,*
- (7)  *$X$  is a simple closed curve.*

**Proof.** The implication (4)  $\Rightarrow$  (6) is obvious. We show the inverse. If (6) is assumed, then  $n = 2$ , because for every  $n > 2$  there are a graph  $X$  and a mapping  $f$  defined on  $X$  which is exactly  $n$ -to-one and such that  $f(X)$  is not a graph even (see [6], 3.2, p. 829). Thus (4) is equivalent to (6).

We show now the following circle of implications: (7)  $\Rightarrow$  (4)  $\Rightarrow$  (5)  $\Rightarrow$  (7).

Indeed, assume (7), i.e., put  $X = \{z \in \mathbb{R}^2 : |z| = 1\}$ . Then  $z \rightarrow z^2$  is the needed mapping, and for every two-to-one mapping  $f$  on  $X$  the image  $f(X)$  is again a simple closed curve (see [6], 4.5, p. 833). Thus (4) follows. The implication from (4) to (5) is obvious. Now assume (5) and let  $f: X \rightarrow f(X)$  be the two-to-one mapping on the graph  $X$  for which  $f(X)$  and  $X$  are homeomorphic. Denote by  $E(X)$  and  $E(f(X))$  the set of all end points of  $X$  and  $f(X)$  respectively. Since for every end point  $y \in E(f(X))$  the two-point set  $f^{-1}(y)$  consists of end points of  $X$  only ([6], 2.4, p. 825), we have

$$2 \cdot \text{card } E(f(X)) \leq \text{card } E(X),$$

and since  $X$  and  $f(X)$  are homeomorphic by assumption, we have  $\text{card } E(f(X)) = \text{card } E(X)$ , which leads to  $\text{card } E(X) = 0$ , i.e., we conclude that  $X$  has no end point. Further, suppose  $X$  contains a ramification point. Since  $f(X)$  is homeomorphic to  $X$  by hypothesis,  $f(X)$  has a ramification point, and — being a graph — it contains a ramification point of a maximal order  $m$ . Denote by  $R(X)$  and  $R(f(X))$  the set of all points of the (maximal) order  $m$  of  $X$  and  $f(X)$  respectively. Two cases are possible. If, for every point  $y$  in  $R(f(X))$ , we have  $f^{-1}(y) \subset R(X)$  (i.e., if

the two-point set  $f^{-1}(y)$  consists of points of order  $m$  only), then  $2 \cdot \text{card } R(f(X)) \leq \text{card } R(X)$ . Since  $\text{card } R(f(X)) = \text{card } R(X)$  by the same argument as previously for the set of end points, we conclude  $R(X) = \emptyset$ , a contradiction. Hence, the other case holds, i.e., there are a point  $y \in R(f(X))$  and a point  $x_1 \in f^{-1}(y)$  such that  $\text{ord}_{x_1} X < m$ . Recall that for every point  $y$  in  $f(X)$  we have

$$2 \cdot \text{ord}_y f(X) = \text{ord}_{x_1} X + \text{ord}_{x_2} X,$$

where  $\{x_1, x_2\} = f^{-1}(y)$  (see [6], 4.4, p. 833). Therefore  $\text{ord}_{x_2} X = 2m - \text{ord}_{x_1} X > m$ , which contradicts the definition of  $m$ . The contradiction shows that  $X$  has no ramification point. Hence  $X$  is composed of points of order 2 exclusively which implies that it is a simple closed curve ([8], §51, V, Theorem 6, p. 294). Thus the proof is complete.

The hypothesis that the space  $X$  under consideration is a graph is necessary in the above proposition and it cannot be weakened in a reasonable way, as one can see from the following example (due to W. J. Charatonik). In the polar coordinates  $(r, \varphi)$  in the plane let

$$\varphi_n \in \{(2k+1) \cdot 2^{-n} \cdot \pi : k \in \{0, 1, 2, \dots, 2^{n-1} - 1\}\}$$

and put

$$X = \{(1, \varphi) : 0 \leq \varphi \leq 2\pi\} \cup \bigcup_{n=1}^{\infty} \{(r, \varphi) : 1 \leq r \leq 1 + 2^{-n} \text{ and } \varphi \in \{\varphi_n, \varphi_n + \pi\}\}.$$

Thus  $X$  is a curve which is regular in the sense of the theory of order, with  $\text{ord}_x X \leq 3$  for each point  $x \in X$ , having countably many ramification points. Define a mapping  $f$  from  $X$  into the plane putting  $f(r, \varphi) = (r, 2\varphi)$ . It can be easily observed that  $f$  is of degree two and that  $f(X)$  is homeomorphic to  $X$ . This example shows that the result obtained in Proposition 3 is the best possible in a sense.

Remark that condition (6) obviously implies the following one:

- (8) there is a natural number  $n > 1$  such that there exists an  $n$ -to-one mapping defined on  $X$  and having the property that the image of  $X$  under this mapping is homeomorphic to  $X$ ,

which is a weaker form of (6) and is related to (6) exactly in the same way as (5) is related to (4). By Proposition 3 a simple closed curve satisfies (8). The authors are not able to answer the following

**Problem 3.** Is every graph  $X$  satisfying (8) a simple closed curve?

Note that taking  $n=2$  in (8) we get (5), which is equivalent to (7) by Proposition 3. Thus it is enough to discuss Problem 3 with  $n > 2$  in (8) instead of  $n > 1$ .

In the next proposition, well — known characterizations of the simple closed curve via local homeomorphisms are presented. We recall these characterizations in order to generalize them replacing local homeomorphisms by the ones in the large sense (see Proposition 6 at the end of the paper). Concerning a proof of these

equivalences, one can show them applying covering space techniques (see [7], 6.5—6.8, p. 247—265), in particular using the following two known facts:

- (a) A local homeomorphism  $f: X \rightarrow Y$  from a graph  $X$  onto a graph  $Y$  is a covering projection (cf. [13], Corollary, p. 199 and [7], p. 247);
- (b) If  $f: X \rightarrow Y$  is a covering projection of the degree  $n$ , then  $\chi(X) = n \cdot \chi(Y)$ , where  $\chi$  denotes the Euler-Poincaré characteristic (cf. [7], 6.8.6, p. 265).

We present here another proof, which seems to be much more geometrical and which makes no use of concepts of algebraic topology.

**Proposition 4.** *The following are equivalent for a graph  $X$ :*

- (9) *for every natural number  $n$  there exists a local homeomorphism of the degree  $n$  of  $X$  onto itself;*
- (10) *there exists a local homeomorphism of the degree two of  $X$  onto itself;*
- (11) *there is a natural number  $n > 1$  and a local homeomorphism of the degree  $n$  of  $X$  onto itself;*
- (12) *for every natural number  $m$  there exists a local homeomorphism of the degree  $n > m$  of  $X$  onto itself;*
- (7)  *$X$  is a simple closed curve.*

**Proof.** (7)  $\Rightarrow$  (9). Put  $X = \{z: |z| = 1\}$ . where  $z$  denotes a complex number, fix an arbitrary natural  $n$ , and observe that the mapping  $f: X \rightarrow X$  defined by  $f(z) = z^n$  is a local homeomorphism of the degree  $n$ .

The implications (9)  $\Rightarrow$  (10) and (10)  $\Rightarrow$  (11) are trivial.

(11)  $\Rightarrow$  (12). Let  $n_0 > 1$  be a natural number such that there exists a local homeomorphism  $f$  of the degree  $n_0$  from  $X$  onto itself, and let  $m$  be an arbitrary natural number. Then there is a natural  $k$  such that  $k \cdot n_0 > m$ . Put  $n = k \cdot n_0$  and define  $g = f^k$  as the  $k$ -th iteration of  $f$ . Thus  $g: X \rightarrow X$  is a local homeomorphism of the degree  $n$  of  $X$  onto itself.

(12)  $\Rightarrow$  (7). Let  $m$  be the cardinality of the set of all ramification points and of all end points of  $X$ , and let  $n > m$  be such a natural number that there exists a local homeomorphism  $f$  of the degree  $n$  of  $X$  onto itself. Consider an arbitrary point  $y$  of  $X$ . Since  $n > m$ , there exists in  $f^{-1}(y)$  a point  $x$  of order 2. Let  $U$  be an open neighbourhood of  $x$  as in the definition of the local homeomorphism  $f$ . Since  $f|U$  is a homeomorphism and since  $f(U)$  is a neighbourhood of  $y = f(x)$ , we conclude that  $y$  is of order 2 in  $X$ . Hence (see [8], §51, V, Theorem 6, p. 294)  $X$  is a simple closed curve. The proof is finished.

Observe that the local homeomorphism  $f$  of the degree  $n$  considered in conditions (9) through (12) of Proposition 4 is assumed to be from  $X$  onto itself, i.e., the assumption  $f(X) = X$  is made on  $f$  in (9), (10), (11) and (12). One can note that this condition is essential in the implication from (11) to (12) because of the iteration  $f^k$  considered in the corresponding part of the proof above. Really, one can find two topologically different graph:  $X$  and  $Y$  and a local homeomorphism  $f$

of the degree  $n > 1$  from  $X$  onto  $Y$  such that  $X$  is not a simple closed curve (see e.g. [13], Example, p. 189).

However, in some other implications the hypothesis  $f(X) = X$  can be dispensed with. Namely one can modify Proposition 4 as follows.

**Proposition 5.** *The following are equivalent for a graph  $X$ :*

- (13) *for every natural number  $n$  there is a local homeomorphism of the degree  $n$  defined on  $X$ ;*
- (14) *for every natural number  $m$  there is a local homeomorphism of the degree  $n > m$  defined on  $X$ ;*
- (7)  *$X$  is a simple closed curve.*

Really, the proofs of the two implications  $(7) \Rightarrow (13) \Rightarrow (14)$  run exactly in the same way as in the proof of Proposition 4. To show that (14) implies (7) note that assuming (14) and repeating the arguments from the corresponding part of the proof of the implication. (12)  $\Rightarrow$  (7) of Proposition 4 we can show that  $Y = f(X)$  is a simple closed curve. By Theorem (1.1) of [13], p. 182 the graph  $X$  contains a simple closed curve, and by Corollary (7.31) of [13], p. 147,  $X$  has no end point. Since no open neighbourhood of a ramification point can be homeomorphically mapped into  $Y$ , we conclude that  $X$  contains no ramification point. Thus  $X$  is either an arc or a simple closed curve; since  $X$  contains a simple closed curve, it is one.

We remark here that one can prove Proposition 5 using again covering space techniques, in particular facts (a) and (b), as it could be done for Proposition 4.

Observe further that the assumption that the space  $X$  considered in Proposition 5 is a graph is essential and it cannot be relaxed to the assumption that  $X$  is a regular curve. It can be seen by the same example of a regular curve  $X$  which has been used previously (defined just after the proof of Proposition 3) and by a mapping  $f$  of  $X$  into the plane defined by  $f(r, \varphi) = (r, n \cdot \varphi)$ , where  $n$  is a natural number. Then  $f$  is a local homeomorphism of the degree  $n$ , and  $f(X)$  is homeomorphic to  $X$ , while  $X$  is not a simple closed curve.

It is natural to ask if one can neglect openness of the mappings under consideration in Propositions 4 and 5, i.e., replace local homeomorphisms by local homeomorphisms in the large sense. An answer to this question is affirmative. Namely observe that — in contrast to local homeomorphisms — those in the large sense need not be of a constant degree. Hence this last condition should be additionally assumed. But under this assumption any local homeomorphism in the large sense defined on a compact space becomes simply a local homeomorphism. In fact, it is proved in [11], Theorem 2, that if a continuous surjection  $f: X \rightarrow Y$  defined on a compact metric space  $X$  is locally one-to-one (this means that each point in  $X$  has an open neighbourhood  $U$  such that the restriction  $f|U$  is one-to-one) and of a constant degree, then  $f$  is open, which implies by Theorem 1

of [9], p. 54, that  $f$  is a local homeomorphism (cf. also [10], Proposition, p. 64). Therefore we have the following

**Proposition 6.** *The words “local homeomorphism” in conditions (9), (10), (11) and (12) of Proposition 4, and in conditions (13) and (14) of Proposition 5 can be replaced by “local homeomorphism in the large sense” or — equivalently — by a “locally one-to-one mapping”.*

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## ЛОКАЛЬНЫЕ ГОМЕОМОРФИЗМЫ И СООТВЕТСТВУЮЩИЕ ОТОБРАЖЕНИЯ НА ГРАФАХ

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### Резюме

В работе рассматриваются локальные гомеоморфизмы линейных графов. Получается несколько характеристик дуги и простой замкнутой кривой в классе графов, которые сформулированы на языке открытых отображений, локальных гомеоморфизмов, локальных гомеоморфизмов в более широком смысле и отображений с постоянной мощностью прообразов точек. Эти результаты проиллюстрированы примерами, показывающими, что в этих характеристиках класс графов не может быть увеличен. Кроме этого работа содержит открытые вопросы.