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TWO CONSTRUCTIONS OF GEODETIC GRAPHS

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1. Introduction

The main purpose of this paper is to present two constructions of geodetic graphs. A graph is *geodetic* if two arbitrary points are connected by a unique shortest path. The problem of characterizing all geodetic graphs was first proposed by Ore [6, p. 105].

There are several results concerning geodetic graphs. We shall mention some of them. A graph G is *strongly geodetic* if any two points of G are connected by at most one path of length not exceeding the diameter of G. These graphs were studied by Bosák, Kotzig and Znám [2]. Every connected strongly geodetic graph is obviously geodetic and it is either a tree or a Moore graph [2]. A *Moore graph* can be defined as a graph with a diameter d and girth $2d+1$. Such a graph must be regular [2, 7]. Hoffman and Singleton [5] have shown that there are at most 4 possible degrees of Moore graphs with the diameter 2, namely 2, 3, 7, and 57. For each of the first three degrees there is the only Moore graph, but the existence or uniqueness of a Moore graph with degree 57 is an open question. Bannai and Ito [1] and independently Damerell [3] have proved that any Moore graph with diameter greater than 2 is an odd cycle.

The class of planar geodetic graphs was characterized by Stemple and Watkins [10]. Some of their results will be mentioned in the sequel.

Skala [8] has investigated in fact a special class of geodetic graphs with the diameter 2. The last graphs were studied by Stemple [9] and Zelinka [11].

In this paper, except as otherwise indicated, the notation and terminology are based on Harary [4]. Given a graph G, $V(G)$ and $E(G)$ denote its point set and line set, respectively. The distance between the points $u, v \in V(G)$ is denoted by $d_G(u, v)$. A shortest $u-v$ path is called *geodesic*. The supremum of all distances in G is the diameter of G and is denoted by $d(G)$. Given an even cycle Z of G (i.e., Z has an even length), we say that points $x, y \in V(Z)$ are *Z-opposite* if $d_Z(x, y) = d(Z)$. Let $u, v \in V(Z)$, $u \neq v$. There are exactly two paths in Z joining u and v. One of them (in our figures usually the right-hand segment of Z) is denoted by $Z[u, v]$ and the other by $Z[v, u]$.

There are only few general results on geodetic graphs. Two of them are the following lemmas.

Lemma 1 (Stemple and Watkins [10, Th. 2]). *A connected graph G is geodetic if and only if G contains no even cycle Z such that for some Z-opposite pair of points x, y,* $d_G(x, y) = d(Z)$.

This lemma is simple, but as we shall see, a useful criterion.

Since the block-cutpoint graph of any nontrivial connected graph is a tree [4, Th. 4.4], the following lemma follows immediately.

Lemma 2 (Stemple and Watkins [10, Th. 3]). *A connected graph G is geodetic if and only if every block of G is geodetic.*

Thus it is often sufficient to study the geodetic blocks only.

2. Two constructions

Sometimes, it is convenient to have several examples of geodetic graphs. In this section, we give two classes of such examples.

Firstly, for a given integer $d \geq 1$, we construct a graph WP_d (widespread Petersen graph) with diameter d as follows (see Fig. 1):

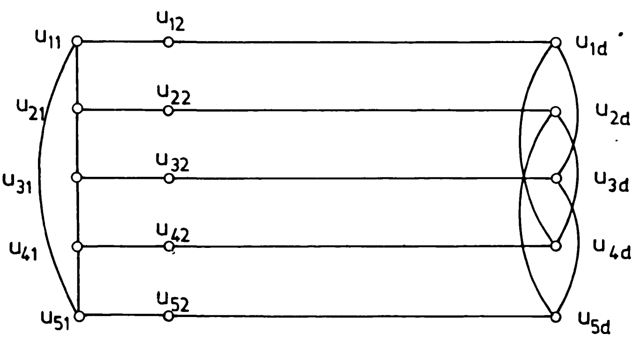

Fig. 1

$$V(WP_d) = \{u_{ij} \mid i = 1, 2, \dots, 5;\ j = 1, 2, \dots, d\},$$

$$E(WP_d) = \{u_{i1}u_{j1} \mid i - j \equiv 1 \pmod 5\} \cup \{u_{id}u_{jd} \mid i - j \equiv 2 \pmod 5\} \cup \bigcup_{i=1}^{5} \{u_{ij}u_{i,j+1} \mid j = 1, 2, \dots, d-1\}.$$

Note that the graph WP_2 is the Petersen graph and WP_1 is the complete graph K_5.

Theorem 1. *For any integer $d \geq 1$ the WP_d is a geodetic graph with diameter d.*

Proof. One can easily verify that WP_d has the diameter d and contains no even cycle with a length less than $2d+1$. Then the proof follows from Lemma 1.

Now we shall describe the second construction. We say that a graph G_1 is an *extension* of a graph G at a point $v \in V(G)$ if G_1 is formed from G by subdividing each line incident with v into two through insertion of one new point. We also say that G was *extended* at v to form G_1. Given a complete graph K_n $(n \geq 2)$, its points will be called *basic points*. We say that a graph G_1 is of the type $K_n^{(i)}$, where $i \geq 0$ is an integer, if either $i=0$ and $G_1 = K_n$ or $i \geq 1$ and there is a graph G of the type $K_n^{(i-1)}$ and a basic point v of G such that G_1 is the extension of G at v. The graph G_1 has the same basic points as G. In general, a $K_n^{(i)}$ has n basic points and $i(n-1)$ nonbasic points. Obviously, any $K_n^{(i)}$ and K_n are homeomorphic. Further, we see that the number i does not determine a $K_n^{(i)}$ uniquely.

Theorem 2. *Any $K_n^{(i)}$ with $n \geq 2$ and $i \geq 0$ is a geodetic graph.*

Proof. Let G be a given $K_n^{(i)}$. According to Lemma 1, it is sufficient to verify that for any even cycle Z and any pair of *Z-opposite points* x, y, $d_G(x, y) < d(Z)$. Note that any even cycle contains an even number of basic points (at least 4). There are two cases to consider.

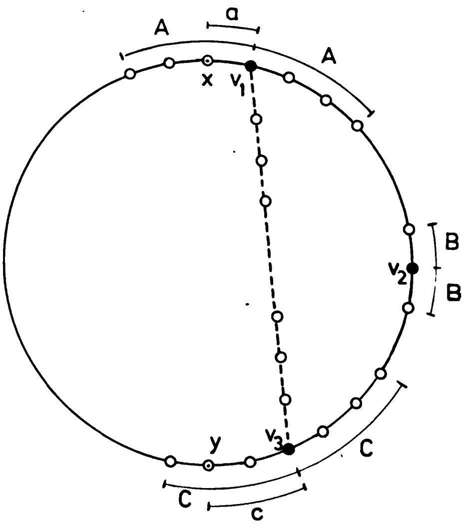

Fig. 2

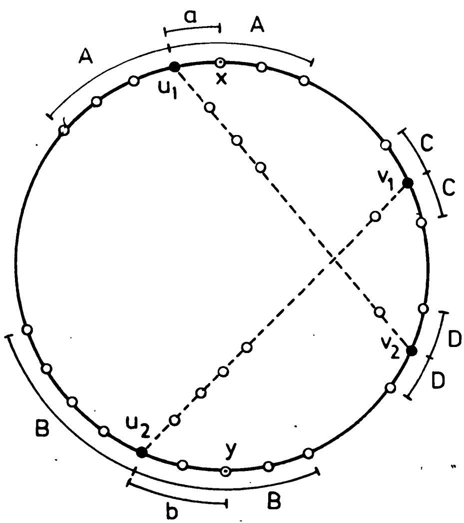

Fig. 3

Case 1. $Z[x, y]$ or $Z[y, x]$ contains at least 3 basic points. Then $d_G(x, y) < d(Z)$. This can be easily seen with the aid of Fig. 2, where we have drawn the case when $Z[x, y]$ contains at least 3 basic points. (In the Fig. 2 as well as in Figs. 3 and 4, the full or the empty small circles mean basic or nonbasic points of G, respectively. The capitals denote how many times the complete graph K_n was extended at a respective basic point. The letters a, b, c, d denote the situation of x and y with respect to basic points.) As the path $x - v_1 - v_3 - y$ has the length $a + A + 1 + C + c$ and $Z[x, y]$ has the length at least $a + A + 1 + 2B + 1 + C + c$, the assertion follows.

Case 2. Both the paths $Z[x, y]$ and $Z[y, x]$ contain less than 3 basic points. Since Z contains at least 4 basic points, both the $Z[x, y]$ and the $[y, x]$ contain exactly 2 basic points. Owing to the symmetry, it is sufficient to consider only the two subcases illustrated in Figs. 3 and 4.

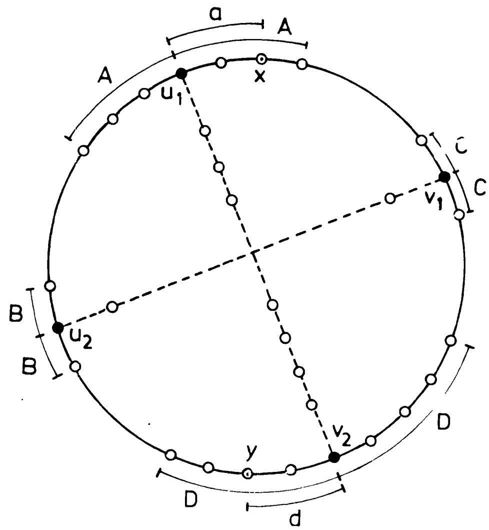

Fig. 4

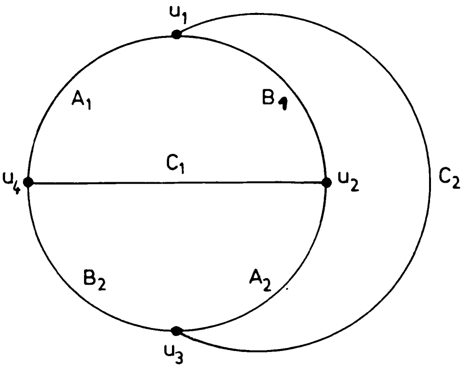

Fig. 5

In the first subcase (Fig. 3) we assert that at least one of the paths $x - u_1 - v_2 - y$ and $x - v_1 - u_2 - y$ has length less than $d(Z)$ $(d(Z) = a + A + 1 + B + b = A - a + 1 + 2C + 1 + 2D + 1 + B - b)$. Otherwise it would be $a + A + 1 + 2D + 1 + B - b > a + A + 1 + B + b$ and $A - a + 1 + 2C + 1 + B + b > A - a + 1 + 2C + 1 + 2D + 1 + B - b$, because the equalities are excluded by the different parities. Summing up the last two inequalities, we obtain a contradiction.

In the second subcase (Fig. 4) at least one of the paths $x - u_1 - v_2 - y$ and $x - v_1 - u_2 - y$ has length less than $d(Z)$ $(d(Z) = A - a + 1 + 2C + 1 + D + d = a + A + 1 + 2B + 1 + D - d)$. In the opposite case it would be $a + A + 1 + D + d > A - a + 1 + 2C + 1 + D + d$ and $A - a + 1 + 2C + 1 + 2B + 1 + D - d > a + A + 1 + 2B + 1 + D - d$. Summing up these two inequalities, we get a contradiction. This completes the proof.

3. Planar geodetic graphs

In this section we reformulate a result of Stemple and Watkins [10]. At first we give some necessary notions.

Let G be a graph homeomorphic to K_4. Let u_1, u_2, u_3, and u_4 be the four points of degree 3 and let A_1, A_2, B_1, B_2, C_1, *and* C_2 be lengths of the 6 segments, i.e., $u_i - u_j$ paths corresponding to the 6 lines of K_4, in accordance with Fig. 5. Then G is said to be a *canonical wheel* if the following conditions are satisfied:

(i) *Each of the* 6 *segments is a unique geodesic of* G *joining its ends.*
(ii) $A_1 + A_2 = B_1 + B_2 = C_1 + C_2$.
(iii) *Each cycle consisting of* 3 *segments is odd.*

Lemma 3 (Stemple and Watkins [10, Th. 1]). *A planar connected graph* G *is geodetic if and only if each block of* G *is one of the following:*

(a) K_2,
(b) *an odd cycle,*
(c) *a canonical wheel.*

This result fully characterizes the planar geodetic graphs. We shall show that it can be expressed as follows.

Theorem 3. *A planar connected graph* G *is geodetic if and only if each block of* G *is a* $K_n^{(i)}$ *with* $2 \leqslant n \leqslant 4$.

Proof. According to Lemma 3 and Theorem 2, it is sufficient to prove that any canonical wheel is a $K_4^{(i)}$.

Let G be a canonical wheel different from K_4. We shall prove that there is a canonical wheel $\mathcal{J}'$ and its point v of degree 3 such that $\mathcal{J}$ is the extension of G' at v. Then the proof will follow immediately by induction on the number of points of degree 2. In other words, we have to find a point u_i at which $\mathcal{J}$ can be reduced to receive a canonical wheel. Obviously, such a point u_i needs to be incident only with segments of length at least 2. Suppose that each point u_i is incident with some segment of length 1. Then at least one of the following two possibilities occurs.

1. There is a point, say, u_1 such that all segments incident with it are of length 1, i.e., $A_1 = B_1 = C_2 = 1$. Then (i) implies $A_2 = B_2 = C_1 = 1$, consequently, G is K_4 which contradicts our assumption.

2. There are two independent segments (say those corresponding to A_1 and A_2) of length 1. Then by (ii) G is K_4, a contradiction.

Thus there is a point, say, u_1 such that all segments incident with it are of length at least 2 (A_1, B_1, $C_2 \geqslant 2$). Then we can reduce G at u_1 (i.e., we shorten each segment at u_1 by 1) to obtain a graph G' homeomorphic to K_4. The corresponding parameters of G' are related to those of G as follows: $A_1' = A_1 - 1$, $A_2' = A_2$, $B_1' = B_1 - 1$, $B_2' = B_2$, $C_1' = C_1$, $C_2' = C_2 - 1$. It can be easily verified that G' fulfills the conditions (ii) and (iii). Now we are going to consider the condition (i). Instead of (i), it is sufficient to verify "the strict triangle inequality" for each "triangle". (E.g., if $C_2 < A_1 + B_2$ and $C_2 < A_2 + B_1$, then by (ii) also $C_2 < A_1 + C_1 + A_2$ and $C_2 < B_1 + C_1 + B_2$, hence the $u_1 - u_3$ segment of length C_2 is the unique $u_1 - u_3$ geodesic.) By the assumption, $C_2 < A_1 + B_2$ and $C_2 < B_1 + A_2$. This implies $C_2' < A_1' + B_2'$ and $C_2' < B_1' + A_2'$. Analogously, we see that $B_1' < A_2' + C_2'$ and $B_1' < A_1' + C_1'$, $A_1' < B_1' + C_1'$ and $A_1' < B_2' + C_2'$. Also $C_1' < A_2' + B_2'$, $B_2' < C_1' + A_2'$, $A_2' < C_1' + B_2'$.

Nevertheless, it can happen that some of the other three desired strict triangle inequalities is not true. Without a loss of generality, we can suppose that $C_1' > A_1' + B_1'$, i.e., $C_1 \geqslant A_1 + B_1 - 2$. By the assumption (i) on G, we have $C_1 \leqslant A_1 + B_1 - 1$. By (iii) $A_1 + B_1 + C_1$ is odd, so $C_1 = A_1 + B_1 - 1$. Using (ii), the last inequality implies

$$A_1 + B_1 + C_2 = m + 1\,, \tag{1}$$

where $m = A_1 + A_2 = B_1 + B_2 = C_1 + C_2$ (see(ii)). Thus we have proved that if the reduction at a point u_i results in a graph which is no canonical wheel, then the sum of the lengths of the three segments at u_i gives $m + 1$.

Now we shall prove that if we had no success with the reduction of G at u_1, then the reduction of G at u_2 results in a canonical wheel.

Firstly, we have to show that A_2, B_1, $C_1 \geqslant 2$. By the assumption on u_1, $B_1 \geqslant 2$. If $A_2 = 1$, then by (ii) $A_1 = m - 1$, which being substituted into (1) gives $B_1 = C_2 = 1$, *a contradiction. Analogously, the assumption* $C_1 = 1$ implies that $A_1 = B_1 = 1$, a contradiction again.

The reduction of G at u_2 gives a graph G'' homeomorphic to K_4. If G'' is a canonical wheel, then there is nothing to prove. In the opposite case, we can (as above on G') prove that

$$A_2 + B_1 + C_1 = m + 1\,. \tag{2}$$

Summing up (1) and (2) and using (ii), we obtain $B_1 = 1$, which is impossible.

This completes the proof of Theorem 3.

4. Problems

1. We conjecture that any geodetic graph homeomorphic to K_n ($n \geqslant 2$) is a $K_n^{(i)}$. This conjecture is true for $n \leqslant 4$ (see Theorem 3).
2. It would be interesting to find a geodetic block with the diameter at least 3 and different from any WP_d and $K_n^{(i)}$. Note that there are such graphs with diameter 2 (see [8] or [9]).

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ДВЕ КОНСТРУКЦИИ ГЕОДЕЗИЧЕСКИХ ГРАФОВ

Ян Плесник

Резюме

Неориентированный граф называется геодезическим графом если для каждых двух вершин существует единственная кратчайшая цепь между ими. Автор давает две конструкции этих графов. Первая (рис. 1) представляет натяжение графа Петерсена. Вторая состоит в натяжении полного графа при каждой из выбранных вершин на единицу или больше. Такой граф гомеоморфен полному графу. Тоже показывается (теорема 3), что эта конструкция охватывает геодезические плоские графы (охарактеризованные в [10]). Автор предлагает гипотезу: Вторая конструкция охватывает каждый геодезический граф, который гомеоморфен полному графу.