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SOMEWHAT CONTINUITY AND SOME OTHER GENERALISATIONS OF CONTINUITY

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1. Introduction. One of the notions generalising the notion of continuity is the almost continuity. It has been defined differently by Stallings [23], Frolik [3], Husain [6], and Singal and Singal [21]. Long and McGehene Jr. [10], Deb [1], Long and Carnahan [9], and Herrington, Hunsaker, Lindgren and Naimpally [5] have shown that the four definitions of the almost continuity are independent of one another. Also Singal and Singal [21], Deb [1], and Papp [17] gave examples to show that the class of the θ-continuous mappings contained properly the class of the almost continuous mappings of Singal and Singal, and was itself contained properly in the class of the weakly continuous mappings. In 1971, Gentry and Hoyle [4] introduced another weaker form of continuity called the "somewhat continuity". Frolik [3], Neubrunnová [14], Neubrunn [15] studied interrelations between the somewhat continuity, the quasi continuity, the semi continuity and some types of the almost continuity. In the present paper we further investigate the relationship of the somewhat continuity with other types of the almost continuity, the θ-continuity and weak continuity.

Also the authors in their papers [19, 20] gave conditions for which the almost continuity of Singal and Singal, and the weak continuity, respectively, imply the almost continuity of Husain. Now we shall obtain conditions (Theorem 6) under which the almost continuity of Husain would imply the almost continuity of Singal and Singal and hence θ-continuity. Our Theorem 5 extends the results of a theorem of Wilansky [24, Theorem 3.5] for the class of continuous functions to the class of weakly continuous functions.

2. Notations and definitions. Let A be a subset of a topological space X. The interior of A, the closure of A, and the complement of A in X are denoted by $\mathrm{Int}(A)$, $\mathrm{Cl}(A)$, and $(X-A)$, respectively. A is semi-open if there exists an open set O in X such that $O \subset A \subset \mathrm{Cl}(O)$. $\boldsymbol{S}\cdot\boldsymbol{O}\cdot(X)$ denotes the class of all semi-open sets in X. A is regular open if $A = \mathrm{Int}(\mathrm{Cl}(A))$.

Definition 1 [Stallings, 23]. *A mapping $f: X \to Y$ is said to be almost continuous, if for every open set W containing the graph of f, there exists a continuous mapping $g: X \to Y$ such that the graph of g is a subset of W.*

Definition 2 [Frolik, 3]. *A mapping $f: X \to Y$ is said to be almost continuous if for every open subset V of Y, $f^{-1}(V) \subset \mathrm{Cl}(\mathrm{Int}(f^{-1}(V)))$.*

Definition 3 [Husain, 6]. *A function $f: X \to Y$ is said to be almost continuous at $x \in X$ if for each open $V \subset Y$ containing $f(x)$, $\mathrm{Cl}(f^{-1}(V))$ is a neighborhood of x. If f is almost continuous at each point of X, then f is called almost continuous.*

Definition 4 [Singal and Singal, 21]. *A mapping $f: X \to Y$ is said to be almost continuous at a point $x \in X$, if for every neighborhood M of $f(x)$, there is a neighborhood N of x such that $f(N) \subset \mathrm{Int}(\mathrm{Cl})(M))$. f is said to be almost continuous if it is almost continuous at each point x of X.*

Definition 5 [7]. *A mapping $f: X \to Y$ is said to be weakly continuous if for each point $x \in X$ and each neighborhood V* of *$f(x)$, there exists a neighborhood U of x such that $f(U) \subset \mathrm{Cl}(V)$.*

Definition 6 [8]. *A mapping $f: X \to X^*$ is semi-continuous iff for O^* open in X^*, $f^{-1}(O^*) \in \boldsymbol{S} \cdot \boldsymbol{O} \cdot (X)$.*

Definition 7 [2]. *A mapping $f: X \to Y$ is said to be θ-continuous if for each point $x \in X$ and each neighborhood V of $f(x)$ there is a neighborhood U of x such that $f(\mathrm{Cl}(U)) \subset \mathrm{Cl}(V)$.*

Definition 8 [11, 12]. *A mapping $f: X \to Y$ is said to be quasicontinuous if for every point $x \in X$ and for every open set V containing x and every open set U containing $f(x)$ there exists a non-empty open set W such that $W \subset V$ and $f(W) \subset U$.*

Definition 9 [4]. *Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. A function $f: (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is said to be somewhat continuous provided that if $U \in \mathcal{T}_2$ and $f^{-1}(U) \neq \emptyset$, then there is a $V \in \mathcal{T}_1$ such that $V \neq \emptyset$ and $V \subset f^{-1}(U)$.*

Definition 10 [24]. *If $f: X \to Y$ is a one to one mapping, then it is almost open iff for every open $G \subset Y$, $f^{-1}(\mathrm{Cl}(G)) \subset \mathrm{Cl}(f^{-1}(G))$.*

Definition 11 [22]. *A space X is almost regular if for each point $x \in X$ and each regular open set V containing x, there exists a regular-open set U such that $x \in U \subset \mathrm{Cl}(U) \subset V$.*

3. Somewhat continuity and other weaker forms of continuity.

Theorem 1. *The almost continuity of Stallings and the somewhat continuity are two independent notions.*

Proof. We prove it by giving two examples:

Example 1. Consider the mapping $f:(X, \mathcal{T}_1)\to(X, \mathcal{T}_2)$, where $X=\{a, b\}$, $\mathcal{T}_1=\{\emptyset, \{a\}, X\}$, $\mathcal{T}_2=\{\emptyset, X, \{a\}, \{b\}\}$, and $f(a)=a$, $f(b)=b$. Then f is almost continuous in the sense of Stallings [13, Ex. 3.1], but f is not somewhat continuous because $U=\{b\}\in\mathcal{T}_2$ and $f^{-1}(U)=\{b\}\neq\emptyset$, but no $V\in\mathcal{T}_1$, $V\neq\emptyset$ is such that $V\subset f^{-1}(U)$. This shows that the almost continuity of Stallings does not imply the somewhat continuity.

Example 2. Let $X=[0, 1]$, $Y=\{a, b, c\}$, $\mathcal{T}_1$ is the usual topology for X, and $\mathcal{T}_2=\{Y, \emptyset, \{c\}, \{a\}, \{a, c\}\}$. Consider the mapping $f:(X, \mathcal{T}_1)\to(Y, \mathcal{T}_2)$ defined by

$$f(x)=A, \quad \text{if} \quad 0\leqslant x\leqslant\tfrac{1}{2}$$
$$f(x)=c, \quad \text{if} \quad \tfrac{1}{2}<x\leqslant 1.$$

Then f is not almost continuous in the sense of Stallings, but f is somewhat continuous. From this example we assert that the somewhat continuity does not imply the almost continuity of Stallings.

Theorem 2. *The almost continuity of Singal and Singal and the somewhat continuity are two independent notions.*

Proof. Consider the following example.

Example 3. Let $i:(R, \mathcal{T}_1)\to(R, \mathcal{T}_2)$ be the identity mapping, where R is the set of reals, $\mathcal{T}_1$ is the usual topology on R, $\mathcal{T}_2$ the co-countable topology. Then i is almost continuous in the sense of Singal and Singal, but it is not somewhat continuous. Thus the almost continuity of Singal and Singal does not imply the somewhat continuity.

Example 4. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces, where

$$X=\{a, b, c, d\}, \quad Y=\{p, q, r\}$$
$$\mathcal{T}_1=\{\emptyset, X, \{a, c\}, \{d\}, \{c\}, \{c, d\}, \{a, c, d\}\}$$
$$\mathcal{T}_2=\{\emptyset, Y, \{r\}, \{q\}, \{r, q\}\}.$$

Define $f: X\to Y$ by

$$f(a)=f(d)=p, \quad f(b)=f(c)=r.$$

Then f is somewhat continuous but not almost continuous in the sense of Singal and Singal. So the somewhat continuity does not imply the almost continuity of Singal and Singal.

Theorem 3. *The almost continuity of Husain and the somewhat continuity are two independent notions.*

Proof. We give the following example:

Example 5. Consider the mapping $f:(R, \mathscr{T}_1)\to(Y, \mathscr{T}_2)$, Where R = set of reals, $Y=\{a, b\}$, $\mathscr{T}_1=\{A\subset R|(R-A)$ is finite$\}$, $\mathscr{T}_2=\{y, \emptyset, \{a\}\}$, and f is defined by

$$f(x)=a \quad \text{if} \quad x \quad \text{is irrational;}$$
$$f(x)=b \quad \text{if} \quad x \quad \text{is rational.}$$

Then f is almost continuous in the sense of Husain, but f is not somewhat continuous. Thus the almost continuity of Husain does not imply the somewhat continuity.

Next consider Example 4. The mapping defined there is somewhat continuous but not almost continuous in the sense of Husain. So the somewhat continuity does not imply the almost continuity of Husain.

The referee has brought to the notice of the authors that the above theorem has also been proved with the help of different examples in the thesis of A. Neubrunnová.

Theorem 4. *The somewhat continuity is independent of the weak continuity and the θ-continuity.*

Proof. The fact that neither the θ-continuity nor the weak continuity implies the somewhat continuity follows from Example 3.

The converse is proved by the mapping defined in Example 4 which is somewhat continuous but is neither θ-continuous nor weakly continuous.

We make use of the following lemma in our next theorem.

Lemma 1 [16, 18]. *Let $f: X\to Y$ be a weakly continuous function. Then for each open $V\subset Y$, $\mathrm{Cl}(f^{-1}(V))\subset f^{-1}(\mathrm{Cl}(V))$.*

Theorem 5. *Let $f: X\to Y$ be one to one and weakly continuous. Then the following are equivalent.*

(1) *f is almost open.*

(2) *For every open set $V\subset Y$, $f^{-1}(\mathrm{Cl}(V))=\mathrm{Cl}(f^{-1}(V))$.*

Proof. From Definition 10 and Lemma 1 we get the required result.

This improves upon a result of Wilansky [24, Theorem 3.5] by replacing the continuity of f by its weak continuity.

For our next theorem we need the following lemmas.

Lemma 2 [18, Theorem 5.5; 16]. *If $f: X\to Y$ is almost continuous in the sense of Husain such that for each open $V\subset Y$, $\mathrm{Cl}(f^{-1}(V))\subset f^{-1}(\mathrm{Cl}(V))$. Then f is weakly* continuous.

Lemma 3 [1]. *If $f: X\to Y$ is a weakly continuous mapping and Y is almost* *regular, then f is almost continuous in the sense of Singal and Singal (hence* θ-continuous).

Theorem 6. *Let Y be an almost regular space. Let $f: X \to Y$ be a mapping such that for every open set $V \subset Y$, $\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$. If f is almost continuous in the sense of Husain, then f is almost continuous in the sense of Singal and Singal (and hence θ-continuous).*

Proof. By combining Lemma 2 and Lemma 3 we get the assertion of the theorem.

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