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# THREE POINT VALUE PROBLEM FOR THIRD-ORDER LINEAR DIFFERENTIAL EQUATION

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## 1. Introduction

In this paper we shall be concerned with the existence of the solution of the three-point value problem for the differential equation

$$y''' + B(x, \tau)y' + C(x, \tau)y = 0, \tag{A}$$

where $B(x, \tau)$, $C(x, \tau)$ and $B'(x, \tau) = \dfrac{\partial B(x, \tau)}{\partial x}$ are continuous functions in the interval $D = (0, \infty) \times (t, T)$. The results of this paper generalize the results of the papers [1] and [4].

A solution of (A) is said to be oscillatory in $(0, \infty)$ iff it has an infinity of zeros in each interval $(a, \infty)$, $a > 0$. The differential equation (A) is said to be oscillatory iff it has at least one (nontrivial) oscillatory solution, and nonoscillatory if it has no (nontrivial) oscillatory solution.

## 2. Preliminary results

We shall need the following two theorems which were proved in [2].

**Theorem (i).** *Let us consider the differential equations*

$$y''' + B(x)y' + C(x)y = 0, \tag{1}$$

$$z''' + b(x)z' + c(x)z = 0, \tag{2}$$

*where $B'(x)$, $C(x)$, $b'(x)$, $c(x)$ are continuous functions in $(0, \infty)$. Suppose that the coefficients of (1) and (2) satisfy the following assumptions*

$$B(x) \geq b(x),\ 2C(x) - B'(x) \geq 2c(x) - b'(x),\ 2C(x) - B'(x) \geq 0.$$

*Let $\alpha$, $\beta$ be two consecutive zeros of a solution $z(x)$ of (2). Let $\alpha$ be a double zero of $z(x)$. Then the solution $y(x)$ of (1) with a single zero at $\beta$ has a zero in the interval $(\alpha, \beta]$.*

**Theorem (ii).** *If the coefficients of (1) satisfy the conditions*

$$2C(x)-B'(x)\geq 0 \quad and \quad B(x)\geq p,$$

*where $p$ is a positive constant, or the conditions*

$$2C(x)-B'(x)\geq q>\frac{4}{3\sqrt{3}}|-p|^{\frac{3}{2}} \quad and \quad B(x)\geq p,$$

*where $p$, $q$ are constants, then the equation (1) is oscillatory.*

*If the coefficients of (1) satisfy the conditions*

$$0\leq 2C(x)-B'(x)\leq\frac{4}{3\sqrt{3}}(-p)^{\frac{3}{2}} \quad and \quad B(x)\leq p\leq 0,$$

*where $p$, $q$ are constants, or the conditions*

$$0\leq 2C(x)-B'(x)\leq\frac{4}{3\sqrt{3}x^3}(1-p)^{\frac{3}{2}} \quad and \quad B(x)\leq\frac{p}{x^2},$$

*where $p\leq 1$ is a constant, then the equation (1) is nonoscillatory.*

**Definition 1.** *A solution of (1) is said to be of class $D(k)$ in an interval $[a, \infty)$ iff the distance between any two consecutive zeros of $y(x)$ in $[a, \infty)$ is less than the number $k$.*

**Theorem 1.** *Let $2C(x)-B'(x)\geq 0$ in $(0, \infty)$. Suppose that there exists an oscillatory solution of (1) which is of class $D(k)$ in an interval $[a, \infty)$, $a>0$. Then there exists a number $K$ such that every solution of (1) is of class $D(K)$ in $[a, \infty)$.*

Proof. Let $y(x)$ be a solution of (1) which is oscillatory and of class $D(k)$ in $[a, \infty)$. Let $a$ be a zero of $y(x)$. Suppose that $a$ is a single zero of $y(x)$. Let $z(x)$ be a solution of (1) with a double zero at the point $a$. Because of Theorem 1 in [2], every solution of (1) with a zero is oscillatory, therefore $z(x)$ is an oscillatory solution of (1). Let $a\leq x_1<x_2<x_3$ be consecutive zeros of $y(x)$. Then the solution $z(x)$ must have a zero in $(x_1, x_3]$. Indeed, if $z(x)$ is positive in $(x_1, x_3]$, then it is positive in $[x_2, x_3]$. Then there exist numbers $c$ and $\tau\in(x_2, x_3)$ (see Lemma 2 in [2]) such that the solution $w(x)=z(x)-cy(x)$ of (1) has a double zero at $\tau$ and a single zero at $a$ which contradicts the identity

$$(3) \qquad [ww''-\tfrac{1}{2}w'^2+\tfrac{1}{2}Bw^2]_a^\tau=-\tfrac{1}{2}\int_a^\tau[2C(x)-B'(x)]w^2\,dx.$$

Thus $z(x)$ has a zero in $(x_1, x_3]$. Since $x_1$, $x_2$, $x_3$ are three arbitrary consecutive zeros of $y(x)$, the solution $z(x)$ is of class $D(3k)$ in $[a, \infty)$.

Now let $u(x)$ be a solution of (1) with a single zero at $a$. It follows from Theorem (i) that the zeros of $u(x)$ and $z(x)$ interlace in $(a, \infty)$ in the sense that if $\alpha, \beta$ are two consecutive zeros of $z(x)$, than $u(x)$ has a zero in $[\alpha, \beta]$. From this fact it follows that the solution $u(x)$ of (1) is of class $D(6k)$ in $[a, \infty)$.

At the beginning we assumed that $y(x)$ had a single zero at $a$. If $y(x)$ has a double zero at $a$, then by a method analogous to the one used before we find that every solution of (1) with the zero at $a$ is of class $D(6k)$ in $[a, \infty)$.

Now let $v(x)$ be a solution of (1) with a zero at a point $b$, and let $b$ be different from the zeros of $y(x)$. Then there is a solution $w(x)$ of (1) such that $w(a)=w(b)=0$. Since $w(x)$ and $y(x)$ have one common zero $a$, then $w(x)$ is of class $D(6k)$ in $[a, \infty)$. However, $v(x)$ has the common zero $b$ with $w(x)$, therefore the solution $v(x)$ of (1) is of class $D(36k)$ in $[a, \infty)$. If we put $36k=K$, then Theorem 1 is proved completely, since every solution of (1) with a zero is oscillatory.

**Lemma 1.** *Let $p, q \geq 0$ be numbers. Let the equation*

$$z''' + pz' + \frac{q}{2} z = 0 \tag{4}$$

*be oscillatory. Then every solution of* (4) *is of class $D\left(\frac{K}{\beta}\right)$ in $[a, \infty)$, where $a$ is an arbitrary positive number, $K$ is a number indepedent of $p$ and $q$, and*

$$\beta = (-q+d)^{\frac{1}{3}} + (q+d)^{\frac{1}{3}}, \quad \textit{where} \quad d = (q^2 + \tfrac{16}{27} p^3)^{\frac{1}{2}}. \tag{5}$$

Proof. If the equation (4) is oscillatory, then the auxiliary equation associated with (4) has the roots

$$x_1 = u + v, \quad x_{2,3} = \alpha \pm \beta' \frac{\sqrt{3}}{2} \mathrm{i},$$

where

$$u = \{-\tfrac{1}{4}q + [(\tfrac{1}{4}q)^2 + (\tfrac{1}{3}p)^3]^{\frac{1}{2}}\}^{\frac{1}{3}},$$

$$v = \{-\tfrac{1}{4}q - [(\tfrac{1}{4}q)^2 + (\tfrac{1}{3}p)^3]^{\frac{1}{2}}\}^{\frac{1}{3}}$$

and $\alpha = -\frac{1}{2}(u+v)$, $\beta' = u - v$. Since $(\frac{1}{4}q)^2 + (\frac{1}{3}p)^3 > 0$, the numbers $u, v, \alpha, \beta'$ are real ones. Denote $d = (q^2 + \frac{16}{27}p^3)^{\frac{1}{2}}$.
Then we can rewrite $\beta'$ in the form

$$\beta' = \frac{1}{\sqrt[3]{4}} [(-q+d)^{\frac{1}{3}} + (q+d)^{\frac{1}{3}}].$$

If the number in the bracket is denoted by $\beta$, then for the roots $x_{2,3}$ it yields

$$x_{2,3} = \alpha \pm \frac{\sqrt[3]{6}}{4} \beta i.$$

Then one solution of (4) is $y(x) = c_1 e^{\alpha x} \sin \left( \frac{\sqrt{6}}{4} \beta x + c_2 \right)$. From the form of $y(x)$ it follows that for every positive number $a$ there exists a solution of (4) of class $D\left(\frac{4\pi}{\sqrt[3]{6}} \cdot \frac{1}{\beta}\right)$ in $[a, \infty)$. Because of Theorem 1, every solution of (4) is of class

$D\left(36 \cdot \frac{4\pi}{\sqrt[3]{6}} \cdot \frac{1}{\beta}\right) = D\left(\frac{K}{\beta}\right)$ in $[a, \infty)$.

### 3. Oscillation theorems

**Theorem 2.** *Assume that* (A) *satisfies the conditions:*
(i) *There exists a number $p$ such that*
*$B(x, \tau) \geq p$ for every $(x, \tau) \in D$ and*

$$\lim_{\tau \to T} [2C(x, \tau) - B'(x, \tau)] = \infty \quad \textit{uniformly in} \quad x \in (0, \infty), \quad \textit{or}$$

(ii) $2C(x, \tau) - B'(x, \tau) \geq 0$ *for every* $(x, \tau) \in D$ *and*

$$\lim_{\tau \to T} B(x, \tau) = \infty \quad \textit{uniformly in} \quad x \in (0, \infty).$$

*Let $[a, b] \subset (0, \infty)$ be an arbitrary interval. Let $y(x)$ be a solution of* (A) *with the property $y(a, \tau) = 0$. Then with the increasing $\tau \to T$ also the number of zeros of the solution $y(x, \tau)$ in $[a, b]$ increases to infinity and at the same time the distance between every consecutive zeros of $y(x, \tau)$ converges to zero.*

Proof. (i). Let the conditions (i) be valid. Then for every $q > 0$ there is a number $\tau_0$ such that $\tau > \tau_0$ implies

$$2C(x, \tau) - B'(x, \tau) > q \quad \text{for all} \quad x \in (0, \infty).$$

Let $q$ be such that the differential equation (4) is oscillatory. Then the solution $z(x)$ of (4) with the properties $z(a) = z'(a) = 0$, $z''(a) \neq 0$ is oscillatory by Theorem 1 in [2]. From Lemma 1 it follows that $z(x)$ is of class $D\left(\frac{K}{\beta}\right)$ in $[a, \infty)$, where $\beta$ is defined by (5). If $2C(x, \tau) - B'(x, \tau)$ diverges to infinity uniformly in $x \in (0, \infty)$, then $q \to \infty$ and because of (5), $\beta \to \infty$. So the the number of zeros of the solution $z(x)$ in $[a, b]$ increases to infinity and at the same time the distance of every two neighbouring zeros of $z(x)$ converges to zero.

Let $y(x, \tau)$ be a solution of (A) with a single zero at the point $a$. Let $a < x_1$ be two consecutive zeros of the solution $z(x)$ of (4). Since all $(x, \tau) \in (0, \infty) \times (\tau_0, T)$ satisfy

$$B(x, \tau) \geq p, \qquad 2C(x, \tau) - B'(x, \tau) \geq q > 0,$$

then, by Theorem (i), the solution $y(x, \tau)$ of (A) has a zero $a_1$ in the interval $(a, x_1]$. Then for the distance of the zeros $a$ and $a_1$ of $y(x, \tau)$ yields $|a - a_1| \leq |x_1 - a| < \frac{K}{\beta}$.

Now let $z_1(x)$ be a solution of (4) such that $z_1(a_1) = z_1'(a_1) = 0$, $z_1''(a_1) \neq 0$. Let $a_1 < x_2$ be two consecutive zeros of $z_1(x)$. Then again, by Theorem (i) we have that the solution $y(x, \tau)$ of (A) has a zero $a_2$ in $(a_1, x_2]$. For the distance between $a_1$ and $a_2$ there follows $|a_1 - a_2| \leq |a_1 - x_2| \leq \frac{K}{\beta}$. By induction we obtain that the distance between every two consecutive zeros of $y(x, \tau)$ is less than $\frac{K}{\beta}$.

Let us note that if a is a single zero of the solution $y(x, \tau)$ of (A), then the condition $2C(x, \tau) - B'(x, \tau) \geq 0$ results in every zero of $y(x, \tau)$ in $(0, \infty)$ is being a single one, and therefore Theorem (i) is applied to each zero of $y(x, \tau)$. If $y(x, \tau)$ is a solution of (A) with the property $y(a, \tau) = y'(a, \tau) = 0$, then from Lemma 1 it follows that the distance between every two consecutive zeros of $y(x, \tau)$ is less than $\frac{K_1}{\beta}$, $K_1 = 6K$.

Consequently the distance between every two consecutive zeros of the solution $y(x, \tau)$ of (A) with the property $y(a, \tau) = 0$, $a > 0$ is less then $\frac{K_1}{\beta}$. From this fact and from the condition

$$\lim_{\tau \to T} [2C(x, \tau) - B'(x, \tau)] = \infty \quad \text{uniformly in} \quad (0, \infty)$$

we have $\beta \to \infty$ and so with the increasing $\tau \to T$ also the number of zeros of the solution $y(x, \tau)$ of (A) in $[a, b]$ increases to infinity, and at the same time the distance between every consecutive zeros of $y(x, \tau)$ converges to zero.

(ii) Let $2C(x, \tau) - B'(x, \tau) \geq 0$ and $\lim_{\tau \to T} B(x, \tau) = \infty$

uniformly in $(0, \infty)$. Then for every $p > 0$ there is a number $\tau_0$ such that $\tau > \tau_0$ implies $B(x, \tau) > p$ for all $x \in (0, \infty)$. Then the equation

$$(6) \qquad z''' + pz' = 0$$

is oscillatory and for $p \to \infty$ the distance between every two consecutive zeros of the solution $z(x)$ of (6) with the properties $z(a) = z'(a) = 0$, $z''(a) \neq 0$ converges to zero. Then in a way analogous to that in part (i) we show that the solution $y(x, \tau)$ of (A) such that $y(a, \tau) = 0$ has the property that with the increasing $\tau \to T$ the number of zeros of $y(x, \tau)$ in $[a, b]$ increases to infinity and at the same time the distance of every two neighbouring zeros converges to zero.

Remark. The part (i) of Theorem 2 generalizes Greguš's oscillatory theorem in [1], in which the assumption $|B(x, \tau)| \leq K_1$, $|B'(x, \tau)| \leq K_2$ in $D$, $K_1$, $K_2$ are constants, are required in addition.

The part (ii) of Theorem 2 generalizes Sansone's oscillatory theorem in [4], in which the assumption $B(x, \tau) < 0$ is required in addition.

Theorem 2 is included in the following more general theorem.

**Theorem 3.** *Let for every $\tau \in (t, T)$ the function $B(x, \tau)$ be bounded below in $(0, \infty)$. Let $2C(x, \tau) - B'(x, \tau) \geq 0$ in $D$. Denote*

$$p(\tau) = \inf_{x \in (0, \infty)} B(x, \tau), \quad q(\tau) = \inf_{x \in (0, \infty)} [2C(x, \tau) - B'(x, \tau)],$$

$$(7) \qquad d(\tau) = q^2(\tau) + \tfrac{16}{27} p^3(\tau)$$

$$(8) \qquad \beta(t) = [-q(\tau) + d(\tau)]^{\frac{1}{3}} + [q(\tau) + d(\tau)]^{\frac{1}{3}}.$$

*If*

$$\lim_{\tau \to T} \beta(\tau) = \infty,$$

*then the conclusion of Theorem 2 is valid.*

Proof. From the assumption $\lim_{\tau \to T} \beta(\tau) = \infty$ it follows that there is $\tau_0 \in (t, T)$ such that $\tau \in (\tau_0, T)$ implies $\beta(\tau) > 0$. From this fact it follows that $d(\tau) > 0$ in $(\tau_0, T)$ because $d(\tau) \leq 0$ implies $\beta(\tau) \leq 0$ by definition (8). Then from the condition $d(\tau) > 0$ in $(\tau_0, T)$ it follows that the differential equation

$$(9) \qquad z''' + p(\tau) z' + \frac{q(\tau)}{2} z = 0$$

is oscillatory in $(0, \infty)$ for every $\tau \in (\tau_0, T)$. Since $\lim_{\tau \to T} \beta(\tau) = \infty$, then the distance between every two consecutive zeros of $z(x, \tau)$ of (9) with the properties $z(a, \tau) = z'(a, \tau) = 0$, $z''(a, \tau) \neq 0$, converges to zero if $\tau \to T$.

From the definition of $p(\tau)$ and $q(\tau)$ it follows

$$B(x, \tau) \geq p(\tau), \qquad 2C(x, \tau) - B'(x, \tau) \geq q(\tau) \geq 0.$$

Then the assumption of Theorem (i) are fulfilled, therefore every solution of (A) with the properties $y(a, \tau)=y'(a, \tau)=0$, $y''(a, \tau)\neq 0$ has a zero in $(a, x_1]$, where $x_1$ is another zero of $z(x, \tau)$. The proof continues in the same way as in Theorem 2.

The following lemma gives a class of functions which satisfy Theorem 3 and the function $B(x, \tau)$ is unbounded below in $D$.

**Lemma 2.** *Let $p(\tau)$, $q(\tau)$ be continuous functions in $(t, T)$ and satisfy the* conditions

$$\lim_{\tau\to T} p(\tau)=-\infty, \quad q(\tau)\geq 0, \quad d(\tau)=K|-p(\tau)|^{2+\varepsilon}, \tag{10}$$

*where $K$ is a positive constant and $d(\tau)$ is defined by* (7). *Then there exists $\tau_0\in(t, T)$ such that for every $\tau\in(\tau_0, T)$ the equation*

$$r^3+p(\tau)r+\frac{q(\tau)}{2}=0 \tag{11}$$

*has the complex roots $\alpha(\tau)\pm\frac{\sqrt[3]{6}}{4}\beta(\tau)\mathrm{i}$, where $\beta(\tau)$ is defined by* (8) *and*

$$\lim_{\tau\to T}\beta(\tau)=\infty, \quad \textit{if} \quad \varepsilon>0,$$

$$\lim_{\tau\to T}\beta(\tau) \quad \textit{exists, if} \quad \varepsilon\leq 0.$$

Proof. From the assumption $\lim_{\tau\to T} p(\tau)=-\infty$ it follows that there is a number $\tau_0\in(t, T)$ such that $\tau\in(\tau_0, T)$ implies $p(\tau)<0$ and so because of (10), the equation (11) has complex roots for every $\tau\in(\tau_0, T)$. Now we calculate $\lim\beta(\tau)$ for $\tau\to T$.

In order to obtain a simple notation we put

$$q(\tau)=a(\tau), \quad [\tfrac{16}{27}p^3(\tau)]^{\frac{1}{3}}=b(\tau), \quad k=K(\tfrac{16}{27})^{-\frac{2+\varepsilon}{3}}.$$

Then from the conditions (7) and (10) we obtain

$$b(\tau)\to-\infty, \quad a(\tau)\to-\infty, \quad \text{if} \quad \tau\to T \quad \text{and}$$

$$a^2(\tau)=-b^3(\tau)+k|b(\tau)|^{2+\varepsilon}, \quad k>0. \tag{12}$$

In a simpe way we can rewrite $\beta(\tau)$ in the form

$$\beta(\tau)=\frac{2[a^2(\tau)+b^3(\tau)]^{\frac{1}{2}}}{[-a(\tau)+(a^2(\tau)+b^3(\tau))^{\frac{1}{2}}]^{\frac{2}{3}}-b(\tau)+[a(\tau)+(a^2(\tau)+b^3(\tau))^{\frac{1}{2}}]^{\frac{2}{3}}}.$$

Substituting here $a(\tau)$ because of (12), and multiplying the dividend and divisor by $[-b(\tau)]^{-1-\varepsilon/2}$ we obtain

$$[\beta(\tau)]^{-1}-\frac{1}{2\sqrt{k}}\cdot[-b(\tau)]^{-\frac{\varepsilon}{2}}=$$

$$=\frac{1}{2\sqrt{k}}\{[-b(\tau)]^{-\frac{3}{2}\varepsilon}+2k[-b(\tau)]^{-1-\frac{\varepsilon}{2}}-2\sqrt{k}([-b(\tau)]^{-1}+k[-b(\tau)]^{\varepsilon-2})^{\frac{1}{2}}\cdot$$

$$\cdot[-b(\tau)]^{-\varepsilon}\}^{\frac{1}{3}}+\frac{1}{2\sqrt{k}}\{[-b(\tau)]^{-\frac{3}{2}\varepsilon}+2k[-b(\tau)]^{-1-\frac{\varepsilon}{2}}+$$

$$+2\sqrt{k}([-b(\tau)]^{-1}+k[-b(\tau)]^{\varepsilon-2})^{\frac{1}{2}}[-b(\tau)]^{-\varepsilon}\}^{\frac{1}{3}}.$$

Let $\varepsilon>0$. Since $\lim_{\tau\to T}[-b(\tau)]=\infty$, then from the last equlity we obtain $\lim_{\tau\to T}[\beta(\tau)]^{-1}=0$ and so $\lim_{\tau\to T}\beta(\tau)=\infty$, because $\beta(\tau)>0$ in $(\tau_0, T)$. If $\varepsilon=0$, then $\lim_{\tau\to T}\beta(\tau)=\frac{2\sqrt{k}}{3}$.

Now let $\varepsilon<0$. The sum of the expressions which are on the right-hand side of the last equality is positive for all $\tau\in(\tau_0, T)$. Therefore $\lim_{\tau\to T}\beta(\tau)=0$, if $\varepsilon<0$.

**Corrolary 1.** *Let $p(\tau)$ and $q(\tau)$ be defined as in Theorem 3. Let $\lim_{\tau\to T}p(\tau)=-\infty$, $q(\tau)\geq\{\frac{16}{27}[-p(\tau)]^3+K[-p(\tau)]^{2+\varepsilon}\}^{1/2}$, where $K$, $\varepsilon$ are positive numbers. Then the conclusion of Theorem 2 is valid.*

### 4. Boundary value problem

In this section we need the following lema proved in [1].

**Lemma 3.** *Let $y(x, \tau)$ be a solution of* (A) *with the property $y(a, \tau)=0$, $a>0$. Then the zeros of $y(x, \tau)$ lying to the right of a are a continuous function of the parameter $\tau$.*

**Theorem 4.** *Let the coefficients* (A) *satisfy the assumptions:*

(i) *$B(x, \tau)$ is bounded below in $D$, and*

$$\lim_{\tau\to T}[2C(x, \tau)-B'(x, \tau)]=\infty \quad \textit{uniformly in} \quad x\in(0, \infty), \quad \textit{or}$$

(ii) $\lim_{\tau\to T}B(x, \tau)=\infty$ *uniformly in* $x\in(0, \infty)$, *and for all* $\tau\in(t, T)$,

*$2C(x, \tau) - B'(x, \tau) \geq 0$ in $(0, \infty)$ where the sign of equality does not hold in any subinterval of $(0, \infty)$, or*

(iii) *$\lim_{\tau \to T} \beta(\tau) = \infty$, and for all $\tau \in (t, T)$, $2C(x, \tau) - B'(x, \tau) \geq 0$ in $(0, \infty)$, where the sign of equality does not hold in any subinterval of $(0, \infty)$.*

*Assertion:* (a) *There exists a nonpositive integer $\delta$ and a sequence of the parameter $\tau$*

$$\tau_{\delta+1}, \tau_{\delta+2}, \ldots, \tau_{\delta+n}, \ldots$$

*tending to $T$ for which the boundary value problem*

$$\begin{aligned} & y(a, \tau) = 0 \\ (13) \qquad & \alpha_1 y(b, \tau) - \alpha y'(b, \tau) = 0, \quad |\alpha_1| + |\alpha| \neq 0, \\ & \beta_1 y(c, \tau) - \beta y'(c, \tau) = 0 \quad |\beta_1| + |\beta| \neq 0 \end{aligned}$$

*of equation* (A) *has a solution determined up to a multiplicative constant.*

(b) *The corresponding solutions $y(x, \tau_{\delta+n})$ have exactly $\delta + n$ zeros in the interval $(a, c)$.*

Proof. We note that under the assumptions of this theorem there exists a parameter $\tau_0 \in (t, T)$ such that $2C(x, \tau) - B'(x, \tau) \geq 0$ for all $\tau \in (\tau_0, T)$ and the sign $=$ does not hold in any subinterval of $(0, \infty)$. From now on we shall consider only $\tau \in (\tau_0, T)$. For those values of parameter $\tau$ every solution of (A) with double zero at a point $a$ has no zeros in $(0, a)$. Also, the assumptions of this theorem give that oscillatory theorems 2 and 3 hold.

Let $y_i(x, \tau)$ be solutions of (A), which satisfy the initial conditions $y_i^{(k)}(a, \tau) = \delta_{i,k}$, $i, k = 0, 1, 2$, $\delta_{i,k}$ is the Kronecker $\delta$. Let $y(x, \tau)$ be a solution of (A) which satisfies the condition $y(a, \tau) = 0$. Then $y(x, \tau)$ [see [1], Teorem 2] can be expressed in the form

$$y(x, \tau) = c_1 y_1(x, \tau) + c_2 y_2(x, \tau),$$

where $c_1$, $c_2$ are arbitrary numbers. Choose the constants $c_1$, $c_2$ such that

$$\alpha_1 y(b, \tau) - \alpha y'(b, \tau) = 0, \quad b > a,$$

which can be written as

$$\begin{aligned} & c_1 y_1(b, \tau) + c_2 y_2(b, \tau) = \bar{c}\alpha \\ (14) \qquad & c_1 y_1'(b, \tau) - c_2 y_2'(b, \tau) = \bar{c}\alpha_1, \quad \bar{c} \neq 0 \text{ arbitrary.} \end{aligned}$$

The determinant of the system (14) is $w(b, \tau) = y_1(b, \tau) \cdot y_2'(b, \tau) - y_1'(b, \tau) \cdot y_2(b, \tau)$. The function $w(x, \tau)$ is a solution of the adjoint equation to

(A). From the properties of the adjoint equations it follows that $w(x, \tau)$ has no zero in $(a, \infty)$, since $w(x, \tau)$ has the double zero at the point $a$. Then $w(b, \tau) \neq 0$ and so the system (14) has a unique solution. Thus the solution $y(x, \tau)$ is unique up to a multiplicative constant, and we may assume $\bar{c}=1$. If $a=b$, we put $y(x, \tau)=y_2(x, \tau)$.

For $\tau=\tau_0$ the solution $y(x, \tau_0)$ of (A) has exactly $\delta$ zeros in $(a, c)$, $c>a$ and $\delta$ is a nonpositive integer ($\delta=0$, if $y(x, \tau)$ has no zeros in $(a, c)$). For $(\delta+1)$st zero of $y(x, \tau_0)$ there follows

$$c<x_{\delta+1}(\tau_0).$$

From the oscillatory Theorems 2 and 3 it follows that there is $\bar{\tau}>\tau_0$ such that $x_{\delta+1}(\bar{\tau})<c$ and there is no $\tau>\bar{\tau}$ such that $x_{\delta+1}(\tau)=c$.

Because of Lemma 3, $x_{\delta+1}(\tau)$ is a continuous function of $\tau$, so there is the largest parameter $\bar{\tau}_\delta \in (\tau_0, \bar{\tau})$ for which $y(c, \bar{\tau}_\delta)=0$ and $y(x, \bar{\tau}_\delta)$ has exactly $\delta$ zeros in $(a, c)$. For the parameter $\tau=\bar{\tau}_\delta$ it follows

$$x_{\delta+1}(\bar{\tau}_\delta)=c<x_{\delta+2}(\bar{\tau}_\delta).$$

From the Theorems 2 and 3 it follows that there is $\tau^*>\bar{\tau}_\delta$ such that $x_{\delta+2}(\tau^*)<c$ and there is no $\tau>\tau^*$ such that $x_{\delta+2}(\tau)=c$.

Since $x_{\delta+2}(\tau)$ is a continuous function of $\tau$, then there is the greatest parameter $\bar{\tau}_{\delta+1} \in (\bar{\tau}_\delta, \tau^*)$ for which $y(c, \bar{\tau}_{\delta+1})=0$ and $y(x, \bar{\tau}_{\delta+1})$ has exactly $\delta+1$ zeros in $(a, c)$. For the parameter $\tau=\bar{\tau}_{\delta+1}$ it follows

$$x_{\delta+2}(\bar{\tau}_{\delta+1})=c<x_{\delta+2}(\bar{\tau}_{\delta+1}).$$

By induction we obtain that there exists a sequence of values of the parameter $\tau$

$$\bar{\tau}_\delta, \bar{\tau}_{\delta+1}, \bar{\tau}_{\delta+2}, \ldots, \bar{\tau}_{\delta+n-1}, \ldots$$

such that the corresponding solutions $y(x, \bar{\tau}_{\delta+n-1})$ of (A) satisfy the condition $y(c, \bar{\tau}_{\delta+n-1})=0$ and $y(x, \bar{\tau}_{\delta+n-1})$ have exactly $\delta+n-1$ zeros in $(a, c)$, $\delta \geq 0$, $n=1, 2, \ldots$

If $\beta=0$ in the third condition of the boundary value problem (13), then it is sufficient to put

$$\tau_{\delta+1}=\bar{\tau}_{\delta+1}, \quad \tau_{\delta+2}=\bar{\tau}_{\delta+2}, \quad \ldots, \quad \tau_{\delta+n}=\bar{\tau}_{\delta+n}, \quad \ldots$$

and so the corresponding solutions $y(x, \tau_{\delta+n})$ satisfy the conditions (13) and have exactly $\delta+n$ zeros in $(a, c)$, $\delta \geq 0$, $n=1, 2, \ldots$

If $\beta \neq 0$, then from the relations

$$\lim_{\tau \to \bar{\tau}^{\delta+}{}_{n-1}} \frac{y'(c, \tau)}{y(c, \tau)}=\infty, \quad \lim_{\tau \to \bar{\tau}^{\delta+}{}_{n}} \frac{y'(c, \tau)}{y(c, \tau)}=-\infty$$

it follows that for the number $\frac{\beta_1}{\beta}$ there are the numbers $\tau_{\delta+n} \in (\bar{\tau}_{\delta+n-1}, \bar{\tau}_{\delta+n})$ such that

$$\frac{y'(c, \tau_{\delta+n})}{y(c, \tau_{\delta+n})}=\frac{\beta_1}{\beta},$$

i.e. the third condition of (13) holds and at the same time the corresponding solutions of (A), $y(x, \tau_{\delta+n})$ have exactly $\delta+n$ zeros in $(a, c)$, where $\delta$ is a nonpositive integer and $n$ is a positive one. Theorem 4 is proved completely.

**Lemma 4.** *Suppose the coefficients of* (A) *satisfy* $2C(x, \tau)-B'(x, \tau)\leq 0$ *and the sign* = *does not hold in any interval, and* $B(x, \tau)\leq 0$. *Let equation* (A) *be oscillatory. Then the solution* $y(x, \tau)$ *of* (A), *for which* $y'(a, \tau)=0$, *or* $y''(a, \tau)=0$, *is oscillatory.*

Proof. Let $y(x, \tau)$ be a solution of (A) such that $y'(a, \tau)=0$. Suppose on the contrary that $y(x, \tau)$ is not oscillatory. Because of Theorem 15 in [2], $y(x, \tau)$ is without zeros and it is positive and nonincreasing. Let $y(a, \tau)=d>0$, $y'(a, \tau)=0$, $y''(a, \tau)=b\leq 0$. The solution $y_1(x, \tau)$ of (A) such that $y_1(a, \tau)=0$, $y_1'(a, \tau)=1$, $y_1''(a, \tau)=0$ is oscillatory since it has a zero. Then there exists a number $\gamma>a$ and a number $c>0$ such that $y(x, \tau)-cy_1(x, \tau)$ has a double zero at $\gamma$ and a single zero at $a$, which contradicts the identity (3).

Similarry we prove that $y(x, \tau)$ with the property $y''(a, \tau)=0$ is oscillatory in $(0, \infty)$.

Because of Theorem 4.11 in [5], there are between two consecutive zeros of any solution of (A) at most two zeros of any other solution. From that fact and Lemma 4 we obtain.

**Theorem 5.** *Suppose the assumption* (i) *or* (iii) *of Theorem 4 hold. Let* $B(x, \tau)\leq 0$ *in D. Let* $y(x, \tau)$ *be a solution of* (A) *such that* $y^{(i)}(a, \tau)=0$, $i=0, 1, 2$. *Then the conclusion of Theorem 2 holds.*

**Theorem 6.** *Suppose the assumptions* (i) *or* (iii) *of Theorem 4 hold. Let* $B(x, \tau)\leq 0$ *and* $C(x, \tau)\geq 0$. *Then there exist a nonpositive integer* $\delta$ *and a sequence* $\{\tau_{\delta+n}\}$ *of values of* $\tau$ *tending to* $T$ *for which the boundary value problem*

$$
\begin{aligned}
& y^{(i)}(a, \tau)=0, \qquad i=0, 1, 2, \\
(15)\qquad & \alpha_1 y(b, \tau)-\alpha y'(b, \tau)=0, \qquad |\alpha_1|+|\alpha|>0, \\
& \beta_1 y(c, \tau)-\beta y'(c, \tau)=0, \qquad |\beta_1|+|\beta|>0
\end{aligned}
$$

*has a unique solution up to a multiplicative constant. The corresponding solution* $y(x, \tau_{\delta+n})$ *has exactly* $\delta+n$ *zeros in the interval* $(a, c)$.

Proof. For $i=0$ the conclusion of this theorem follows from Theorem 4. If $i=1$, then every solution of (A) which satisfies the first condition of (15) can be expressed in the form $y(x, \tau)=c_0y_0(x, \tau)+c_2y_2(x, \tau)$. Applying Theorem 7 in [1] it follows that $w(b, \tau)=y_0(b, \tau)\cdot y_2'(b, \tau)-y_0'(b, \tau)\cdot y_2(b, \tau)\neq 0$, if $B(x, \tau)\leq 0$. Thus $y(x, \tau)$ satisfying the first and second condition of (15) is unique up to a multiplicative constant.

If $i=2$, then every solution of (A) which satisfies the first condition of (15) can be expressed in the form $y(x,\tau)=c_0y_0(x,\tau)+c_1y_1(x,\tau)$. Since $w(b,\tau)=y_0(b,\tau)\cdot y_1'(b,\tau)-y_0'(b,\tau)\cdot y_1(b,\tau)\neq 0$ if $B(x,\tau)\leq 0$ and $C(x,\tau)\geq 0$, then $y(x,\tau)$ satisfying the first and the second condition of (15) is unique up to a multiplicative constant.

The next step of the proof to satisfy the third condition of (15) is the same as in Theorem 4.

## 5. Disconjugacy and existence of the number $\delta=0$, resp. $\delta=1$ in $\{\tau_{\delta+n}\}$

In this last section we show that under the additional assumptions for the coefficients of (A) we can choose $\delta=0$, resp. $\delta=1$ in Theorem 4, and so give a theorem which is an analogy to the Sturm oscillatory theorem for differential equations of the second order ([3], p. 168).

For a linear equation with constant coefficients there holds.

**Lemma 5.** *A third order differential equation with constant coefficients is nonoscillatory in* $(0,\infty)$ *if and only if it is disconjugate, i.e. if its every solution has at most two single zeros, or one double zero in* $(0,\infty)$.

*Simillary, there holds that the Euler equation*

$$z'''+\frac{p}{x^2}z'+\frac{\frac{\varepsilon}{2}-p}{x^3}z=0$$

*is nonoscillatory in* $(0,\infty)$ *if and only if it is disconjugate in* $(0,\infty)$.

**Theorem 7.** *Suppose the coefficients of* (A) *satisfy*

1. $$B(x,\tau)\leq p,\quad 0\leq 2C(x,\tau)-B'(x,\tau)\leq q,$$

*where* $p\leq 0$, $q\leq\frac{4}{3\sqrt{3}}(-p)^{\frac{3}{2}}$ *are constants, or*

2. $$B(x,\tau)\leq\frac{p}{x^2},\quad 0\leq 2C(x,\tau)-B'(x,\tau)\leq\frac{\varepsilon}{x^3},$$

*where* $p\leq 1$, $\varepsilon\leq\frac{4}{3\sqrt{3}}(1-p)^{\frac{3}{2}}$ *are constants.*

*Then the equation* (A) *is disconjugate in* $(0,\infty)$.

Proof. Because of Theorem (ii), the equation (A) is nonoscillatory. From the relations between $p$ and $q$, resp. $p$ and $\varepsilon$, it follows that the equation (4), resp. the Euler equation is nonoscillatory and, by Lemma 5, disconjugate. Let $a$ be an arbitrary positive number. Let $y(x, \tau)$ be a solution of (A) for which $y(a, \tau)=y'(a, \tau)=0$, $y''(a, \tau)\neq 0$. This solution has no zero in $(0, \infty)$. Indeed, if $y(b, \tau)=0$, $b>a$, then because of Theorem (i), every solution of (4), or the Euler equation with a single zero at $a$ has a zero in $(a, b]$. This is a contradiction to the fact that there is a solution of (4), resp. of the Euler equation, which has a zero at $a$ and has no zero in $(a, \infty)$.

By the identity (3) it follows that every solution of (A) with a single zero at $a$ has at most one zero in $(0, \infty)$. Since $a$ is an arbitrary point of $(0, \infty)$, every solution of (A) has at most one double zero, or two single zeros in $(0, \infty)$.

**Theorem 8.** *Suppose the coefficients of* (A) *satisfy the condition* $2C(x, \tau)-B'(x, \tau)\geq 0$ *and at the same time the sign* = *does not hold in any interval. Furthemore let*

a) *there exist numbers* $K_1$, $K_2$ *such that*

$$K_1\leq B(x, \tau)\leq K_2<0 \quad \textit{for all} \quad (x, \tau)\in D,$$

$$\lim_{\tau\to T}[2C(x, \tau)-B'(x, \tau)]=\infty \quad \textit{uniformly in} \quad x\in(0, \infty),$$

$$\lim_{\tau\to t}[2C(x, \tau)-B'(x, \tau)]=0 \quad \textit{uniformly in} \quad x\in(0, \infty),$$

*or*

b) *there exists a number* $p<0$ *such that*

$$B(x, \tau)\leq\frac{p}{x^2} \quad \textit{for all} \quad (x, \tau)\in(a, \infty)\times(t, T), \quad a>0,$$

$$\lim_{\tau\to T}[2C(x, \tau)-B'(x, \tau)]=\infty \quad \textit{uniformly in} \quad x\in(0, \infty),$$

$$\lim_{\tau\to t}[2C(x, \tau)-B'(x, \tau)]x^3=0 \quad \textit{uniformly in} \quad x\in(0, \infty),$$

*or*

c) *the assumptions* (iii) *of Theorem 4 hold,*

$$B(x, \tau)\leq K_2<0 \quad \textit{for all} \quad (x, \tau)\in D,$$

$$\lim_{\tau\to t}[2C(x, \tau)-B'(x, \tau)]=0 \quad \textit{uniformly in} \quad x\in(0, \infty).$$

*Then the conclusions of Theorem 4 hold, where* $\delta=0$ *if* $\beta=0$, *and* $\delta=1$ *if* $\beta\neq 0$.

Proof. Since the assumptions of Theorem 4 are included in this theorem, the conclusion of Theorem 4 holds. Now we are to show that $\delta=0$ if $\beta=0$, and $\delta=1$ if $\beta\neq 0$ in (13). First of all we consider that the conditions a) and b) to be fulfilled.

Since

$$\lim_{\tau \to t} [2C(x, \tau) - B'(x, \tau)] = 0 \quad \text{uniformly in} \quad x \in (0, \infty).$$

then for $\varepsilon = \frac{4}{3\sqrt{3}}(-K_2)^{\frac{3}{2}} > 0$ there is $\tau_0$ such that

$$0 \leq 2C(x, \tau_0) - B'(x, \tau_0) < \frac{4}{3\sqrt{3}}(-K_2)^{\frac{3}{2}},$$

where $K_2$ is a number for which $B(x, \tau) \leq K_2$. Because of Theorem 7, the equation (A) is disconjugate for $\tau = \tau_0$. Since $y(x, \tau_0)$ is the solution of (A) having a zero at $a$, then this solution has at most one zero in $(a, c)$. Because the zeros of $y(x, \tau)$ are a continuous function of $\tau$, then

$$\lim_{\tau \to T} [2C(x, \tau) - B'(x, \tau)] = \infty \quad \text{uniformly in} \quad x \in (0, \infty)$$

implies that there is $\bar{\tau}$ such that $y(x, \bar{\tau})$ has exactly one zero in $(a, c)$.

Now let the conditions b) hold. From the condition

$$\lim_{\tau \to t} [2C(x, \tau) - B'(x, \tau)]x^3 = 0 \quad \text{uniformly in} \quad x \in (a, \infty)$$

it follows that for a number $\varepsilon > 0$, satisfying

$$\varepsilon \leq \frac{4}{3\sqrt{3}}(1-p)^{\frac{3}{2}},$$

there is a $\tau_0$ such that

$$0 \leq [2C(x, \tau_0) - B'(x, \tau_0)]x^3 < \varepsilon,$$

i.e.

$$0 \leq 2C(x, \tau_0) - B'(x, \tau_0) \leq \frac{\varepsilon}{x^3}, \quad \text{where} \quad \varepsilon \leq \frac{4}{3\sqrt{3}}(1-p)^{\frac{3}{2}}.$$

Then, because of Theorem 7, the equation (A) is disconjugate for $\tau = \tau_0$. The condition

$$\lim_{\tau \to T} [2C(x, \tau) - B'(x, \tau)] = \infty \quad \text{uniformly in} \quad x \in (a, \infty)$$

implies that there is a number $\bar{\tau}$ such that the corresponding solution $y(x, \bar{\tau})$ of (A) has the single zero at $a$ and exactly one zero in $(a, c)$.

Hence we conclude that in each case there is a number $\bar{\tau}$ such that $y(x, \bar{\tau})$ has a zero at $a$ and exactly one zero in $(a, c)$. If we follow the proof of Theorem 4, we obtain: If $\beta=0$, then there exists a sequence $\{\tau_n\}_{n=1}^{\infty}$ of values $\tau$ such that the corresponding solutions $y(x, \tau_n)$ of (A) have exactly $n$ zeros in $(a, c)$. If $\beta \neq 0$, then there exists a sequence $\{\tau_n\}_{n=2}^{\infty}$ of values $\tau$ such that the corresponding solutions $y(x, \tau_n)$ of (A) have exactly $n$ zeros in $(a, c)$, $n \geqslant 2$.

## REFERENCES


[1] GREGUŠ, M.: Uber die lineare homogene Differentialgleichung dritter Ordnung. Wiss. Z. Univ. Halle, Math.-Nat. XII/3, s. 265—286, 1963.

[2] ROVDER, J.: Oscillation criteria for third-order linear differential equations. Mat. Čas. 25, 1975, 231—244.

[3] RUBINSTEIN, Z.: A Course in Ordinary and Partial Differential Equations. Academic Press, New York and London 1969.

[4] SANSONE, G.: Studi sulle equazini differenziali lineari omogenee di terzo ordine nel campo reale. Revista Matem. y Fizica Teorica, Serie A, 1948, 195—253.

[5] SWANSON, C. A.: Comparison and Oscillation Theory of Linear Differential Equation. New York and London 1968.

[6] ŠVEC, M.: Einige asymptotische und oszillatorische Eigenschaften der Differntialgleichung $y''' + A(x)y' + B(x)y = 0$. Czech. Mat. J. 15 (90) 1965, 378—393.


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## ТРЁХТОЧЕЧНАЯ ЗАДАЧА ДЛЯ ЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ТРЕТЬЕГО ПОРЯДКА

Йосеф Ровдер

### Резюме


**При помощи теорем сравнения для дифференциального уравнения (1) доказаны теоремы о колебании для дифференциального уравнения (A). При помощи этих теорем доказано, что существует последовательность $\{\tau_{\delta+n}\}$ собственных значений, для которых краевая задача (13) имеет решение определенное с точностью до произвольного постоянного множителя. Решение $y(x, \tau_{\delta+n})$ обращается в нуль на интервале $(a, c)$ равно $\delta+n$ раз.**

**В последней части доказано, что при дальнейших предположениях о коэффициентах уравнения (A) существует $\delta=0$ если $\beta=0$, и $\delta=1$ если $\beta\neq 0$.**