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TOLERANCES AND THE CHINESE REMAINDER THEOREM

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By a *tolerance* on an algebra $\mathfrak{A}=(A, F)$ is meant a reflexive and symmetric binary relation T on A satisfying the Substitution Property with respect to each $f\in F$, i.e. $\langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)\rangle \in T$ for each n-ary $f\in F$ provided $\langle a_i, b_i\rangle \in T$ for $i=1, \ldots, n$. The set $LT(\mathfrak{A})$ of all tolernces on $\mathfrak{A}$ is an algebraic lattice with respect to the set inclusion, see [1], [2]. Despite $\Theta\in LT(\mathfrak{A})$ for every congruence Θ on $\mathfrak{A}$, the congruence lattice is not a sublattice of $LT(\mathfrak{A})$ in a general case, see [1]. However, some congruence properties can be also generalized for tolerances. The object of this paper is to characterize the solvability of systems of tolerances in the sense of the Chinese Remainder Theorem [4].

Let $\mathfrak{A}=(A, F)$ be an algebra and $\emptyset\neq M\subseteq A$. Denote by $T(M)$ or $\Theta(M)$ the least tolerance or congruence, respectively, on $\mathfrak{A}$ collapsing all elements of M, i.e. $T(M)=\cap\{T\in LT(\mathfrak{A});\ \langle a, b\rangle\in T$ for each $a, b\in M\}$, and analogously for $\Theta(M)$. If $M=\{a_1, \ldots, a_n\}$, write briefly $T(a_1, \ldots, a_n)$ or $\Theta(a_1, \ldots, a_n)$.

By a *system of n tolerances over* $\mathfrak{A}$ is meant

$$\text{(S)} \qquad p_i(x_1, \ldots, x_m)T_iq_i(x_1, \ldots, x_m) \quad \text{for } i=1, \ldots, n,$$

where p_i, q_i are either m-ary polynomials over $\mathfrak{A}$ or elements of A and $T_i\in LT(\mathfrak{A})$ for each $i=1, \ldots, n$. A system (S) is *solvable* (over $\mathfrak{A}$) is there exists a sequence $\langle a_1, \ldots, a_m\rangle$ $(a_i\in A)$, the so called *solution* of (S), such that

$$\langle p_i(a_1, \ldots, a_m), q_i(a_1, \ldots, a_m)\rangle \in T_i$$

is valid for $i=1, \ldots, n$.

In this terminology, an algebra $\mathfrak{A}=(A, F)$ satisfies the Chinese Remainder Theorem (see [4]) if every system of congruences

$$a_i\Theta_ix \quad \text{for } i=1, \ldots, n\geq 3$$

is solvable for $\Theta_i\supseteq\cap\{\Theta(a_i, a_j);\ j=1, \ldots, i-1, i+1, \ldots, n\}$.

This formulation can be generalized also for tolerances.

Definition. *Let $n \geq 3$ be an integer. An algebra $\mathfrak{A} = (A, F)$ is said to satisfy the n-Tolerance Chinese Remainder Theorem if every system of tolerances*

$$a_i T_i x \quad \text{for } i = 1, \ldots, n$$

is solvable for $T_i \supseteq \cap\{T(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n);\ j = 1, \ldots, i-1, i+1, \ldots, n\}$.

A variety $\mathcal{V}$ of algebras satisfies the *n-Tolerance Chinese Remainder Theorem* if each $\mathfrak{A} \in \mathcal{V}$ satisfies the n-Tolerance Chinese Remainder Theorem. We give the following Mal'cev type characterization of this property.

Theorem. *Let $n \geq 3$ be an integer. For every variety $\mathcal{V}$ of algebras the following conditions are equivalent:*

(1) $\mathcal{V}$ satisfies the n-Tolerance Chinese Remainder Theorem

(2) There exists an n-ary polynomial p of $\mathcal{V}$ such that

$$x = p(x, \ldots, x, y) = p(x, \ldots, x, y, x) = \ldots = p(y, x, \ldots, x).$$

Proof. (1) $\Rightarrow$ (2): Let $F_n(x, \ldots, x_n) = \mathfrak{A} = (A, F)$ be a free algebra of $\mathcal{V}$ with free generators $x_1, \ldots, x_n$. Since $\Theta_i = \cap\{\Theta(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n);\ j = 1, \ldots, i-1, i+1, \ldots, n\} \supseteq \cap\{T(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n);\ j = 1, \ldots, i-1, i+1, \ldots, n\}$, the system $x_i \Theta_i x$ $(i = 1, \ldots, n)$ has by (1) a solution $b \in A$. Since $b \in F_n(x_1, \ldots, x_n)$, there exists an n-ary polynomial p of $\mathcal{V}$ such that $b = p(x_1, \ldots, x_n)$. However,

$$\langle x_i, b \rangle \in \Theta_i \subseteq \Theta(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n),$$

thus $x_i = p(x_i, \ldots, x_i, x_j, x_i, \ldots, x_i)$ is true in $\mathfrak{A}/\Theta_i$ for each $j \neq i$. Since $\mathfrak{A}/\Theta_i$ is a free algebra of $\mathcal{V}$ with two free generators, this identity is true in $\mathcal{V}$. Varying $i = 1, \ldots, n$ and $j \in \{1, \ldots, i-1, i+1, \ldots, n\}$, we obtain all the identities of (2).

(2) $\Rightarrow$ (1): Let $\mathcal{V}$ satisfy (2) and

$$a_i T_i x \quad \text{for } i = 1, \ldots, n$$

be a system of tolerances ($T_i \in LT(\mathfrak{A})$ for some $\mathfrak{A} \in \mathcal{V}$ such that $T_i \supseteq \cap\{T(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n);\ i = 1, \ldots, i-1, i+1, \ldots, n\}$. Put $b = p(a_1, \ldots, a_n)$ for p satisfying (2). Hence, for $i = 1$ and varying j from 2 to n,

$$\langle a_1, b \rangle = \langle p(a_1, a_2, a_1, \ldots, a_1), p(a_1, a_2, \ldots, a_n) \rangle \in T(a_1, a_3, a_4, \ldots, a_n)$$
$$\langle a_1, b \rangle = \langle p(a_1, a_1, a_3, a_1, \ldots, a_1), p(a_1, a_2, \ldots, a_n) \rangle \in T(a_1, a_2, a_4, \ldots, a_n)$$
$$\vdots$$
$$\langle a_1, b \rangle = \langle p(a_1, \ldots, a_1, a_n), p(a_1, a_2, \ldots, a_n) \rangle \in T(a_1, a_2, \ldots, a_{a-1}),$$

i.e. $\langle a_1, b \rangle \in T_1$. Analogously it can be proved that $\langle a_i, b \rangle \in T_i$ for $i = 1, 2, \ldots, n$; thus $\mathcal{V}$ satisfies the n-Tolerance Chinese Remainder Theorem.

Example. An example of a variety satisfying the n-Tolerance Chinese Remainder Theorem for each $n \geq 3$ is the variety of all lattices. It suffices to take

$$p(x_1, \ldots, x_n) = (x_1 \wedge x_2) \vee (x_2 \wedge x_3) \vee \ldots \vee (x_{n-1} \wedge x_n) \vee (x_n \wedge x_1).$$

This example is not trivial because there exist tolerances on lattices different from congruences, see [1].

Corollary. *Let $\mathcal{V}$ be a variety of algebras. The following conditions are equivalent:*

(1) *$\mathcal{V}$ satisfies the 3-Tolerance Chinese Remainder Theorem*

(2) *$\mathcal{V}$ satisfies the Chinese Remainder Theorem*

(3) *There exists a ternary polynomial p of $\mathcal{V}$ such that* $x = p(x, x, y) = p(x, y, x) = p(y, x, x)$.

Proof. (1) $\Leftrightarrow$ (3) by the Theorem, (3) $\Leftrightarrow$ (2) by [3] or by Theorem 6.6 in [4].

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ТОЛЕРАНТНОСТИ И КИТАЙСКАЯ ТЕОРЕМА ОБ ОСТАТКАХ

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Резюме

Толерантностей на алгебре $\mathfrak{A} = (A, F)$ называется рефрексивное и симметричное бинарное отношение на A удовлетворяющее свойству субституции относительно F. В работе обобщается Китайская Теорема об остатках для систем толерантностей и дается Мальцевское условие для многообразий алгебр, удовлетворяющих этому свойству.