Xiaofeng Wu; Yi Zhao; Muhong Wang

Global synchronization of chaotic Lur'e systems via replacing variables control

Kybernetika, Vol. 44 (2008), No. 4, 571--584

Persistent URL: http://dml.cz/dmlcz/135875

KYBERNETIKA — VOLUME *44* (2008), NUMBER 4, PAGES 571 – 584

GLOBAL SYNCHRONIZATION OF CHAOTIC LUR'E SYSTEMS VIA REPLACING VARIABLES CONTROL

Xiao-Feng Wu, Yi Zhao and Mu-Hong Wang

Finding sufficient criteria for synchronization of master-slave chaotic systems by replacing variables control has been an open problem in the field of chaos control. This paper presents some recent works on the subject, with emphasis on chaos synchronization of both identical and parametrically mismatched Lur'e systems by replacing variables control. The synchronization schemes are formally constructed and two classes of sufficient criteria for global synchronization, linear matrix inequality criterion and frequency-domain criterion, are reviewed and discussed.

Keywords: chaos, synchronization, Lur'e system, replacing variables control

AMS Subject Classification: 37D45, 74H65, 93D05

1. INTRODUCTION

In 1990, Pecora and Carroll [4, 5] published their pioneering work on synchronization of chaotic systems by replacing variables control. Since then, synchronization of chaotic systems has received considerable attention and has been studied in several master-slave synchronization schemes related to various control techniques, e. g. feedback control [3, 7 – 10, 12, 14], and impulsive control [11, 18], etc. Many sufficient criteria for chaos synchronization have been proposed over the last decade.

However, sufficient criterion for chaos synchronization by replacing variables control has received less attention and theoretical advances on the subject has seldom been reported. A synchronization criterion used in some references [4, 5] is to analyze the negativity of conditional Lyapunov exponents of the slave system. However, this criterion is only necessary but not sufficient for synchronization [12].

This paper summarizes our recent works [13, 15, 16] on the master-slave Lur'e chaos synchronization by replacing variables control. Both identical and parametrically mismatched synchronization schemes are introduced and linked to the absolute stability of the corresponding error systems. Two classes of sufficient global synchronization criteria, linear matrix inequality criterion and frequency-domain criterion, are reviewed and discussed. Chua's circuits are used as an example to show the effectiveness of the criteria.

The rest of the paper is organized as follows. In the next section, basic theory on the absolute stability is introduced. Chaos synchronization for identical M-S Lur'e systems and the one with parameter mismatch is discussed in Section 3 and Section 4, respectively. Some synchronization results for Chua's circuits are given in Section 6. The final section presents some concluding remarks.

2. DEFINITIONS AND LEMMAS

Consider the following Lur'e system:

$$\dot{y} \;=\; Ay + Bf(Cy), \tag{1}$$

where $y \in \mathbb{R}^n, f : \mathbb{R}^{n_h} \to \mathbb{R}^{n_h},\ A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times n_h},\ C \in \mathbb{R}^{n_h\times n}$.

Definition 1. For $0 < \mu < +\infty$, $f = \{f_1, f_2, \cdots, f_{n_h}\}$ is said to be belonging to sector $[0, \mu]$, or $f \in F_\mu$, if for $i = 1, 2, \ldots, n_h$,

$$f_i \in F_\mu \;=\; \{\varphi \in \mathbb{R} : \varphi \text{ continues and } \varphi(0) = 0, 0 \le \sigma\varphi(\sigma) \le \mu\sigma^2, \sigma \ne 0\}. \tag{2}$$

Definition 2. System (1) is said to be absolutely stable in the function set F_μ, if for any $f \in F_\mu$, the zero solution of the system is globally asymptotically stable.

Definition 3. System (1) is said to be absolutely non-asymptotically stable in F_μ, if for any $f \in F_\mu$, $y(t) \not\to 0$ as $t \to \infty$.

Definition 4. System (1) is called a principal case, if all the eigenvalues of the matrix A have negative real parts, i. e. $\operatorname{Re}\lambda(A) < 0$. While a particular case occurs in system (1) if apart from some eigenvalues having negative real parts, there are also some eigenvalues with zero real parts in A. When A has only a simple zero eigenvalue and the rest have negative real parts, system (1) is called the simplest particular case.

Definition 5. The complex-valued function $T(z)$ belongs to a class of strictly positive real functions ($T(z) \in \{SPR\}$) if for real values of z, this function is real, and if $\operatorname{Re} z \ge 0$ then this function always has $\operatorname{Re} T(z) > 0$.

Lemma 1 (Aizerman and Gantmacher [1]) The complex-valued function $T(z) \in \{SPR\}$ if and only if the following conditions are satisfied:

(i) For real values of z, the function $T(z)$ takes on real values only.

(ii) The function $T(z)$ has no poles in $\operatorname{Re} z > 0$.

(iii) On the imaginary axis, the function $T(z)$ can have only simple poles with positive residues.

(iv) The inequality $\operatorname{Re} T(j\omega) > 0$ holds for $\forall\, \omega \in \mathbb{R} \cup \{\infty\}$.

Lemma 2. (Meyer, Kalman and Yacubovic [17]) Suppose $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $F \in \mathbb{R}^n$, $\mathrm{R}e\,\lambda(A) < 0$, and the constant $r > 0$. Then, the following matrix equations for real symmetric matrices $P \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{n\times n}$, and vector $q \in \mathbb{R}^n$,

$$\begin{cases} PA + A^T P = -qq^T - D, \\ PB - F = \sqrt{r}q \end{cases}$$

have solutions $P > 0$ and $D > 0$ if and only if

$$T(z) = r + 2F^T A(z)^{-1} B \in \{\mathrm{SPR}\},$$

where $A(z) = zI - A$.

3. CHAOS SYNCHRONIZATION FOR IDENTICAL MASTER–SLAVE LUR'E SYSTEMS

3.1. Synchronization scheme

Consider the following master Lur'e system:

$$M : \dot{x} = Ax + B\sigma(Cx), \tag{3}$$

where the state vector $x \in \mathbb{R}^n$ is divided into $(x_d, x_r)^T$ with the driving (control) vector $x_d \in \mathbb{R}^{n_d}$ and the responsive vector $x_r \in \mathbb{R}^{n_r}$, $n_d + n_r = n$. The nonlinear function $\sigma(\cdot) : \mathbb{R}^{n_h} \to \mathbb{R}^{n_h}$ belongs to sector $[0, k]$. The system matrix is

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathbb{R}^{n\times n},$$

with $A_{11} \in \mathbb{R}^{n_d\times n_d}$, $A_{12} \in \mathbb{R}^{n_d\times n_r}$, $A_{21} \in \mathbb{R}^{n_r\times n_d}$, $A_{22} \in \mathbb{R}^{n_r\times n_r}$, $B = (B_d, B_r)^T \in \mathbb{R}^{n\times n_h}$, $B_d \in \mathbb{R}^{n_d\times n_h}$, $B_r \in \mathbb{R}^{n_r\times n_h}$, $C = (C_d, C_r) \in \mathbb{R}^{n_h\times n}$, $C_d \in \mathbb{R}^{n_h\times n_d}$, $C_r \in \mathbb{R}^{n_h\times n_r}$.

Hence, system (3) can be represented as

$$M : \begin{cases} \dot{x}_d = A_{11}x_d + A_{12}x_r + B_d\sigma(Cx), \\ \dot{x}_r = A_{21}x_d + A_{22}x_r + B_r\sigma(Cx). \end{cases} \tag{4}$$

Now, assume the signals of the control vector x_d can be extracted and used to control the following identical slave system:

$$S : \dot{z} = Az + B\sigma(Cz). \tag{5}$$

The slave system has a similar form of

$$S : \begin{cases} \dot{z}_d = A_{11}z_d + A_{12}z_r + B_d\sigma(Cz), \\ \dot{z}_r = A_{21}z_d + A_{22}z_r + B_r\sigma(Cz), \end{cases} \tag{6}$$

where the vectors z, z_d and z_r are of dimension n, n_d and n_r, respectively. The slave system is controlled by replacing variables as follows:

$$z_d(t) = x_d(t), \quad \forall\, t \geq 0. \tag{7}$$

Our aim is to choose the control vector x_d such that the responsive subsystem (x_r, z_r), for any initial system state $(x(0), z(0))$, satisfies

$$\lim_{t\to\infty} \|x_r(t) - z_r(t)\|_2 = 0, \tag{8}$$

where $\|\cdot\|_2$ denotes the Euclidean norm.

Define the error variable $e_r(t) = x_r(t) - z_r(t)$. Then, for the synchronization controller (7), we obtain a dynamical error system,

$$\dot{e}_r = A_{22}e_r(t) + B_r\eta(C_re_r, Cz), \tag{9}$$

where the nonlinear function $\eta : \mathbb{R}^{n_h} \times \mathbb{R}^{n_h} \to \mathbb{R}^{n_h}$, and

$$\eta(C_re_r, Cz) = \sigma(Cx) - \sigma(Cz) = \sigma(C_re_r + Cz) - \sigma(Cz). \tag{10}$$

Suppose η belongs to F_k for C_re_r, $0 < k < \infty$. Hence, for $i = 1, 2, \ldots, n_h$ and nonzero $C_{ri} \in \mathbb{R}^{l\times n_h}$,

$$0 \leq C_{ri}e_r\eta(C_{ri}e_r, Cz) \leq k(C_{ri}e_r)^2,\ \forall\, z. \tag{11}$$

For a diagonal constant matrix $0 \leq \Lambda \in \mathbb{R}^{n_h\times n_h}$, the following inequality holds:

$$\eta^T\Lambda(\eta - kC_re_r) \leq 0,\ \forall\, e_r, z. \tag{12}$$

Since $\eta = 0$ if $e_r = 0$, $e_r = 0$ is an equilibrium point of the error system (9). Obviously, synchronization in the sense of (8) is equivalent to the global asymptotic stability of the error system (9) at the equilibrium point $e_r = 0$.

Let us further discuss the relations between the absolute stability of the error system and the synchronization. If the error system (9) is absolutely stable in F_k, then for any $\eta \in F_k$, the actual error system is globally asymptotically stable at $e_r = 0$, so the actual M-system (3) and S-system (5) globally synchronize in the sense of (8). If the error system (9) is absolutely non-asymptotically stable in F_k, then the error system with an actual $\eta \in F_k$ is either stable, in the Lyapunov sense, under the condition of the non-asymptotical stability, or is unstable, at $e_r = 0$. The unstable error system implies that the actual system (3) and system (5) don't synchronize. The fact that the error system is stable under the condition of the non-asymptotical stability implies that the actual system (3) and system (5) robustly synchronize, but don't synchronize in the sense of (8). Hence, we obtain the relations between the absolute stability of the error system and the synchronization of the M-S systems as follows.

Error system (9) *is absolutely stable in* $F_k \Rightarrow$ *systems* (3) *and* (5) *with an actual* $\eta \in F_k$ *globally synchronize in the sense of* (8).

Error system (9) *is absolutely non-asymptotically stable in* $F_k \Rightarrow$ *systems* (3) *and* (5) *with an actual* $\eta \in F_k$ *don't synchronize in the sense of* (8).

Obviously, the reverse of the above propositions doesn't hold in general.

3.2. Linear matrix inequality criteria

Using a quadratic Lyapunov function, $V(e_r) = e_r^T P e_r,\ P = P^T > 0$, the following sufficient criteria for global synchronization or non-synchronization can be obtained by means of Lyapunov's direct method.

Theorem 1. (Wu, Zhao, and Huang [15], Wu, Zhao and Zhou [16]) If there exists a constant matrix $0 < P = P^T \in \mathbb{R}^{n_r \times n_r}$ and a constant diagonal matrix $0 \le \Lambda \in \mathbb{R}^{n_h \times n_h}$, such that the following inequalities is satisfied, then systems (3) and (5) globally synchronize in the sense of (8):

$$1) \qquad Y_1 = \begin{pmatrix} A_{22}^T P + P A_{22} & P B_r + k C_r^T \Lambda \\ B_r^T P + k \Lambda C_r & -2\Lambda \end{pmatrix} < 0, \tag{13}$$

$$2) \qquad Y_2 = A_{22}^T P + P A_{22} + P B_r B_r^T P + k^2 C_r^T C_r < 0. \tag{14}$$

Theorem 2. (Wu, Zhao, and Huang [15]) If there exists a constant matrix $0 < P = P^T \in \mathbb{R}^{n_r \times n_r}$, such that

$$Y_3 = A_{22}^T P + P A_{22} - P B_r B_r^T P - k^2 C_r^T C_r > 0, \tag{15}$$

then error system (9) is absolutely unstable in F_k, so systems (3) and (5) don't synchronize in the sense of (8).

Remark 1. Both criterion (14) and criterion (15) are independent of the matrix $\Lambda \in \mathbb{R}^{n_h \times n_h}$.

3.3. Frequency-domain criteria

We now consider Lur'e systems with single non-linearity, i. e. $n_h = 1$. In this case, the matrix Λ may be replaced by a positive constant β. Using Lemmas 1 and 2, we can prove that the above LMI criteria are equivalent to the following frequency-domain criteria.

It is first considered that the error system (9) belongs to the principal case, i. e. $\operatorname{Re}\lambda(A_{22}) < 0$. The following frequency-domain criterion is related to the linear matrix inequality criterion (13).

Theorem 3. (Wu, Zhao and Zhou [16]) Let $A_1(z) = zI - A_{22}$, $W_1(z) = C_r A_1(z)^{-1} B_r$, and z be a complex variable. If $\operatorname{Re}\lambda(A_{22}) < 0$ and

$$1 - k \operatorname{Re} W_1(j\omega) > 0,\ \forall\, \omega \in \mathbb{R} \cup \{\infty\}, \tag{16}$$

then systems (3) and (5) globally synchronize in the sense of (8).

Now, we consider the case that A_{22} has a simple zero eigenvalue. In this case, we are not allowed to directly apply Lemma 2 to the error system (9) and must transform the error system to be an available equivalent system.

To obtain the equivalent system, we first determine the eigenvector $\nu_0 \in \mathbb{R}^{n_r}$ associated with the zero eigenvalue of A_{22} by

$$A_{22}\nu_0 = 0. \tag{17}$$

Then, we construct a nonsingular matrix $T \in \mathbb{R}^{n_r \times n_r}$, the last column vector of which equals ν_0.

Make a transform $e_r = Ty,\ y \in \mathbb{R}^{n_r}$, so as to obtain $u = C_r e_r = C_r Ty$. Consequently, we obtain a dynamical system of $n_r + 1$ dimensions as follows:

$$\dot{y} = (T^{-1}A_{22}T)y + T^{-1}B_r\eta(u, Cz), \tag{18}$$

$$\dot{u} = (C_r A_{22}T)y + C_r B_r \eta(u, Cz), \tag{19}$$

which is equivalent to (9).

By (17), one has

$$\hat{A}_{22} = T^{-1}A_{22}T = \begin{pmatrix} A_0 & 0 \\ a & 0 \end{pmatrix} \in \mathbb{R}^{n_r \times n_r},\ A_0 \in \mathbb{R}^{(n_r-1)\times(n_r-1)},\ a^T \in \mathbb{R}^{n_r-1}, \tag{20}$$

$$G^T = C_r A_{22} T = (G_0, 0),\ G_0^T \in \mathbb{R}^{n_r-1}. \tag{21}$$

Also, let

$$\hat{B}_r = T^{-1}B_r = (B_0^T, b)^T \in \mathbb{R}^{n_r}, B_0 \in \mathbb{R}^{n_r-1}. \tag{22}$$

$$\rho = C_r B_r \in \mathbb{R}. \tag{23}$$

Suppose $y = (y_1, y_2, \ldots, y_{n_r})^T = (y_0, y_{n_r})^T$, where $y_0 = (y_1, y_2, \ldots, y_{n_r-1})^T \in \mathbb{R}^{n_r-1}$. Then, an n_r-dimensional dynamical system consisting of the first equations of (18) and equation (19) is obtained as follows:

$$\begin{cases} \dot{y}_0 = A_0 y_0 + B_0\eta(u, Cz), \\ \dot{u} = G_0 y_0 + \rho\ \eta(u, Cz). \end{cases} \tag{24}$$

Obviously, system (24) is equivalent to system (9) in stability. When A_{22} belongs to the simplest particular case, we have $\operatorname{Re}\lambda(A_0) < 0$. When $-A_{22}$ belongs to that, we have $\operatorname{Re}\lambda(A_0) > 0$.

Make a new transform,

$$\xi = \frac{1}{r}(G_0 A_0^{-1} y_0 - u) \in \mathbb{R}, \tag{25}$$

where

$$r = G_0 A_0^{-1} B_0 - \rho. \tag{26}$$

It can be proved that if system (24) is globally asymptotically stable, then $r \neq 0$.

By substituting the variable ξ for the variable u, one has

$$\dot{\xi} = \frac{1}{r}(G_0 A_0^{-1}\dot{y}_0 - \dot{u}) = \frac{1}{r}[G_0 A_0^{-1}(A_0 y_0 + B_0\eta) - G_0 y_0 - \rho\ \eta] = \eta(u, Cz), \tag{27}$$

where

$$u = G_0 A_0^{-1} y_0 - r\xi. \tag{28}$$

We obtain a dynamical system of n_r dimensions, which is equivalent to system (9) in stability, as follows:

$$\begin{cases} \dot{y}_0 = A_0 y_0 + B_0 \eta(u, Cz), \\ \dot{\xi} = \eta(u, Cz). \end{cases} \tag{29}$$

To analyze the absolute stability of system (29), we take the following Lyapunov functional:

$$V(y_0, \xi) = y_0^T P_0 y_0 + \alpha (r\xi)^2, \tag{30}$$

where $0 < P_0 = P_0^T \in \mathbb{R}^{(n_r-1)\times(n_r-1)},\ \alpha > 0,\ r \neq 0$.

Hence, we have the following theorem, using Lyapunov's direct method.

Theorem 4. (Wu, Zhao and Zhou [16]) Suppose A_{22} has at most one simple zero eigenvalue, and the real parts of the rest eigenvalues are not equal to zero. The parameters $A_0,\ G_0,\ B_0,\ \rho$ and r are defined by (20) – (23) and (26), respectively. If $r = 0$, then system (9) is absolutely non-asymptotically stable. If $r > 0$, and there exists a constant matrix $0 < P_0 = P_0^T \in \mathbb{R}^{(n_r-1)\times(n_r-1)}$ and a constant $\alpha > 0$, such that

$$Y_4 = \begin{pmatrix} A_0^T P_0 + P_0 A_0 & P_0 B_0 + (G_0 A_0^{-1})^T ar \\ B_0^T P_0 + \alpha r G_0 A_0^{-1} & -2\frac{ar}{k} \end{pmatrix} < 0, \tag{31}$$

then systems (3) and (5) globally synchronize in the sense of (8).

Based on Lemmas 1 and 2, we can give a frequency-domain criterion equivalent to inequality (31), in the following.

Theorem 5. (Wu, Zhao and Zhou [16]) Let $A_2(z) = zI - A_0,\ W_2(z) = G_0 A_0^{-1} A_2(z)^{-1} B_0,\ z$ be a complex variable, and $A_0,\ G_0,\ B_0,\ \rho$ and r be defined by (20) – (23) and (26), respectively. If A_{22} belongs to the simplest particular case, $r > 0$, and

$$1 - k\,\mathrm{Re}\,W_2(j\omega) > 0,\ \forall \omega \in \mathbb{R} \cup \{\infty\}, \tag{32}$$

then systems (3) and (5) globally synchronize in the sense of (8).

The following frequency-domain criteria are equivalent to the linear matrix inequality criterion (14).

Theorem 6. (Wu, Zhao, and Huang [15]) Let

$$A_3(z) = zI - A_{22} - kB_r C_r,\ W_3(z) = C_r A_3(z)^{-1} B_r,$$

$$A_4(z) = zI - A_{22} + kB_r C_r,\ W_4(z) = C_r A_4(z)^{-1} B_r,$$

and let z be a complex variable. Then, systems (3) and (5) globally synchronize in the sense of (8), provided that either of the following condition holds:

$$1) \qquad \operatorname{Re}\lambda(A_{22}+kB_rC_r)<0 \text{ and } 1+2k\operatorname{Re}W_3(j\omega)>0,\ \forall\,\omega\in\mathbb{R}\cup\{\infty\}, \quad (33)$$

$$2) \qquad \operatorname{Re}\lambda(A_{22}-kB_rC_r)<0 \text{ and } 1-2k\operatorname{Re}W_4(j\omega)>0,\ \forall\,\omega\in\mathbb{R}\cup\{\infty\}. \quad (34)$$

According to the non-synchronization criterion (15), we have the following result.

Theorem 7 (Wu, Zhao, and Huang [15]) Let

$$A_5(z)=zI+A_{22}-kB_rC_r,\ W_5(z)=C_rA_5(z)^{-1}B_r,$$

$$A_6(z)=zI+A_{22}+kB_rC_r,\ W_6(z)=C_rA_6(z)^{-1}B_r,$$

and z be a complex variable. Then, systems (3) and (5) don't synchronize in the sense of (8), provided that either of the following condition holds:

$$1) \qquad \operatorname{Re}\lambda(kB_rC_r-A_{22})<0 \text{ and } 1+2k\operatorname{Re}W_5(j\omega)>0,\quad \forall\,\omega\in\mathbb{R}\cup\{\infty\}, \quad (35)$$

$$2) \qquad \operatorname{Re}\lambda(-kB_rC_r-A_{22})<0 \text{ and } 1-2k\operatorname{Re}W_6(j\omega)>0, \forall\,\omega\in\mathbb{R}\cup\{\infty\}. \quad (36)$$

Remark 2. The above frequency domain criteria belong to the algebraic inequality criteria. So, compared to LMI criterion, these frequency-domain criteria can conveniently be applied to design the control variable for the synchronization and to analyze the influence of the system parameters on the synchronization, as in [15, 16].

4. CHAOS SYNCHRONIZATION FOR THE MASTER–SLAVE LUR'E SYSTEMS WITH PARAMETER MISMATCH

4.1. Synchronization scheme

Consider the master-slave Lur'e systems as follows:

$$M:\dot{x}=A'x+B'\sigma(C'x), \quad (37)$$

$$S:\dot{z}=Az+B\sigma(Cz), \quad (38)$$

where it is assumed that there exist parameter mismatch between the two systems, i.e. $\|A'-A\|_2$, $\|B'-B\|^2$ and $\|C'-C\|_2$ do not equal zero simultaneously. The state vectors $x,z\in\mathbb{R}^n$ are divided into $x=(x_d,x_r)^T$ and $z=(z_d,z_r)^T$ with the replacing vector $x_d\in\mathbb{R}^{n_d}$, the replaced vector $z_d\in\mathbb{R}^{n_d}$, and the responsive vectors $x_r,z_r\in\mathbb{R}^{n_r}$, $n_d+n_r=n$. The nonlinear function $\sigma(\cdot):\mathbb{R}^{n_h}\to\mathbb{R}^{n_h}$ is diagonal and continuous, with $\sigma_i(\cdot)$ belonging to sector $[0,k]$, for $i=1,2,\ldots,n_h$.

The system matrices $A',A\in\mathbb{R}^{n\times n}$ are decomposed as

$$A'=\begin{pmatrix} A'_{11} & A'_{12}\\ A'_{21} & A'_{22}\end{pmatrix},\qquad A=\begin{pmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix},$$

with $A'_{11},A_{11}\in\mathbb{R}^{n_d\times n_d}$, $A'_{12},A_{12}\in\mathbb{R}^{n_d\times n_r}$, $A'_{21},A_{21}\in\mathbb{R}^{n_r\times n_d}$, $A'_{22},A_{22}\in\mathbb{R}^{n_r\times n_r}$,

$B', B \in \mathbb{R}^{n\times n_h}$, $B' = (B'_d, B'_r)$, $B = (B_d, B_r)$, $B'_d, B_d \in \mathbb{R}^{n_d\times n_h}$, $B'_r, B_r \in \mathbb{R}^{n_r\times n_h}$, $C', C \in \mathbb{R}^{n_h\times n}$, $C' = (C'_d, C'_r)$, $C = (C_d, C_r)$, $C'_d, C_d \in \mathbb{R}^{n_h\times n_d}$, $C'_r, C_r \in \mathbb{R}^{n_h\times n_r}$.

The coupling between the master and the slave systems is carried out by replacing variables control $z_d(t) = x_d(t)$, $\forall\, t \geq t_0$. Hence, the master-slave synchronization scheme with both parameter mismatch and replacing variables control can be described as

$$\begin{array}{c} M : \left\{ \begin{array}{l} \dot{x}_d = A'_{11}x_d + A'_{12}x_r + B'_d\sigma(C'x), \\ \dot{x}_r = A'_{21}x_d + A'_{22}x_r + B'_r\sigma(C'x). \end{array} \right. \\ S : \left\{ \begin{array}{l} \dot{z}_d = A_{11}z_d + A_{12}z_r + B_d\sigma(Cz), \\ \dot{z}_r = A_{21}z_d + A_{22}z_r + B_r\sigma(Cz), \end{array} \right. \\ \Theta : z_d(t) = x_d(t), \end{array} \tag{39}$$

with a replacing variables controller Θ.

Define the responsive error variable $e_r(t) = x_r(t) - z_r(t)$. From scheme (39), we can obtain a dynamical responsive error system, as

$$\begin{aligned} \dot{e}_r(t) &= \dot{x}_r(t) - \dot{z}_r(t) \\ &= (A'_{21} - A_{21})x_d + (A'_{21} - A_{21})x_r \\ &\quad + B'_r\sigma(C'x) - B_r\sigma(Cx) + A_{22}e_r(t) + B_r\eta(C_re_r, Cz) \\ &= W(x) + A_{22}e_r(t) + B_r\eta(C_re_r, Cz). \end{aligned} \tag{40}$$

In formula (40),

$$W(x) = \Delta A_2 x + B'_r\sigma(C'x) - B_r\sigma(Cx) \in \mathbb{R}^{n_r},$$

where $\Delta A_2 = (A'_{21} - A_{21}, A'_{22} - A_{22}) \in \mathbb{R}^{n_r\times n}$, and

$$\eta(C_re_r, Cz) = \sigma(Cx) - \sigma(Cz) = \sigma(C_re_r + Cz) - \sigma(Cz).$$

The term $W(x)$ reflects the influence of parameter mismatch between the master and slave systems on the responsive error system. It has been proved [13] that there always exists a real constant $\mu > 0$ such that

$$\|W(x)\|_2 \leq \mu\|x\|_2, \quad \forall\, x \in \mathbb{R}^n, \tag{41}$$

where the constant μ represents a measure for parameter mismatch between the master and the slave systems.

Our aim is to choose the replacing vector x_d such that $\|e_r\|_2 \to 0$ as $t \to \infty$. However, a zero error $e_r(t)$ can not be achieved for the synchronization scheme with non-identical master-slave systems. Therefore, a new concept of synchronization, *synchronization with finite L_2-gain*, is introduced here.

Definition 6. Synchronization scheme (4) achieves global synchronization with a finite L_2-gain, if for any finite initial states $(x(0), z(0))$, there exist a real constant $d > 0$ and a $T \geq 0$ such that

$$\|x_r(t) - z_r(t)\|_2 = \|e_r(t)\|_2 \leq d\mu, \ \forall\, t \geq T. \tag{42}$$

This definition suggests a new concept of robust synchronization which limits the synchronization error bound in terms of the measure μ for the parameter mismatch case, differing from Definitions 1 of Ref. [6] for which the synchronization error bound is independent of parameter mismatch. Moreover, this definition implies global synchronization instead of the local synchronization presented in Definitions 1 of Ref. [6]

In order to derive some sufficient criteria for the global synchronization with finite L_2-gain, the following assumption is needed.

Assumption 1. The trajectory of the master system (37) is bounded, i. e. there exists a real constant $\delta > 0$ such that for any initial condition x_0, there exists a time $T(x_0)$ such that

$$\|x(t, x_0)\|_2 \leq \delta, \ \forall\, t \geq T(x_0).$$

Clearly, this assumption is based on the boundary feature of chaotic attractors [6].

4.2. Linear matrix inequality criteria

Based on a quadratic Lyapunov function, $V(e_r) = e_r^T P e_r, \ P = P^T > 0$, we have the following result.

Theorem 8. (Wu and Wang [13]) If there exists a constant matrix $0 < P = P^T \in \mathbb{R}^{n_r \times n_r}$, a constant diagonal matrix $0 \leq \Lambda \in \mathbb{R}^{n_r \times n_r}$, and a positive real constant α, such that either of the following inequalities is satisfied, then synchronization scheme (39) achieves global synchronization with a finite L_2-gain, where the synchronization error bound is

$$d = \frac{2\delta\lambda_{\max}(P)}{\alpha}\frac{\sqrt{\lambda_{\max}(P)}}{\sqrt{\lambda_{\min}(P)}},$$

and $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ are the maximum and minimum eigenvalues of the matrix P, respectively:

$$1) \qquad Y_5 = \begin{pmatrix} A_{22}^T P + P A_{22} + \alpha I & P B_r + k C_r^T \Lambda \\ B_r^T P + k \Lambda C_r & -2\Lambda \end{pmatrix} < 0, \tag{43}$$

$$2) \qquad Y_6 = A_{22}^T P + P A_{22} + \alpha I + P B_r B_r^T P + k^2 C_r^T C_r < 0. \tag{44}$$

4.3. Frequency-domain criteria

For Lur'e systems with single nonlinearity (i. e. $n_h = 1$), the following frequency-domain criteria for synchronization scheme (39) can be derived from inequalities (43) and (44) by means of Lemmas 1 and 2.

Theorem 9. (Wu and Wang [13]) Let $A_7(z) = zI - A_{22},\ I \in \mathbb{R}^{n_r \times n_r}$ be a unit matrix and z be a complex variable. Suppose $\mathrm{R}e\,\lambda(A_{22}) < 0$. Then, the inequality (43) with $n_h = 1$ holds if there exist two positive real constants α and β such that

$$1 - \mathrm{Re}\, W_7(j\omega) > 0,\ \forall\, \omega \in \mathbb{R} \cup \{\infty\}, \tag{45}$$

where

$$W_7(z) = \left(\frac{1}{\beta} G_1 B_r + k C_r^T\right)^T A_7(z)^{-1} B_r \in \mathbb{R},$$

and $0 < G_1 = G_1^T \in \mathbb{R}^{n_r \times n_r}$ satisfies

$$A_{22}^T G_1 + G_1 A_{22} = -\alpha I. \tag{46}$$

Theorem 10. (Wu and Wang [13]) Let $A_r = A_{22} - kB_r C_r,\ A_8(z) = zI - A_r$, and z be a complex variable. Suppose $\mathrm{Re}\,\lambda(A_r) < 0$. Then, the matrix inequality (44) with $n_h = 1$ holds if there exists a constant $\alpha > 0$ such that

$$1 - 2\mathrm{Re}\, W_8(j\omega) > 0,\ \forall\, \omega \in \mathbb{R} \cup \{\infty\}, \tag{47}$$

where

$$W_8(z) = (G_2 B_r + k C_r^T)^T A_8(z)^{-1} B_r,$$

and

$$A_r^T G_2 + G_2 A_r = -\alpha I. \tag{48}$$

5. AN EXAMPLE: CHUA'S CIRCUITS

Generally speaking, not all replacing variables can make the master-slave Lur'e systems synchronize in the sense of (8) or (42). This is because the choice of x_d determines the system matrices $A_{22},\ B_r$ and C_r in the error system (9), or matrices (A'_{22}, A_{22}), (B'_r, B_r) ,(C'_r, C_r) and vector $W(x)$ in the error system (40). Hence, designing suitable replacing variables is an important task for synchronization schemes.

A method to design the replacing variables is to minimize the dimension n_d of the replacing variables such that some obtained synchronization criteria can be satisfied.

Another work related to synchronization is how to determine the ranges of the system parameters, in which the master-slave Lur'e systems coupled by the designed replacing variables achieve global synchronization or non-synchronization. In the following, we give some results for synchronization of master-slave Chua's circuits coupled by a single replacing variable [13, 16].

Consider the identical M-S Chua's circuits:

$$\begin{cases} \dot{x}_1 = a(x_2 - m_1 x_1) - a(m_0 - m_1)\sigma(x_1), \\ \dot{x}_2 = x_1 - x_2 + x_3, \\ \dot{x}_3 = -bx_2, \end{cases}$$

$$\begin{cases} \dot{z}_1 = a(z_2 - m_1 z_1) - a(m_0 - m_1)\sigma(z_1), \\ \dot{z}_2 = z_1 - z_2 + z_3, \\ \dot{z}_3 = -bz_2, \end{cases}$$

where $a \geq 0$, $b \geq 0$, the nonlinear function $\sigma(x) = \frac{1}{2}(|x+d| - |x-d|)$ belongs to sector $[0,1]$.

Using frequency-domain criteria (16) and (32), we can obtain the parameter ranges corresponding to global synchronization or non-synchronization of the master-slave Chua's systems coupled by a single replacing variable, as summarized in Table. More details for the derivation and illustrating examples can be found in [16].

Table. Ranges of parameters for Chua's circuits to synchronize or not to synchronize.

	Synchronization	Non-Synchronization
$x_d = x_1$	$b > 0, a \geq 0, \forall m_0, \forall m_1$	$b = 0, a \geq 0, \forall m_0, \forall m_1$
$x_d = x_2$		$b \geq 0, a > 0, \forall m_0, m_1 \neq 0$
$x_d = x_1$	$b \geq 0, a > 0, m_0 > 1, m_1 \geq 1$	(i) $b \geq 0, a > 0, \forall m_0, \forall m_1$ (ii) $b \geq 0, a > 0, m_0 = 1, m_1 = 1$

By means of the frequency-domain criteria (45)–(48), parameter ranges of the master-slave Chua's circuits with parameter mismatch can be analytically solved in the sense of synchronization with a finite L_2-gain via replacing singe-variable control. It must be noted that the ranges are same as the case of complete synchronization, as shown in Table. This shows that robustness of synchronization for the master-slave Chua's circuits is maintained in the ranges of the parameters.

The illustrative example has verified that within the parameter ranges it is possible to synchronize the master-slave Chua's circuits by replacing a single variable up to a small synchronization error bound, even though the qualitative behavior of the slave circuit is somewhat different from that of the master one. Simulation on the example has shown that a smaller synchronization error bound corresponds to a smaller parameter mismatch between the master and the slave systems, and vice versa. More details for the derivation can be seen from [13].

6. CONCLUDING REMARKS

Some recent advances in chaos synchronization of the master-slave Lur'e systems coupled by replacing variables control has been surveyed in the paper. Two synchronization schemes are described, respectively, for identical and parametrically mismatched cases. LMI criteria and frequency-domain criteria for the schemes have been systematically discussed. We believe that the results provide a promising approach for synchronization of autonomous and non-autonomous chaotic systems via the simple and effective replacing variables control.

ACKNOWLEDGEMENT

This work was partially supported by the National Nature Science Foundation of China under Grant Nos. 60674049 and 10371136, and the Nature Science Foundation of Guangdong Province of China under Grant No. 021765.

(Received September 30, 2007.)

REFERENCES

[1] M. A. Aizerman and F. R. Gantmacher: Absolute Stability of Regulator Systems. Holder-Day Inc. 1964.

[2] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, and C. S. Zhou: The synchronization of chaotic systems. Phys. Rep. *366* (2002), 1–101.

[3] P. F. Curran and L. O. Chua: Absolute stability theory and the synchronization problem. Internat. J. Bifur. Chaos *7* (1997), 1375–1382.

[4] L. M. Pecora and T. L. Carroll: Synchronization in chaotic systems. Phys. Rev. Lett. *64* (1990), 821–824.

[5] L. M. Pecora and T. L. Carroll: Driving systems with chaotic signals. Phys. Rev. A *44* (1991), 2374–2383.

[6] J. A. K. Suykens, P. F. Curran, and L. O. Chua: Robust synthesis for master-slave synchronization of Lur'e systems. IEEE Trans. Circuits and Systems-I *46* (1999), 841–850.

[7] J. A. K. Suykens, P. F. Curran, T. Yang, J. Vandewalle, and L. O. Chua: Nonlinear synchronization of Lur'e systems: dynamic output feedback case. IEEE Trans. Circuits and Systems-I *44* (1997), 1089–1092.

[8] J. A. K. Suykens, J. Vandewalle, and L. O. Chua: Nonlinear synchronization of chaotic Lur'e system. Internat. J. Bifur. Chaos *7* (1997), 1323–1335.

[9] J. A. K. Suykens and J. Vandewalle: Master-slave synchronization of Lur'e systems. Internat. J. Bifur. Chaos *7* (1997), 665–669.

[10] J. A. K. Suykens, M. E. Yalcin, and J. Vandewalle: Chaotic systems synchronization. In: Chaos and Bifurcation Control: Theory and Application (G. Chen and X. Yu, eds., Lecture Notes in Control and Information Sciences 292). Springer-Verlag, Hong Kong 2004, pp. 117–135.

[11] J. A. K. Suykens, T. Yang, and L. O. Chua: Impulsive synchronization of chaotic Lur's systems by measurement feedback. Internat. J. Bifur. Chaos *8* (1998), 1371–1381.

[12] X. F. Wang, Z. Q. Wang, and G. R. Chen: A new criterion for synchronization of coupled chaotic oscillators with application to Chua's circuits. Internat. J. Bifur. Chaos *9* (1999), 1169–1174.

[13] X. F. Wu and M. H. Wang: Robust synchronization of chaotic Lur'e systems via replacing variables control. Internat. J. Bifur. Chaos *16* (2006), 3421–3434.
[14] X. F. Wu and Y. Zhao: Frequency domain criterion for chaos synchronization of Lur'e systems via state feedback control. Internat. J. Bifur. Chaos *15* (2005), 1145–1154.
[15] X. F. Wu, Y. Zhao, and X. H. Huang: Some new criteria for lag synchronization of chaotic Lur'e systems by replacing variables control. J. Control Theory Appl. *2* (2004), 259–266.
[16] X. F. Wu, Y. Zhao, and S. Zhou: Lag synchronization of chaotic Lur'e systems via replacing variables control. Internat. J. Bifur. Chaos *15* (2005), 617–635.
[17] H. M. Xie: Theory and Application of Absolute stability. Science Press, Beijing 1986.
[18] T. Yang and L. O. Chua: Impulsive stabilization for control and synchronization of chaotic systems: theory and application to secure communication. IEEE Trans. Circuits and Systems-I *44* (1997), 976–988.

Xiao-Feng Wu, Department of Mathematics, Zhangzhou Teachers College, Zhangzhou 363000 and Guangzhou Navy Academy, Guangzhou 510430. China.
e-mail: wuxiaof@21cn.com

Yi Zhao, Department of Mathematics, Zhongshan University, Guangzhou 510275. China.
e-mail: stszy@zsu.edu.cn

Mu-Hong Wang, Guangzhou Navy Academy, Guangzhou 510430. China.
e-mail: binbo2100@163.com