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SELF-COMPLEMENTARY VERTEX-TRANSITIVE UNDIRECTED GRAPHS

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We consider undirected graphs without loops and multiple edges. If G is an undirected graph, then by $\bar{G}$ we denote the graph with the same vertex set as G in which two distinct vertices are adjacent if and only if they are not adjacent in G. The graph $\bar{G}$ is called the complement of G. If $\bar{G}$ is isomorphic to G, we say that G is a self-complementary graph. These graphs were studied by G. Ringel [1] and H. Sachs [2].

A vertex-transitive graph is a graph G with the property that to any two vertices u and v of G there exists an automorphism φ of G such that $\varphi(u)=v$. (Some authors call these graphs "transitive" or "symmetric".)

In this paper we shall present some results on graphs which are simultaneously self-complementary and vertex-transitive.

Theorem 1. *If n is the number of vertices of a regular finite self-complementary graph, then $n\equiv 1 \pmod 4$.*

Proof. In [1] and [2] it was proved that for the number n of vertices of a finite self-complementary graph either $n\equiv 0 \pmod 4$, or $n\equiv 1 \pmod 4$ holds. The number of edges of a self-complementary graph G with n vertices is equal to one half of the number of edges of K_n, namely $\frac{1}{4}n(n-1)$. If G is regular of the degree r, we have

$$\tfrac{1}{2}nr=\tfrac{1}{4}n(n-1),$$

which implies

$$r=\tfrac{1}{2}(n-1).$$

As r is an integer, n must be odd, which excludes the case $n\equiv 0 \pmod 4$. Hence $n\equiv 1 \pmod 4$, which was to be proved.

As a vertex-transitive graph is always regular, we have a corollary.

Corollary. *If n is the number of vertices of a finite self-complementary vertex-transitive graph, then $n\equiv 1 \pmod 4$.*

Before proving a further theorem, we shall state two lemmas.

Lemma 1. *Let $n \geqq 3$ be a prime number. Then either* $2^{1/2(n-1)} \equiv 1 \pmod{n}$, *or* $2^{1/2(n-1)} \equiv -1 \pmod{n}$.

Proof. According to Fermat's Theorem we have $2^{n-1} \equiv 1 \pmod{n}$. This means that $2^{n-1}-1$ is divisible by n. The number $2^{n-1}-1$ can be written in the form $(2^{1/2(n-1)}+1)\,(2^{1/2(n-1)}-1)$. As n is prime, it must divide one of the numbers $2^{1/2(n-1)}+1$, $2^{1/2(n-1)}-1$, which implies the assertion.

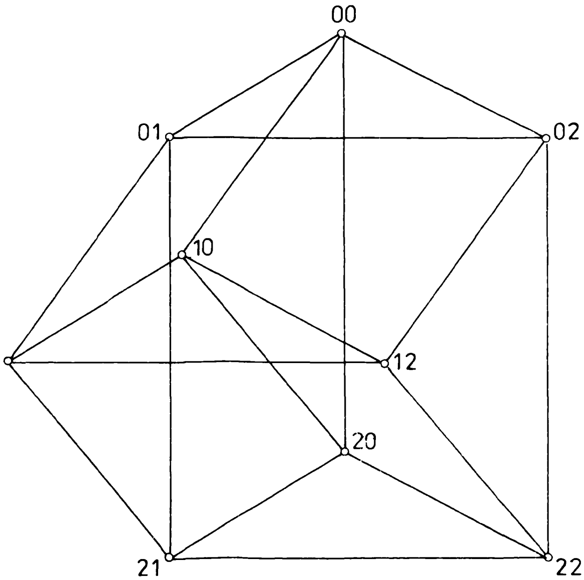

Fig. 1

Lemma 2. *Let n be a prime number,* $n \equiv 1 \pmod 4$ *and let n have the property that for any m such that $1 \leqq m < \frac{1}{2}(n-1)$ the conditions* $2^m \not\equiv 1 \pmod{n}$ *and* $2^m \not\equiv -1 \pmod{n}$ *hold. Then to any two distinct numbers x, y of the set* $\{1, 2, \ldots, n\}$ *a unique number $\beta(x, y)$ can be assigned with the property that* $1 \leqq \beta(x, y) \leqq \frac{1}{2}(n-1)$ *and either* $x - y \equiv 2^{\beta(x,y)} \pmod{n}$, *or* $y - x \equiv 2^{\beta(x,y)} \pmod{n}$.

Proof. Suppose that there exist numbers p, q of the set $S = \{1, 2, \ldots, \frac{1}{2}(n-1)\}$ such that $2^p \equiv 2^q \pmod{n}$ and $p > q$. Then the difference $2^p - 2^q = 2^{p-q}(2^q - 1)$ is divisible by n. As n is a prime number and 2^{p-q} is not divisible by any prime number other than 2, the number $2^q - 1$ is divisible by n and thus $2^q \equiv 1 \pmod{n}$. But we have $1 \leqq q < p \leqq \frac{1}{2}(n-1)$ and thus this is a contradiction with the conditions of the lemma. Therefore the numbers 2^m for $m \in S$ are pairwise non-congruent modulo n. Further suppose that $2^p \equiv -2^q \pmod{n}$ for some p, q from the set $\{1, 2, \ldots, n\}$, $p > q$. Then analogously we obtain $2^q \equiv -1 \pmod{n}$, which is again a contradiction. Therefore to each number $t \in S$ there exists exactly one number $\gamma(t)$ of this set such that either $2^{\gamma(t)} \equiv t \pmod{n}$, or $2^{\gamma(t)} \equiv -t \pmod{n}$. If for t we take that one (uniquely determined) of the numbers $x-y$, $y-x$ which is in S, we have $\beta(x, y) = \gamma(t)$.

Theorem 2. *Let n be a prime number, $n \equiv 1 \pmod{4}$ and let n have the property that for any m such that $1 \leqq m < \frac{1}{2}(n-1)$ the conditions $2^m \not\equiv 1 \pmod{n}$ and $2^m \not\equiv -1 \pmod{n}$ hold. Then there exists a self-complementary vertex-transitive graph with n vertices.*

Proof. The notation is the same as in Lemma 2 and in its proof. We construct the graph G as follows. The vertex set of G is $V = \{1, 2, \ldots, n\}$. Two distinct vertices x, y of V are adjacent in G if and only if $\beta(x, y)$ is odd. Now for each integer a we define the mapping $\varphi_a \colon V \to V$ so that

$$\varphi_a(x) \equiv x + a \pmod{n}$$

for each $x \in V$. Evidently each φ_a is a bijection and $\varphi_a(x) - \varphi_a(y) \equiv x - y \pmod{n}$ for each x and y, hence each φ_a is an automorphism of G. If $u \in V$, $v \in V$, then $v = \varphi_{v-u}(u)$ and therefore G is vertex-transitive. Further let the mapping $\psi \colon V \to V$ be defined so that

$$\psi(x) \equiv 2x \pmod{n}$$

for each $x \in V$. If $x - y \equiv 2^{\beta(x,y)} \pmod{n}$, then $\psi(x) - \psi(y) \equiv 2x - 2y \equiv 2^{\beta(x,y)+1} \pmod{n}$ and analogously for $y - x$. Therefore

$$\beta(\psi(x), \psi(y)) = \beta(x, y) + 1$$

for $\beta(x, y) < \frac{1}{2}(n-1)$ and

$$\beta(\psi(x), \psi(y)) = 1$$

for $\beta(x, y) = \frac{1}{2}(n-1)$. Hence

$$\beta(\psi(x), \psi(y)) \not\equiv \beta(x, y) \pmod{2}$$

and ψ is an isomorphism of G onto its complement $\bar{G}$. We have proved that G is vertex-transitive and self-complementary.

The simplest example of such a graph is a circuit of the length 5.

Another example of a number fulfilling the conditions of Theorem 2 is the number 13. Examples of prime numbers congruent with 1 modulo 4 which do not fulfil the conditions are the numbers 17 and 257.

The conditions of Theorem 2 are not necessary.

Theorem 3. *There exists a self-complementary vertex-transitive graph with 9 vertices.*

Proof. This graph is $K_3 \times K_3$, i. e. the graph whose vertices are all ordered pairs of the numbers 0, 1, 2 and in which the vertices $[i, j]$, $[k, l]$ are joined by an edge if and only if either $i = k$, or $j = l$, but not both simultaneously. This graph is in Fig. 1; brackets and commas are omitted. As it is a direct product of complete graphs, it is vertex-transitive. An isomorphism ψ of G onto $\bar{G}$ is given as follows (brackets and commas are again omitted):

$$00 \mapsto 00,$$
$$01 \mapsto 11 \mapsto 02 \mapsto 22 \mapsto 01,$$
$$10 \mapsto 21 \mapsto 20 \mapsto 12 \mapsto 10.$$

Problem 1. *Does there exist a self-complementary vertex-transitive graph with n vertices for each $n \equiv 1 \pmod 4$?*

Problem 2. *Are there infinitely many numbers n satisfying the conditions of Theorem 2?*

Now we shall study infinite graphs.

Theorem 4. *There exists a self-complementary vertex-transitive graph with the vertex set of the cardinality $\aleph_0$.*

Proof. By R denote the set of all rational numbers, by N the set of all integers. By $\langle a, b)$ we shall denote the interval consisting of all real numbers x such that $a \leqq x < b$. Denote $M = \bigcup_{k \in N} \langle 2^{2k}, 2^{2k+1})$, $\bar{M} = \bigcup_{k \in N} \langle 2^{2k+1}, 2^{2k+2})$. Evidently $M \cap \bar{M} = \emptyset$, $M \cup \bar{M}$ is the set of all positive real numbers. For each positive real number x we have $2x \in \bar{M}$ if and only if $x \in M$. Now we shall construct the graph G. The vertex set of G is R, two distinct vertices x and y are adjacent if and only if $|x - y| \in \bar{M}$. For each $a \in R$ we define the mapping $\varphi_a \colon R \to R$ so that

$$\varphi_a(x) = x + a$$

for each $x \in R$. Evidently $\varphi_a(x) - \varphi_a(y) = x - y$ for any a, x, y, each φ_a is a bijection, therefore each φ_a is an automorphism of G. If $u \in R$, $v \in R$, then $\varphi_{v-u}(u) = v$ and hence G is vertex-transitive. Further, let the mapping $\psi \colon R \to R$ be defined so that

$$\psi(x) = 2x$$

for each $x \in R$. The mapping ψ is a bijection and $|\psi(x) - \psi(y)| = 2|x - y|$ for any x and y. Therefore for $x \neq y$ the number $|\psi(x) - \psi(y)|$ is in $\bar{M}$ if and only if $|x - y| \in M$, hence ψ is an isomorphism of G onto $\bar{G}$ and G is self-complementary.

Theorem 5. *There exists a self-complementary vertex-transitive graph with the vertex set of the power of continuum.*

Proof is analogous to the proof of Theorem 4; instead of the set of all rational numbers the set of all real numbers is used.

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САМОДОПОЛНИТЕЛЬНЫЕ ВЕРШИННО-ТРАНЗИТИВНЫЕ НЕОРИЕНТИРОВАННЫЕ ГРАФЫ

Богдан Зелинка

Резюме

В статье доказаны следующие теоремы.

Теорема 1. Если n есть число вершин регулярного конечного самодополнительного графа, то $n = 1$ (mod 4).

Теорема 2. Пусть n есть простое число, $n \equiv 1$ (*mod* 4) и пусть n обладает тем свойством, что $2^{1\ 2(n-1)} \equiv 1 \pmod{n}$ или $2^{1/2(n-1)} \equiv -1 \pmod{n}$ и для всякого m такого, что

$$1 \leqq m < \tfrac{1}{2}(n-1),$$

пусть $2^m \not\equiv 1 \pmod{n}$ и $2^m \not\equiv -1 \pmod{n}$. Потом существует самодополнительный вершинно-транзитивный граф с n вершинами.

Теорема 4. Существует самодополнительный вершинно-транзитивный граф с множеством вершин мощности $\aleph_0$.

Теорема 5. Существует самодополнительный вершинно-транзитивный граф с множеством вершин мощности континуума.