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ON SOME PROPERTIES OF THE SOLUTIONS OF THE THIRD ORDER NONLINEAR DIFFERENTIAL EQUATION WITH DELAY

FRANTIŠEK ŠIŠOLÁK

In the present paper we shall consider the differential equation

$$x'''(t)+p(t)x'(t)+f(t, x(h(t)))=0, \tag{a}$$

where $p(t)$, $h(t)\in C(I)$, $I=[t_0, \infty)$, $h(t)\leqq t$ for $t\in I$, $\lim_{t\to\infty} h(t)=\infty$ and $f(t, y)\in C(D)$, $D=I\times R$.

The motivation for the study of this equation comes from J. W. Heidl [3]. Heidel has investigated the behaviour of nonoscillatory solutions and the existence of oscillatory solutions of the differential equation

$$y'''+p(t)y'+q(t)y^r=0,$$

where r was assumed to be the quotient of odd integers.

Other early results somewhat connected with [3] were obtained by M. Greguš [1], M. Hanan [2], A. C. Lazer [5], M. Švec [8], I. Ličko and M. Švec [6], I. Kikuradze [4], I. G. Mikusinski [7].

In this paper some results of Heidel will be generalized.

We shall use the notation (A) for the following assumptions:

a) $p(t)\in C(I)$ and $p(t)\leqq 0$ for $t\in I=[t_0, \infty)$, $t_0>0$

b) $h(t)\in C(I)$, $h(t)\leqq t$ for $t\in I$ and $\lim_{t\to\infty} h(t)=\infty$

c) $f(t, y)\in C(D)$, $D=I\times R$ and $f(t, y)y<0$ for $y\neq 0$ and $t\in I$.

Theorem 1. *Suppose that (A) holds. If $x(t)$ is a nonoscillatory solution of (a) defined on the interval I, then there exists a number $t_1\geqq t_0$ such that either $x(t)x'(t)>0$ or $x(t)x'(t)\leqq 0$ for $t\geqq t_1$.*

Proof. Assume that $x(t)$ is a nonoscillatory solution of (a) such that $x(h(t))>0$ for $t>t_2\geqq t_0$. The function $x'(t)$ has not a finite limit of zero points. If $x'(t)$ has an finite number of zeros, the Theorem is clear.

Suppose that $\{u_n\}$ is an increasing sequence of all zeros of the function $x'(t)$ and $t_2\leqq u_1$. Multiplying (a) by $x'(t)$ and integrating between u_n and u_{n+1} yields

$$(1)\qquad -\int_{u_n}^{u_{n+1}}[x''(t)]^2\,dt+\int_{u_n}^{u_{n+1}}p(t)[x'(t)]^2\,dt+\int_{u_n}^{u_{n+1}}x'(t)f(t,\,x(h(t)))\,dt=0.$$

$n=1, 2, \ldots$. From (1) it follows that $x'(t)<0$ for $t\in(u_n,\,u_{n+1})$, $n=1, 2, \ldots$. Then $x'(t)\leqq 0$ for all $t\in[u_1,\,\infty)$.

If $x(t)<0$, the proof is similar.

Lemma. *Let $y(t)\in C^2(I)$, $I=[t_0,\,\infty)$, $t_0\geqq 0$. Suppose that $y(t)>0$ for $t\in I$ and $\lim_{t\to\infty} y(t)<\infty$ if $y'(t)\geqq 0$.*

Then

$$\liminf_{t\to\infty}|t^\alpha y''(t)-\alpha t^{\alpha-1}y'(t)|=0 \quad \textit{for} \quad \alpha\leqq 2.$$

The proof can be found in [3].

Remark 1. Under the assumptions $y(t)\in C^2(I)$, $y(t)<0$ and $\lim_{t\to\infty} y(t)>-\infty$ if $y'(t)\leqq 0$ the conclusion of the Lemma is also valid.

Theorem 2. *Suppose that (A) holds and $-\infty<-M\leqq p(t)t^\alpha$ for $t\in I$, $\alpha\leqq 2$. Let the function $f(t,\,y)$ be nonincreasing with respect to y and let*

$$L\int_{t_0}^{\infty}s^\alpha f(s,\,L)\,ds=-\infty$$

for every $L\neq 0$. If $x(t)$ is a nonoscillatory solution of (a) defined on the interval I such that $x(t)x'(t)\leqq 0$, then $\lim_{t\to\infty}x(t)=0$.

Proof. Let $x(t)$ be a solution of (a) such that $x(h(t))>0$ and $x'(t)\leqq 0$ for $t>t_1\geqq t_0\geqq 1$. Suppose that $\lim_{t\to\infty}x(t)=L>0$. Clearly, $\lim_{t\to\infty}x(h(t))=L$. It follows from the hypotheses of the Theorem that

$$0\leqq\int_{t_1}^{\infty}p(t)t^\alpha x'(t)\,dt\leqq -M[L-x(t_1)],$$

$$0>\int_{t_1}^{\infty}t^{\alpha-2}x'(t)\,dt>L-x(t_1).$$

Multiplying (a) by t^α, $\alpha\leqq 2$, integrating from t_1 to t and using the last two inequalities, we obtain

$$(2)\qquad t^\alpha x''(t)-\alpha t^{\alpha-1}x'(t)\geqq K-\int_{t_1}^{t}s^\alpha f(s,\,x(h(s)))\,ds,$$

where K is a finite constant. Then by the hypothesis of the Theorem

$$\lim_{t\to\infty}\left[K-\int_{t_1}^{t} s^{\alpha}f(s, x(h(s)))\,ds\right] \geqq \lim_{t\to\infty}\left[K-\int_{t_1}^{t} s^{\alpha}f(s, L)\,ds\right]=\infty.$$

However, by Lemma

$$\liminf_{t\to\infty} |t^{\alpha}x''(t)-\alpha t^{\alpha-1}x'(t)|=0.$$

This contradiction proves the Theorem.

In the case $x(t)<0$ and $x'(t)\geqq 0$ the proof of Theorem 2 is similar.

The proofs of the following three theorems will not be given here. They are similar to those in [3].

Theorem 3. *Suppose that (A) holds and let*

$$\int_{t_0}^{\infty} sp(s)\,ds>-\infty.$$

If $x(t)$ is a nonoscillatory solution of (a) defined on the interval I, then there is a number $t_1\geqq t_0$ such that $x(t)x'(t)>0$ for all $t\in[t_1, \infty)$.

Theorem 4. *Suppose that (A) holds and let*

$$-\frac{2}{t^2}\leqq p(t)\leqq 0$$

for $t\in I$. If $x(t)$ is a nonoscillatory solution of (a) defined on the interval I, then there exists a number $t_1\geqq t_0$ such that $x(t)x'(t)>0$ for all $t\geqq t_1$.

Theorem 5. *Suppose that (A) holds and the function $f(t, y)$ is nonincreasing with respect to y.*

Let $L\int_{t_0}^{\infty} s^2 f(s, L)\,ds=-\infty$ for every $L\neq 0$. If $x(t)$ is a nonoscillatory solution of (a) defined on the interval I such that $x(t)x'(t)>0$, then $\lim_{t\to\infty}|x(t)|=\infty$.

Theorem 6. *Suppose that (A) holds and the function $f(t, y)$ is nonincreasing with respect to y. Let*

$$L\int_{t_0}^{\infty} f(s, L)\,ds=-\infty, \quad \text{for} \quad L\neq 0.$$

If $x(t)$ is a nonoscillatory solution of (a) defined on the interval I such that

$$x(t)x'(t)\geqq 0, \quad \text{then} \quad \lim_{t\to\infty}|x(t)|=\lim_{t\to\infty}|x'(t)|=\lim_{t\to\infty}|x''(t)|=\infty.$$

Proof. Assume that $x(t)>0$. It follows from the properties of the functions $x(t)$, $x'(t)$ and $h(t)$ that there exists a number $t_1\geqq t_0$ such that $x(h(t))>x(t_0)=L>0$.

By integration (a) from t_1 to t we obtain

$$x''(t)=x''(t_1)-\int_{t_1}^{t} p(s)x'(s)\,ds-\int_{t_1}^{t} f(s,x(h(s)))\,ds \geqq \tag{3}$$

$$\geqq x''(t_1)-\int_{t_1}^{t} f(s,L)\,ds.$$

It follows from (3) that $\lim_{t\to\infty} x''(t)=\infty$. Since $x'''(t)\geqq 0$, $\lim_{t\to\infty} x'(t)=\lim_{t\to\infty} x(t)=\infty$.

Remark 2. If we replace the hypothesis c) in (A) by c') $f(t,y)\in C(D')$, $D'=I\times R^+$ and $f(t,y)<0$ for every $(t,y)\in D'$ then the conclusions of Theorems 1—6 are valid.

The proof of Remark 2 is the same as the proofs of Theorem 1—6. The function $f(t,y)=q(t)y^k$, where $q(t)<0$, $y>0$ for $t\in I$ and $k\in R^+$ satisfies the hypothesis c'). Consequently Theorems 1—6 are valid for positive solutions of the differential equation

$$y'''+p(t)y'+q(t)y^k=0.$$

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О НЕКОТОРЫХ СВОЙСТВАХ РЕШЕНИЙ НЕЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ТРЕТЬЕГО ПОРЯДКА С ЗАПАЗДЫВАЮЩИМ АРГУМЕНТОМ

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Резюме

В работе рассматриваются некоторые свойства неосцилляционных решений дифференциального уравнения

$$x'''(t) + p(t)x'(t) + f(t, x(h(t))) = 0$$

в промежутке $[t_0, \infty)$.