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MOVES WITHOUT FORBIDDEN TRANSITIONS IN A GRAPH

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Throughout the paper we understand by a graph a non-oriented finite graph without loops. Let G be a graph with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Denote by E_i the set of all edges of G incident at v_i and by d_i the degree of v_i (i. e. $d_i = |E_i|$).

A sequence $P = \{x_1, y_1, x_2, y_2, \ldots, y_n, x_{n+1}\}$ of elements of G (where $n > 0$; x_v are vertices; y_j are edges and y_j joins in G the vertices $x_j \neq x_{j+1}$) is called a move of G if any edge of G occurs in P at most once. The move is closed (or open) if $x_{n+1} = x_1$ (or $x_{n+1} \neq x_1$). The moves obtained from P by the reversion of its elements and by a translation of its elements (if P is closed) are not considered here as different from P.

By a transition of P through x_i we mean a triple of elements $\{y_{i-1}, x_i, y_i\}$. In the case when P is a closed move, we consider as a transition through $x_1 = x_{n+1}$ also the triple $\{y_n, x_1, y_1\}$.

Under the decomposition of G into moves we understand such a set $\bar{P}$ of moves of G, that every edge of G belongs exactly to one move from $\bar{P}$. Obviously, we have:

Lemma 1. *The decomposition of G into closed moves exists if and only if every vertex $v_i \in V(G)$ is of an even degree.*

If the decomposition of G into closed moves contains only one element, than we call this closed move a Eulerian line of G. The following lemma is known (see, e. g. [1]):

Lemma 2. *The Eulerian line of G exists if and only if G is connected, contains at least two vertices and every of its vertices is of an even degree.*

Lemma 3. *Let $\bar{P}$ be a decomposition of G into closed moves and let v_i be a vertex of $V(G)$. For the number $p(i)$ of different transitions $\{e_x, v_i, f_x\}$ ($x = 1, 2, \ldots, p(i)$) through v_i we have $p(i) = \frac{1}{2}d_i$. Further we have: $[j \neq k] \Rightarrow [\{e_j, f_j\} \cap \cap \{e_k, f_k\} = \emptyset]$ and $\{e_1, f_1\} \cup \{e_2, f_2\} \cup \ldots \cup \{e_{p(i)}, f_{p(i)}\} = E_i$.*

Proof. The validity of the lemma is evident from the definition of the closed move, of the decomposition of a graph into moves and from the definition of the transition through the vertex.

We call the decomposition $D_i = \{\{e_1, f_1\}, \{e_2, f_2\}, \ldots, \{e_{p(i)}, f_{p(i)}\}\}$ of the set E_i from Lemma 3 the δ-decomposition of E_i formed by the decomposition P and we call the system $\bar{D} = \{D_1, D_2, \ldots, D_n\}$ the δ-system formed in G by $\bar{P}$. We denote the fact that $\bar{D}$ is the δ-system formed by $\bar{P}$ thus: $\bar{D} = \delta(\bar{P})$.

Lemma 4. *Every decomposition of G into closed moves forms exactly one δ-system in G, and to every system $\bar{D} = \{D_1, D_2, \ldots, D_n\}$ of the decompositions of the sets E_i into pairs of edges there exists exactly one decomposition $\bar{P}$ of G into closed moves such that $\bar{D} = \delta(\bar{P})$.*

Proof. The validity of the first assertion of the lemma follows directly from the definition of the δ-system.

Now let $\bar{D} - \{D_1, D_2, \ldots, D_n\}$ be a system of decompositions of the sets E_i into pairs of edges. Let us travel along the elements of G according to the following rules:

(1) If in a travelling we arrive along an edge e to its endpoint (= vertex v_x), then we proceed along that edge which forms with e a pair in the decomposition $D_x \in \bar{D}$ (in other words: the system $\bar{D}$ determines all transitions through the vertices from $V(G)$).

(2) We can continue to travel only along such an edge along which we have not been travelling yet; if we should according (1) continue to travel along an edge which we passed already, our travelling is finished.

Let us begin our travelling along an edge e_1 from one of its endpoints. It is clear that after having passed a finite number of edges our travelling must end and it ends in the vertex from which our travelling started.

We can describe our whole travelling by a certain closed move (denote it by P_1). If P_1 contains all the edges of G, then for $\bar{P} = \{P_1\}$ we have: $\delta(\bar{P}) = \bar{D}$. If in G there exists an edge e_2, not belonging to P_1, we can begin our analogical travelling along the edge e_2. We obtain then a closed move P_2. Then either $P_1 \cup P_2$ contains all the edges of G and for $\bar{P} = \{P_1, P_2\}$ we have $\delta(\bar{P}) = \bar{D}$ or there exists an edge e_3, not belonging to $P_1 \cup P_2$. This consideration can be repeated and after a finite number of steps we obtain such a set $\bar{P} -$ $\{P_1, P_2, \ldots, P_m\}$ of closed moves of G, where every edge from G belongs at least to one of the moves from $\bar{P}$. From the construction of these moves (the departure from the vertex after the given arrival in the vertex is determined uniquely!) it follows that every edge of G belongs only to one move from $\bar{P}$. From this it follows that $\delta(\bar{P}) = \bar{D}$, q. e. d.

Let for every vertex $v_i \in V(G)$ a certain decomposition Q_i of the set E_i into classes of edges (incident to v_i) be given. We shall call a move P_x in G admissible with respect to the system $\bar{Q} = \{Q_1, Q_2, \ldots, Q_n\}$ if and only if for every $v_i \in V(G)$ the following holds: two edges of an arbitrary transition of P_x through v_i belong to different classes of Q_i. If all moves of a certain decomposition $\bar{P}$ of G into moves are admissible with respect to $\bar{Q}$, we call $\bar{P}$ an admissible decomposition into moves with respect to $\bar{Q}$.

Note 1. To the system $\bar{Q}$ there corresponds a set of „forbidden" transitions through the vertices. It is clear that we can construct the set of forbidden transitions also in another way. Some related problems have alrady been solved [2], [3], [4]). The first who formulated the problem to find the necessary and sufficient conditions for the existence of a Eulerian line not-containing the forbidden transitions was Nash—Williams (at Tihany, in September 1966). In this paper we deal with the solution of this problem, and also with that of a more general problem.

Theorem 1. *Let G be a connected graph, and let its every vertex v_i be of an even degree (i. e. let $d_i = 2c_i$, where c_i is a natural number). Let $\bar{Q} = \{Q_1, Q_2, \ldots, Q_n\}$ be a given system of decompositions of the sets (E_i $i = 1, 2, \ldots, n$) into classes of edges. A Eulerian line of G, admissible with respect to $\bar{Q}$ exists if and only if for all $i \in \{1, 2, \ldots, n\}$ we have: the number of elements of an arbitrary class of Q_i is not greater than c_i.*

Proof. If a certain class of a decomposition $Q_i \in \bar{Q}$ has more edges than c_i, then E_i cannot be decomposed into pairs of edges so that two edges of every pair belong to different classes of the decomposition Q_i. Hence the condition for the existence of the Eulerian line admissible with respect to $\bar{Q}$ mentioned in the theorem is a necessary condition.

Let $\bar{Q}$ have the requested property, i. e. for all $i \in \{1, 2, \ldots, n\}$ the following holds: the number of elements of an arbitrary class from Q_i is less than or equal to c_i. Let us denote successively the edges of the set E_i by the symbols $f_i(1), f_i(2), \ldots, f_i(2c_i)$ so that first of all we denote all edges of one class from Q_i and then all edges of the other classes of Q_i. We then construct the decomposition D_i of the set E_i into c_i pairs of edges as follows: $D_i = \{\{f_i(1), f_i(1 + c_i)\}, \{f_i(2), f_i(2 + c_i)\}, \ldots, \{f_i(c_i), f_i(2c_i)\}\}$. It is clear that every pair of D_i contains edges belonging to different classes of Q_i. Let us put $\bar{D} = \{D_1, D_2, \ldots, D_n\}$. According to Lemma 4 there exists exactly one decomposition $\bar{P} = \{P_1, P_2, \ldots, P_m\}$ of G into closed moves so that $\bar{D} = \delta(\bar{P})$. From the construction of the system $\bar{D}$ it follows that $\bar{P}$ is an admissible decomposition into moves with respect to $\bar{Q}$. If $m = 1$, then P_1 is the requested Eulerian line.

Suppose that $m > 1$. As G is a connected graph, there exists at least one such a vertex v_k, which is the common vertex of certain two moves P_r, P_s from $\bar{P}$. Let $\{g_r, v_k, h_r\}$ be an arbitrary transition of P_r through v_k and let $\{g_s, v_k, h_s\}$ be an arbitrary transition of P_s through v_k.

From the construction of D_k it follows that the edges g_r, h_r (and also the edges g_s, h_s) belong to different classes of Q_k. We distinguish the following three cases: the edges of the set $H = \{g_r, h_r, g_s, h_s\}$ belong: (I) to 4 different classes; (II) to 3 different classes; (III) to 2 different classes of Q_k. Without loss of generality it can be supposed that in the case (II) the edges g_r, g_s belong to the same class of Q_k and in the case (III) the edges g_r, g_s belong to one and the edges h_r, h_s belong to another class of Q_k.

Assertion. The moves P_r, P_s can be unified into one closed move P_t so that two edges of an arbitrary transition through an arbitrary vertex v_x belong again to different classes of Q_x. Let us prove it. Consider that we can travel along all the elements of a closed move (we are talking about a travelling mentioned in the proof of Lemma 4) so that the transition of the move does not change for any of the vertices even if we begin our travelling in an arbitrary chosen vertex and if we are travelling in one of the two possible senses.

It is possible to travel through both the moves P_r, P_s in one closed move in such a way that we begin at the vertex v_k along h_r, we travel along the whole move P_r, and after we return into v_k along the last edge of P_r (it is clearly the edge g_r) we continue to travel along the whole move P_s beginning along the edge h_s. If we have finished the whole move P_s our travel comes to an end in v_k, and the last edge along which we have been will be g_s. Instead of the transition $\{g_r, v_k, h_r\}$ from P_r and of the transition $\{g_s, v_k, h_s\}$ from P_s, there will appear in P_t the following two new transitions: $\{g_r, v_k, h_s\}$, $\{g_s, v_k, h_r\}$ and the rest of the transitions through all the vertices of the moves P_r, P_s remain without changes in P_t. Since the edges g_r, h_s (and also the edges g_s, h_r) belong according to the supposition in all three possible cases to different classes of Q_k, P_t has the required properties and the validity of our assertion is proved.

Therefore if $m > 1$, then there exists a decomposition of G into $m - 1$ closed and also with respect to $\bar{Q}$ admissible moves. After the finite number $m - 1$ of such unifications of two moves we obtain the decomposition with only one move, which is a Eulerian line admissible with respect to $\bar{Q}$. This proves the theorem.

Theorem 2. *Let F be a graph with following property: each of its components contains at least one vertex of an odd degree; let $V(F) = \{v_1, v_2, \ldots, v_n\}$ be the set of its vertices, E_i the set of edges incident at v_i. Let $2p$ be the number of all vertices of an odd degree and let $\bar{R} = \{R_1, R_2, \ldots, R_n\}$ be the given system of the decompositions of the sets E_i ($i = 1, 2, \ldots, n$). Such a decomposition of F into p open moves, which is admissible with respect to R, exists if and only if for all $i \in \{1, 2, \ldots, n\}$ the number of elements of a arbitrary class of R_i is not greater than $\frac{1}{2}(1 + |E_i|)$.*

Proof. Let $X(F) = \{v_{x(1)}, v_{x(2)}, \ldots, v_{x(2p)}\}$ be the set of all vertices of odd degree from $V(F)$. Let us construct from the graph F the graph G in such a way that we add to it one new vertex v_0 and $2p$ new edges $e_1, e_2, \dots, e_{2p}$ so that e_i joins v_0 with $v_{x(i)}$. The following holds: G is a connected graph and each of its vertices has an even degree; every vertex from G which has in F an odd degree is incident in G at exactly one edge of the set $E_0 = \{e_1, e_2, \dots, e_{2p}\}$.

We construct from $\bar{R}$ a system of decompositions $\bar{Q} = \{Q_0, Q_1, Q_2, \dots, Q_n\}$ of the sets $E_i (i = 0, 1, 2, \dots, n)$ in the following manner:

(1) every class from Q_0 contains exactly one edge;

(2) if $i > 0$ and v_i does not belong to $X(F)$, then $Q_i = R_i$;

(3) $Q_{x(j)}$ $(j = 1, 2, \dots, 2p)$ contains all classes from $R_{x(j)}$ and another class containing one element and this is e_j.

According to Theorem 1 in G there exists a Eulerian line L, which is admissible with respect to $\bar{Q}$ if and only if for all $k \in \{0, 1, 2, \dots, n\}$ the following holds: none of the classes of Q_k contains more than half of the edges incident at v_k. $\bar{Q}$ has this property if and only if $\bar{R}$ has the following property: for an arbitrary $i \in \{1, 2, \dots, n\}$ we have: the number of edges of any class of $\bar{R}$ is not greater than $\frac{1}{2}(1 + |E_i|)$. Hence L exists if and only if $\bar{R}$ has the property required in the theorem.

It is clear that if we delete from L all edges belonging to E_0 and also the vertex v_0, L splits into p open moves of F, which have the required properties. Conversely: if a decomposition P of F into p open moves with the required properties exists, then there exists in G a Eulerian line, which has the required property. The validity of the theorem is clear.

Note 2. It is known that p from Theorem 2 is the minimal number of open moves into which the graph with $2p$ vertices of odd degree can be decomposed. The reader of this paper will himself easily prove that a graph which can be decomposed in this way cannot include a component which would have none of the vertices of odd degree. So the suppositions of Theorem 2 contain the conditions which the graph must fulfil in order that there exists any decomposition into a minimal number p of open moves. (See [1], [3].)

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