Mathematica Slovaca

Milan Oslej

# Remark on Sturm-Liouville functions

*Mathematica Slovaca*, Vol. 33 (1983), No. 1, 41--44

Persistent URL: http://dml.cz/dmlcz/136317

Math. Slovaca 33, 1983, No. 1, 41—44

# REMARK ON STURM—LIOUVILLE FUNCTIONS

MILAN OSLEJ

Consider a differential equation

$$y'' + q(x)y = 0 \tag{q}$$

where $q(x) \in C_0(a, \infty)$, $a \geqq 0$.

Denote

$$M_k(W, \lambda) = \int_{x_k}^{x_{k+1}} W(x)|y(x)|^\lambda \mathrm{d}x \tag{1}$$

$\lambda > -1$, $k = 1, 2, \ldots$, where $y(x)$ is an arbitrary non-trivial solution of $(q)$, $x_1, x_2, \ldots$ is any finite or infinite sequence of consecutive zeros of any non-trivial solution $z(x)$ of $(q)$, which may or may not be independent of $y(x)$ and the function $W(x) > 0$ fullfils certain conditions concerning higher monotonicity.

L. Lorch, M. E. Muldoon and P. Szego derived in [2] simple sufficient conditions for the sequence (1) to be monotonic of the higher order on $(a, \infty)$. In this paper there will be given an extension of the above mentioned result from [2].

## 1. Definitions and notations

A function $\varphi(x)$ is said to be monotonic of order $n$ or $n$-times monotonic on an interval $I$, if

$$(-1)^i \varphi^{(i)}(x) \geqq 0, \quad i = 0, 1, 2, \ldots, n, \quad x \in I \tag{2}$$

For such a function we write $\varphi(x) \in M_n(I)$ or $\varphi(x) \in M_n(a, b)$ in case that $I$ is an interval $(a, b)$. In case the strict inequality holds throughout (2) we write $\varphi(x) \in M_n^*(I)$.

We say that $\varphi(x)$ is completely monotonic on $I$, if (2) holds for $n = \infty$.

A sequence $\{\mu_k\}_{k=1}^{\infty}$ denoted simply $\{\mu_k\}$ is said to be $n$-times monotonic, if

$$(-1)^i \Delta^i \mu_k \geqq 0, \quad i = 0, 1, \ldots, n, \quad k = 0, 1, \ldots \tag{3}$$

Here $\Delta\mu_k = \mu_{k+1} - \mu_k$, $\Delta^2\mu_k = \Delta(\Delta\mu_k)$ etc. For such a sequence we write $\{\mu_k\} \in M_n$.

In case the strict inequality holds throught (3) we write $\{\mu_k\} \in M_n^*$. $\{\mu_k\}$ is called completely monotonic, if (3) holds for $n = \infty$.

As usual, $\varphi(x) \in C_n(I)$ means that $\varphi(x)$ has continuous derivatives including to the $n$-th order. $D_\xi[\varphi(\xi)]$ denotes the first derivative $\frac{d\varphi(\xi)}{d\xi}$ and $D_\xi^n[\varphi(\xi)]$ denotes the $n$-th derivative $\frac{d^n\varphi(\xi)}{d\xi^n}$.

## 2. New result

**Theorem.** Let differential equation $(q)$ be oscillatory on an interval $(a, \infty)$, let $n \geq 0$ be an integer and let there exists the function $\psi(x) > 0$, $\psi(x) \in C_2(a, \infty)$ satisfying

$$0 < \lim_{x \to \infty} (\psi''\psi^3 + q\psi^4) \leq \infty$$

Let $\psi^2(x) \in M_n(a, \infty)$ and $0 \neq D_x(\psi''\psi^3 + q\psi^4) \in M_n(a, \infty)$. Let $W(x)$ be a function satisfying

$$W(x) > 0, \quad (-1)^n W^{(n)}(x) \geq 0.$$

Let $y(x)$ be an arbitrary non-trivial solution of $(q)$ and $x_1, x_2, \ldots$ any sequence of consecutive zeros of any non-trivial solution $z(x)$ of $(q)$ which may or may not be linearly independent of $y(x)$. Then for $\lambda > -1$

$$\left\{\int_{x_k}^{x_{k+1}} \frac{W(x)}{\psi^2(x)} \left|\frac{y(x)}{\psi(x)}\right|^\lambda dx\right\} \in M_n^* \tag{4}$$

and in special case for $\lambda = 0$

$$(-1)^i \Delta^{i+1} x_k > 0, \quad k = 1, 2, \ldots, \quad i = 0, 1, \ldots, n \tag{5}$$

Remark. Hence, under the hypotheses of the theorem

$$\left\{\int_{x_k}^{x_{k+1}} \bar{W}(x) \left|\frac{y(x)}{\psi(x)}\right|^\lambda dx\right\} \in M_n^* \tag{6}$$

because (4) is still valid when $W(x)$ is replaced by $\bar{W}(x) \cdot \psi^2(x)$, since this last function belongs to $M_n(a, \infty)$.

If $\psi^{2+\lambda}(x) \in M_n(a, \infty)$ holds, then we can write

$$\left\{\int_{x_k}^{x_{k+1}} \bar{W}(x) |y(x)|^\lambda dx\right\} \in M_n^* \tag{7}$$

because (4) is still valid when $W(x)$ is replaced by $\bar{W}(x) \cdot \psi^{2+\lambda}(x)$.

Proof of theorem. Let us have the differential equation $(q)$. The change of variable

$$\xi=\int_a^x \frac{du}{\psi^2(u)} \tag{8}$$

where $\psi>0$, $\psi\in C_2(a,\infty)$ and integral $\int_a^\infty \frac{du}{\psi^2(u)}$ is assumed divergent, transforms $(q)$ into

$$D_\xi^2\eta(\xi)+\varphi(\xi)\eta=0 \tag{9}$$

where $\eta(\xi)=y(x)/\psi(x)$ and $\varphi(\xi)=\psi''(x)\cdot\psi^3(x)+q(x)\psi^4(x)$.

Hence, the mapping (8) takes the $x$-interval $(a,\infty)$ into the $\xi$-interval $(0,\infty)$. Using the change of variable (8) we get

$$\int_{x_k}^{x_{k+1}} W(x)\frac{1}{\psi^2(x)}\left|\frac{y(x)}{\psi(x)}\right|^\lambda dx=\int_{\xi_k}^{\xi_{k+1}} W[x(\xi)]|\eta(\xi)|^2\,d\xi$$

where $\xi_1, \xi_2, \ldots$ are the zeros of solution $\zeta(\xi)$ of (9) corresponding, respectively, to the zeros $x_1, x_2, \ldots$ of $z(x)$ (here $\zeta(\xi)=z(x)$).

In case $n\geqslant 2$ and $x_1>a$ the present theorem will follow from theorem 3.3 of [2] as applied to the equation (9), provided we show that

$$D_\xi[\varphi(\xi)]>0,\quad D_\xi[\varphi(\xi)]\in M_n(0,\infty) \tag{10}$$

and that

$$W[x(\xi)]>0,\quad W[x(\xi)]\in M_n(0,\infty). \tag{11}$$

Now,

$$D_\xi[\varphi(\xi)]=D_x[\psi''\psi^3+q\psi^4]\cdot D_\xi[x(\xi)]=\psi^2 D_x[\psi''\psi^3+q\psi^4]>0.$$

But $\psi^2(x)$ belongs to $M_n(a,\infty)$ so that a slight modification of ([2], lemma 2.2) in which $p'(x)\leqslant 0$ replaces $p(x)<0$ and $\geqslant$ replaces $>$ in (2.7), implies that $D_\xi[x(\xi)]\in M_n(0,\infty)$.

Hence, in wiew of ([2], lemma 2.1), our hypotheses on $W(x)$ show that $W[x(\xi)]\in M_n(0,\infty)$, and (11) holds. Since $D_x[\varphi(\xi)]$, considered as a function of $x$, belongs to $M_n(a,\infty)$ and $D_\xi[x(\xi)]$ belongs $M_n(0,\infty)$, ([2], lemma 2.1) shows that $D_\xi[\varphi(\xi)]\in M_n(0,\infty)$. Hence, (10) holds and the proof of theorem is complete, in case $n\geqslant 2$ and $x_1>a$. The case $n=0$ is obvious. The case $n=1$, $x_1=a$ (for all $n$) we get analogously as in proof of theorem 3.1 of [3]. (5) we get from (4), if $\lambda=0$, $W(x)=\psi^2(x)$.

Example. Let us have a differential equation

$$y''+(e^{2x}-v^2)\cdot y=0 \tag{12}$$

which has solutions in the form $y=C_v(e^x)$, where $C_v$ is Bessel function of order $v$.

It is obvious that the sufficient conditions from [2] give no result on higher monotonicity of sequence $\{M_k\}$ from (1) for differential equation (12).

If we take $\psi(x)=e^{-x/2}$, then $\psi^2(x)\in M_\infty(0, \infty)$ and we get

$$(\psi''\psi^3+q\psi^4)=[1-(v^2-1/4)\cdot e^{-2x}]\in M_\infty(0, \infty)$$

for $|v|>1/2$.

**Result.** If $|v|>1/2$, then

$$\left\{\int_{x_k}^{x_{k+1}} e^x\cdot W(x)|y(x)\cdot e^{x/2}|^\lambda \,\mathrm{d}x\right\}\in M_n^*$$

and

$$\left\{\int_{x_k}^{x_{k+1}} W(x)|y(x)\cdot e^{x/2}|^\lambda \,\mathrm{d}x\right\}\in M_n^*.$$

holds for $\lambda>-1$.

Hence, in wiew of $[e^{(-x/2)\cdot\lambda}]\in M_\infty(0, \infty)$ for $\lambda\geq 0$ and if $W(x)=e^{(-x/2)\cdot\lambda}$, then

$$\left\{\int_{x_k}^{x_{k+1}} |y(x)|^\lambda \,\mathrm{d}x\right\}\in M_\infty^*$$

holds for $|v|>1/2$, $\lambda\geq 0$.

Concluding this paper I should like to express my gratitude to Doc. Dr. M. Háčik, CSc., for his valuable remarks.

## REFERENCES

[1] LORCH, L.—SZEGO, P.: Higher monotonicity properties of certain Sturm-Liouville functions. Acta Math. 109, 1963, 55—73.

[2] LORCH, L.—MULDOON, M. E.—SZEGO, P.: Higher monotonicity properties of certain Sturm-Liouville functions III. Can. J. Math. 22, 1970, 1238—1265.

[3] LORCH, L.—MULDOON, M. E.—SZEGO, P.: Higher monotonicity properties of certain Sturm-Liouville functions IV. Can J. Math. 24, 1972, 349—368.

Received February 6, 1981

*Katedra matematiky*
*Vysoká škola dopravy a spojov*
*Marxa-Engelsa 25*
*010 88 Žilina*

## ЗАМЕТКА О ФУНКЦИЯХ ШТУРМА–ЛИУВИЛЛЯ

Milan Oslej

### Резюме

В этой статье исследуются достаточные условия для того, чтобы последовательности, которые зависят от нулей решения дифференциального уравнения $(q)$, были монотонные высшего порядка в промежутке $(a, \infty)$.