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AN OPTIMAL CONTROL PROBLEM FOR AN ELLIPTIC VARIATIONAL INEQUALITY

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We shal be dealing with an optimal control for an elliptic variational inequality with controls involved both in the operator of the problem and in the right hand side. A similar problem with controls only in the right hand side has been solved in the book [2].

1. The Existence Theorem

Let U with a norm $\|\cdot\|_U$ be a reflexive Banach space of controls, $U_{ad} \subset U$ a set of admissible controls. We assume U_{ad} be convex, closed and bounded in U.

We assume further a reflexive Banach space V with a norm $\|\cdot\|$ and a convex closed subset $K \subset V$. V^* means a dual space of V with a norm $\|\cdot\|_*$ and a duality pairing $[\cdot,\cdot]$ between V^* and V.

Let $\{A(e)\}$, $A(e)\colon K \to V^*$ for every $e \in U_{ad}$, be a family of operators satisfying the following assumptions:

(1) $A(e)$ is for every $e \in U_{ad}$ strongly monotone i.e.
$[A(e)u - A(e)v,\ u - v] > 0$ for every $u,\ v \in K,\ u \neq v,\ e \in U_{ad}$

(2) $A(e)$ is for every $e \in U_{ad}$ hemicontinuous i.e.
$\lim_{t \to 0} [A(e)(u + t(v - u)),\ w] = [A(e)u,\ w]$
for every $e \in U_{ad},\ u,\ v \in K,\ w \in V$

(3) $\{A(e)\}$ is uniformly bounded i.e.
$\|A(e)v\|_* \leq C$, if $\|e\|_U \leq C_1$ and $\|v\| \leq C_2$

(4) $\{A(e)\}$ is uniformly coercive i.e. there exist such $v_0 \in K$ and a real function $r\colon [0, \infty) \to R$, $\lim_{t \to \infty} r(t) = \infty$, that
$[A(e)v,\ v - v_0] \geq \|v\| r(\|v\|)$ for every $v \in K$

(5) $A(\cdot)v\colon U_{ad} \to V^*$ is for every $v \in K$ strengthenly continuous i.e.
$e_n \rightharpoonup e_0$ (weakly) in U implies $A(e_n)v \to A(e_0)v$ (strongly) in V^*.

Let the operator B: $U_{ad} \to V^*$ be strengthenly continuous and $f \in V^*$. Under the assumptions (1), (2), (3) the operator $A(e): K \to V^*$ is pseudomonotone for every $e \in U_{ad}$ (def. in [3]) and then due to the theorem from [3] there exists a unique solution $u(e) \in K$ of a variational inequality

(6) $[A(e)u(e), v-u(e)] \geq [f+B(e), v-u(e)]$
for every $v \in K$

Our aim is to solve the following optimal control problem:

Problem P. To find a control $e_0 \in U_{ad}$ which fulfills:

(7) $[A(e_0)u(e_0), v-u(e_0)] \geq [f+B(e_0), v-u(e_0)]$
for every $v \in K$

(8) $\|Cu(e_0)-z_d\|^2_{\mathscr{H}} = \min_{e \in U_{ad}} \|Cu(e)-z_d\|^2_{\mathscr{H}}$,

where $u(e) \in K$ is a solution of (6), $\mathscr{H}$ is a Hilbert space, $C \in L(V, \mathscr{H})$ is a linear control operator, $z_d \in \mathscr{H}$ is a fixed element.

Theorem 1. *There exists at least one solution $e_0 \in U_{ad}$ of Problem P.*

Proof. We have $J(e) = \|Cu(e)-z_d\|^2_{\mathscr{H}} \geq 0$ for every $e \in U_{ad}$. Hence $\inf_{e \in U_{ad}} J(e) \geq 0$. Let $(e_n)_{n=1}^{\infty}$ be the minimizing sequence for a functional $J(\cdot)$ i.e.

(9) $\lim_{n\to\infty} J(e_n) = \inf_{e \in U_{ad}} J(e)$

As the set U_{ad} is convex and closed in the reflexive space U it is weakly closed in U. Then there exist such a subsequence of $(e_n)_{n=1}^{\infty}$ (we denote it again by $(e_n)_{n=1}^{\infty}$) and the element $e_0 \in U_{ad}$ that

(10) $e_n \rightharpoonup e_0$ (weakly in U)

Denoting $u_n = u(e_n) \in K$, $n = 1, 2, \dots$ we have

(11) $[A(e_n)u_n, v-u_n] \geq [f+b(e_n), v-u_n]$
for every $v \in K$, $n = 1, 2, \dots$

Inserting $v_0 \in K$ in (11) we arrive at

(12) $[A(e_n)u_n, u_n - v_0] \leq [f+B(e_n), u_n - v_0]$

Using the uniform coerciveness of a system $\{A(e)\}$ and the streghten continuity of B we obtain

(13) $\|u_n\| r(\|u_n\|) \leq C_1 \|u_n\| + C_2$

As $\lim_{t\to\infty} r(t) = \infty$ we have

(14) $\|u_n\| \leqslant C, \quad n=1, 2, \ldots$

We can now extract such a subsequence of $(u_n)_{n=1}^{\infty}$ denoted again by $(u_n)_{n=1}^{\infty}$ that

(15) $u_n \rightharpoonup u$ (weakly in V)

Moreover $u \in K$, because $u_n \in K$, $n=1, 2, \ldots$ and K is weakly closed in V.

As a family of operators $\{A(e)\}$ is uniformly bounded (see (2)) we have $\|A(e_n)u_n\|_* \leqslant C$ for $n=1, 2, \ldots$ Then there exists an element $\chi \in V^*$ such that

(16) $A(e_n)u_n \rightharpoonup \chi$ (weakly in V^*)

Monotonicity of $A(e_n)$ implies

(17) $[A(e_n)u_n - A(e_n)v, u_n - v] \geqslant 0$
for every $v \in K$, $n=1, 2, \ldots$

Inserting $v=u$ in (11) we obtain using (10), (15) and the strenghtened continuity of the operator B,

(18) $\limsup \, [A(e_n)u_n, u_n - u] \leqslant 0$

and combining with (16)

(19) $\limsup \, [A(e_n)u_n, u_n] \leqslant [\chi, u]$

Taking into account relations (15), (16), (17), (19) and the strenghtened continuity of the operator $A(\cdot)v$: $U_{ad} \to V^*$ we arrive at

(20) $[\chi - A(e_0)v, u - v] \geqslant 0$ for every $v \in K$

Let $v = u + t(w-u)$, $t \in (0, 1)$, $w \in K$. Then we have

(21) $[\chi - A(e_0)(u + t(w-u)), u - w] \geqslant 0$
for every $w \in K$, $t \in (0, 1)$

Making use of hemicontinuity of $A(e_0)$ we obtain after $t \to 0$ and putting again $w = v$

(22) $[A(e_0)u, u-v] \leqslant [\chi, u-v]$ for every $v \in K$

Puting $v=u$ in (17) we have $[A(e_n)u_n, u_n - u] \geqslant [A(e_n)u, u_n - u]$. The strenghtened continuity of $A(\cdot)u$ and the weak convergence $u_n \rightharpoonup u$ imply immediatly $\lim_{n\to\infty} [A(e_n)u, u_n - u] = 0$ and hence $\liminf \, [A(e_n)u_n, u_n - u] \geqslant 0$. Comparing with (18) we have

(23) $\lim_{n\to\infty} [A(e_n)u_n, u_n - u] = 0$

Relations (16), (22), (23) enable us to estimate

$$(24)\quad [A(e_0)u,\ u-v]\leq \lim_{n\to\infty} [A(e_n)u_n,\ u_n-v]$$

for every $v\in K$

We are coming now to the conclusion that the element $u\in K$ is a solution of a variational inequality

$$(25)\quad [A(e_0)u,\ u-v]\leq[f+B(e_0),\ u-v]\quad \text{for every } v\in K,$$

having used (24), (11), (15) and the strenghtened continuity of B.

Hence we have proved

$$(26)\quad u=u(e_0),\ u(e_n)\rightharpoonup u(e_0)\quad \text{(weakly in } V)$$

what implies

$$(27)\quad Cu(e_n)\rightharpoonup Cu(e_0)\quad \text{(weakly in } \varkappa)$$

A functional g: $\varkappa\to R$, $g(w)=\|w-z_d\|^2_\varkappa$, $w\in\varkappa$ is weakly lower semicontinuous and therefore

$$(28)\quad J(e_0)=\|cu(e_0)-z_d\|^2_\varkappa\leq\liminf\|Cu(e_n)-z_d\|^2_\varkappa- \\ =\liminf J(e_n)=\inf J(e_n)$$

which completes the proof of (8) and of the Theorem.

Remark. It is an opened question to gain further information about the set $X\subset U_{ad}$ of solutions of Problem P. We have only verified that $X\neq\emptyset$. The core of the problem is that ve have been solving the control problem governed by the variational inequality and hence the minimized functional J with respect to $e\in U_{ad}$ is not convex.

2. The Example

We shall investigate the optimal control problem for the thickness function of a thin plate with an obstacle.

Let $\Omega\subset R^2$ be the middle plane of a plate. We assume that Ω has the Lipschitz boundary $\partial\Omega=\Gamma_1\cup\Gamma_2\cup\Gamma_3$. We suppose that a part Γ_1 of the boundary of the plate is clamped, a part Γ_2 is simply supported and a part Γ_3 is free. An obstacle for the deflection of the plate can be described by the function φ: $\bar\Omega\to R$ satisfying the inequality $\varphi(x,y)\leq 0$ on $\Gamma_1\cup\Gamma_2$.

We denote

$$(29)\quad V=\left\{v\in H^2(\Omega)\ v=0 \text{ on } \Gamma_1\cup\Gamma_2, \frac{\partial v}{\partial n}=0 \text{ on } \Gamma_1\right\},$$

where $H^2(\Omega)$ is a Sobolev space of all functions from $L_2(\Omega)$ which have the distributive derivatives up to the 2-nd order from $L_2(\Omega)$. The boundary conditions are satisfied in the sense of traces ([4]).

Functions expressing deflections of the plate belong to the set

(30) $K=\{v \in V,\ v(x, y) \geqslant \varphi(x, y) \text{ a.g. in } \Omega\}$

The thickness functions $e\colon \Omega \to R$ play the role of controls. We assume the set U_{ad} of admissible controls in the form

(31) $U_{ad}=\{e \in H^2(\Omega) \| e\|^2_{H^2(\Omega)} \leqslant M,\ e(x, y) \geqslant m>0 \text{ on } \Omega\}$

Due to the imbedding theorems in the Sobolev space $H^2(\Omega)$, K is a convex closed subset of V and U_{ad} a convex closed and bounded subset of the space $U=H^2(\Omega)$.

The operators $A(e)\colon K \to V^*$, $e \in U_{ad}$, of the problem are of the form

$$[A(e)u, v]=\frac{E}{12(1-\mu^2)} \iint_{\Omega} e^3(x, y) \left[\left(\frac{\partial^2 u}{\partial x^2}+\mu \frac{\partial^2 u}{\partial y^2}\right) \frac{\partial^2 v}{\partial x^2}+\right.$$

(32) $$\left. +2(1-\mu) \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y}+\left(\frac{\partial^2 u}{\partial y^2}+\mu \frac{\partial^2 u}{\partial x^2}\right) \frac{\partial^2 v}{\partial y^2}\right] \mathrm{d}x\, \mathrm{d}y,$$

$$u \in K,\quad v \in V,\quad e \in U_{ad},\quad \mu \in (0, 1)$$

The operators $A(e)$ satisfy the assumptions (1)—(5). If the outer force has the form of linear bounded functional $f \in V^*$, then a deflection of the plate $u(e) \in K$ is a solution of a variational inequality

(33) $[A(e), v-u(e)] \geqslant [f, v-u(e)]$ for every $v \in K$

For simplicity we do not consider the operator $B\colon U_{ad} \to V^*$. A cost functional can be of the form

(34) $$J(e)=\iint_{\Omega} (Tu(e)-z_d)^2 \,\mathrm{d}x\, \mathrm{d}y,\ e \in U_{ad},$$

where $I\colon V \to L_2(\Omega)$ is the identity operator, $z_d \in L_2(\Omega)$, or

(35) $$\hat{J}(e)=\int_{\Gamma_3} (Tu(e)-z_d)^2 \,\mathrm{d}s,\quad e \in U_{ad},$$

where $T\colon V \to L_2(\Gamma_3)$ is the operator of traces, $z_d \in L_2(\Gamma_3)$. The optimality conditions for the functionals (34) and (35) mean the minimizing of the distance betwean the deflection of the plate $u(e)$ and the priscribed function z_d on Ω, or Γ_3. Due to the Theorem 1. there exists the optimal thickness function $e_0\colon \Omega \to R$ which minimizes the functional J or $\hat{J}$ on the set of admissible functions U_{ad}.

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ОДНА ЗАДАЧА ОПТИМАЛЬНОГО УПРАВЛЕНИЯ ДЛЯ ЭЛЛИПТИЧЕСКОГО ВАРИАЦИОННОГО НЕРАВЕНСТВА

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Резюме

В работе рассматривается задача оптимального управления для эллиптического вариационного неравенства с управлениями в операторе и в правой части. Доказывается существование оптимального управления. Показывается пример оптимализации толщины тонкой пластины с препятствием.