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A CATEGORICAL VIEW AT GENERALIZED CONCEPT LATTICES

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We continue in the direction of the ideas from the Zhang's paper [14] about a relationship between Chu spaces and Formal Concept Analysis. We modify this categorical point of view at a classical concept lattice to a generalized concept lattice (in the sense of Krajči [7]): We define generalized Chu spaces and show that they together with (a special type of) their morphisms form a category. Moreover we define corresponding modifications of the image / inverse image operator and show their commutativity properties with mapping defining generalized concept lattice as fuzzifications of Zhang's ones.

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1. MOTIVATION

It is often very useful and inspiring to see the same thing from more different points of view. This general sentence we can apply to Formal Concept Analysis. Guo-Quiang Zhang in this paper [14] has considered a concept lattice in the terms of the category theory. As he says, his paper brings these (originally independent) areas together and establishes fundamental connections among them, leaving open opportunities for the exploration of cross-disciplinary influences. He emphasizes the substantial culture differences among these fields: Formal Concept Analysis focuses on internal properties of and algorithms for concept structures almost exclusively on an individual basis, while the Category Theory mandates that concept structures should be looked at collectively as a whole with appropriate morphisms relating one individual structure to another. (Note that Zhang speaks about the third area, Domain Theory, too, but we are not going to focus on this part of his considerations.)

In this humble contribution we are going to continue in this direction of research. In the papers [7] and [8] we define a new type of fuzzification of Formal Concept Analysis, a so-called generalized concept lattice, which moreover in some sense generalizes some other fuzzy constructions of concept lattices (namely a fuzzy concept lattice, an one-sided concept lattice and a concept lattice with hedges, all mentioned below). A natural questions arise: If a notion of Chu space is a pendant of a classical (crisp, Ganter & Wille's) concept lattice in the category theory, what object will be a categorical counterpart of this generalized concept lattice? And what about morphisms of such objects? We will try to answer to the first question and partially answer to the second one in this paper.

2. CONCEPT LATTICES AND CHU SPACES

By a Chu space ([14]) we understand a triple (A, B, R), where A is a (non-empty) set of attributes, B is a (non-empty) set of objects, and R is a subset of $A \times B$.

It is easy to see that a notion of Chu space corresponds to a notion of context leading to a classical concept lattice ([6]): For a Chu space (A, B, R) define the following mappings $\uparrow : \mathscr{P}(B) \to \mathscr{P}(A)$ and $\downarrow : \mathscr{P}(A) \to \mathscr{P}(B)$:

- If $X \subseteq B$ then $X^{\uparrow} = \{a \in A : (\forall\, b \in X)\langle a, b\rangle \in R\}$.
- If $Y \subseteq A$ then $Y^{\downarrow} = \{b \in B : (\forall\, a \in Y)\langle a, b\rangle \in R\}$.

These two mappings form a *Galois connection* which (in this case) means these four properties:

1a) If $X_1 \subseteq X_2$ then $X_1^{\uparrow} \supseteq X_2^{\uparrow}$ for all $X_1, X_2 \subseteq B$.

1b) If $Y_1 \subseteq Y_2$ then $Y_1^{\downarrow} \supseteq Y_2^{\downarrow}$ for all $Y_1, Y_2 \subseteq A$.

2a) $X \subseteq (X^{\uparrow})^{\downarrow}$ for all $X \subseteq B$.

2b) $Y \subseteq (Y^{\downarrow})^{\uparrow}$ for all $Y \subseteq A$.

By a *concept* we understand a pair $\langle X, Y\rangle$ such that $X^{\uparrow} = Y$ and $Y^{\downarrow} = X$, then X will be the *extent* of $\langle X, Y\rangle$ and Y will be the *intent* of $\langle X, Y\rangle$. The set of all concepts is called a *concept lattice*. It is proven (in the basic book [6]) that it is really a lattice (moreover complete).

Theorems on one single Chu space / concept lattice are rather static, they describe status quo of it. But it is often inspiring to see things dynamically. In this case we can want to ask how concepts will change if a new object is added to context. The following notion brings dynamics to such considerations:

By a *Chu mapping* from a Chu space (A_1, B_1, R_1) to a Chu space (A_2, B_2, R_2) we will understand a pair of functions $\langle p, q\rangle$ with $p : A_2 \to A_1$ and $q : B_1 \to B_2$ (note that indices are interchanged in p) satisfying $\langle p(a_2), b_1\rangle \in R_1$ iff $\langle a_2, q(b_1)\rangle \in R_2$ for all $a_2 \in A_2$ and $b_1 \in B_1$.

In this framework an adding of a new row to a table means the change from one Chu space to another. This construction induces a Chu mapping $\langle p, q\rangle$ from the initial Chu space to the enlarged one such that p and q are identities.

It is easy to see that all Chu spaces and Chu mappings as their morphisms form a category.

In the next we are going to fuzzify the following notions from [14]:

Any function $f : A \to B$ can be lifted to the (crisp) powerset level in the two canonical ways:

$$f^{+} : \mathscr{P}(A) \to \mathscr{P}(B) \quad \text{with} \quad X \mapsto \{f(a) : a \in X\}$$
$$f^{-} : \mathscr{P}(B) \to \mathscr{P}(A) \quad \text{with} \quad Y \mapsto \{a : f(a) \in Y\}$$

f^{+} is the standard (*forward*) *image* operation and f^{-} is the *inverse image* operation.

Lemma 1. (Zhang [14]) Let $\langle p, q\rangle$ is a Chu mapping from (A_1, B_1, R_1) to (A_2, B_2, R_2). Then $\uparrow_1 \circ p^- = q^+ \circ \uparrow_2$, i.e. the diagram

$$\begin{array}{ccc} \mathscr{P}(B_1) & \xrightarrow{\uparrow_1} & \mathscr{P}(A_1) \\ \downarrow q^+ & \# & \downarrow p^- \\ \mathscr{P}(B_2) & \xrightarrow{\uparrow_2} & \mathscr{P}(A_2) \end{array}$$

commutes.

3. L–LIFTINGS

In the next chapters we will need this L-fuzzification of the image and the inverse image of a set:

Let L be (the support of) a complete lattice, S and T be arbitrary sets and $h : S \to T$. Then define the *canonical* L*-liftings* $h^+_L : L^S \to L^T$ (*forward* L*-image*) and $h^-_L : L^T \to L^S$ (*inverse* L*-image*) in the following way:

- If $g : S \to L$ then $h^+_L(g) : T \to L$ is defined by the formula
$$h^+_L(g)(t) = \sup\{g(s) : h(s) = t\}.$$
- If $f : T \to L$ then $h^-_L(f) : S \to L$ is defined by the formula
$$h^-_L(f)(s) = (h \circ f)(s) = f(h(s))$$
(i. e. $h^-_L(f) = h \circ f$).

In the special case $L = \{0, 1\}$ we really obtain the coincidence between $h[X]$ and $h^+_L(\chi_X)$ (or loosely $h^+_L(X) \approx h[X]$) and between $h^{-1}[Y]$ and $h^-_L(\chi_Y)$ (or loosely $h^-_L(Y) \approx h^{-1}[Y]$), namely:

Lemma 2. $h^+_{\{0,1\}}(\chi_X) = \chi_{h[X]}$ for arbitrary $X \subseteq S$, i. e. $\chi_X : S \to \{0, 1\}$.

Proof. $h^+_{\{0,1\}}(\chi_X)(t) = 1$,
iff $\sup\{\chi_X(s) : h(s) = t\} = 1$, iff $\chi_X(s) = 1$ for at least one s such that $h(s) = t$, iff $s \in X$ for at least one s such that $h(s)=t$, iff there exists $s \in X$ such that $h(s)=t$, iff $t \in h[X]$, iff $\chi_{h[X]}(t) = 1$. □

Lemma 3. $h^-_{\{0,1\}}(\chi_Y) = \chi_{h^{-1}[Y]}$ for arbitrary $Y \subseteq T$, i. e. $\chi_Y : T \to \{0, 1\}$.

Proof. $h^-_{\{0,1\}}(\chi_Y)(s) = 1$,

iff $\chi_Y(h(s)) = 1$, iff $h(s) \in Y$, iff $s \in h^{-1}[Y]$, iff $\chi_{h^{-1}[Y]}(s) = 1$. □

The compositions of our liftings $h^+_L \circ h^-_L : L^S \to L^S$ and $h^-_L \circ h^+_L : L^T \to L^T$ fulfill these interesting properties:

Lemma 4.

a1) $(h_L^+ \circ h_L^-)(g) \geq g$ (pointwise) for all $g : S \to L$.

a2) $h_L^+ \circ h_L^-$ is the identity on L^S iff h is an injection.

a3) $(h_L^+ \circ h_L^-)(g) = (h_L^+ \circ h_L^-) \circ (h_L^+ \circ h_L^-)(g)$ for all $g : S \to L$.

a4) If $g_1 \leq g_2$ (pointwise) then $(h_L^+ \circ h_L^-)(g_1) \leq (h_L^+ \circ h_L^-)(g_2)$ for all $g_1, g_2 : S \to L$.

b1) $(h_L^- \circ h_L^+)(f) \leq f$ for all $f : T \to L$.

b2) $h_L^- \circ h_L^+$ is the identity on L^T iff h is a surjection.

b3) $(h_L^- \circ h_L^+)(g) = (h_L^- \circ h_L^+) \circ (h_L^- \circ h_L^+)(f)$ for all $f : T \to L$.

b4) If $f_1 \leq f_2$ (pointwise) then $(h_L^- \circ h_L^+)(f_1) \leq (h_L^- \circ h_L^+)(f_2)$ for all $f_1, f_2 : T \to L$.

Proof.

a) For all $s \in S$ by definitions:

$$\begin{aligned} & \left((h_L^+ \circ h_L^-)(g)\right)(s) = \left(h_L^-(h_L^+(g))\right)(s) \\ = \quad & \left(h \circ (h_L^+(g))\right)(s) = \left(h_L^+(g)\right)(h(s)) = \sup\{g(u) : h(u) = h(s)\}. \end{aligned}$$

Then:

1) Clearly $\left((h_L^+ \circ h_L^-)(g)\right)(s) = \sup\{g(u) : h(u) = h(s)\} \geq g(s)$.

2) If h is one-to-one then there is the only u such that $h(u) = h(s)$, namely s. It follows that $\sup\{g(u) : h(u) = h(s)\} = g(s)$, i.e. $h_L^+ \circ h_L^-$ is the identity.

   If h is not an injection, i.e. $h(s_1) = h(s_2)$ for some $s_1 \neq s_2$, take an arbitrary g such that $g(s_1) = 0_L$ and $g(s_2) = 1_L$. Then $\left((h_L^+ \circ h_L^-)(g)\right)(s_1) = \sup\{g(u) : h(u) = h(s_1)\} \geq g(s_2) = 1_L \neq 0_L = g(s_1)$. It means $(h_L^+ \circ h_L^-)(g) \neq g$, therefore $h_L^+ \circ h_L^-$ is not the identity.

3) Clearly if $h(s_1) = h(s_2)$ then $\left((h_L^+ \circ h_L^-)(g)\right)(s_1) = \left((h_L^+ \circ h_L^-)(g)\right)(s_2)$. Then we have: $\left((h_L^+ \circ h_L^-) \circ (h_L^+ \circ h_L^-)(g)\right)(s)$
   $= \sup\{\left((h_L^+ \circ h_L^-)(g)\right)(u) : h(u) = h(s)$
   $= \sup\{\sup\{g(v) : h(v) = h(u)\} : h(u) = h(s)\}$
   $= \sup\{\sup\{g(v) : h(v) = h(s)\} : h(u) = h(s)\}$
   (because $h(v) = h(u) = h(s)$) $= \sup\{\left((h_L^+ \circ h_L^-)(g)\right)(s) : h(u) = h(s)\} = \left((h_L^+ \circ h_L^-)(g)\right)(s)$ (the expression under sup does not depend on u and at least one such u exists).

4) $\left((h_L^+ \circ h_L^-)(g_1)\right)(s) = \sup\{g_1(u) : h(u) = h(s)\}$
   $\leq \sup\{g_2(u) : h(u) = h(s)\} = \left((h_L^+ \circ h_L^-)(g_2)\right)(s)$.

b) Again by definitions we have for all $t \in T$:
$\left((h_L^- \circ h_L^+)(f)\right)(t) = \left(h_L^+(h_L^-(f))\right)(t) = \sup\{\left(h_L^-(f)\right)(s) : h(s) = t\}$
$= \sup\{f(h(s)) : h(s) = t\} = \sup\{f(t) : h(s) = t\}$,
what is $f(t)$ if $t \in h[S]$, and 0_L elsewhere (note that this is independent on f).

Then we have:

1) We have seen that $\left((h_L^- \circ h_L^+)(f)\right)(t) \leq f(t)$.
2) The equality in the preceding formula holds iff $t \in h[S]$. It means that $(h_L^- \circ h_L^+)(f) = f$ iff $h[S] = T$ (i. e. h is a surjection).
3) If $t \in h[S]$, we have $\left((h_L^- \circ h_L^+)(f)\right)(t) = f(t)$ and
$\left(((h_L^- \circ h_L^+) \circ (h_L^- \circ h_L^+))(f)\right)(t) = \left((h_L^- \circ h_L^+)\left((h_L^- \circ h_L^+)(f)\right)\right)(t)$
$= \left((h_L^- \circ h_L^+)(f)\right)(t) = f(t)$.
If $t \notin h[S]$, we have $\left((h_L^- \circ h_L^+)(f)\right)(t) = 0_L$ and
$\left(((h_L^- \circ h_L^+) \circ (h_L^- \circ h_L^+))(f)\right)(t) = \left((h_L^- \circ h_L^+)\left((h_L^- \circ h_L^+)(f)\right)\right)(t) = 0_L$.
4) $\left((h_L^- \circ h_L^+)(f_1)\right)(t) = \sup\{f_1(t) : h(s) = t\} \leq \sup\{f_2(t) : h(s) = t\}$
$= \left((h_L^- \circ h_L^+)(f_2)\right)(t)$.

We can summarize properties 1), 3) and 4) (a *kernel operator* is the dual notion to a closure operator):

Corollary 1. a) $h_L^+ \circ h_L^-$ is a closure operator. b) $h_L^- \circ h_L^+$ is a kernel operator.

4. A GENERALIZED CONCEPT LATTICE

An idea of defining of a generalized concept lattice arose as an answer to the natural question of looking for a common platform for so far known fuzzifications of a classical (crisp) concept lattice.

Let us recall its definition and basic properties ([7]):
Let P be a poset, C and D be complete lattices. Let $\bullet : C \times D \to P$ be isotone and left-continuous in both their arguments, i. e.

1a) $c_1 \leq c_2$ implies $c_1 \bullet d \leq c_2 \bullet d$ for all $c_1, c_2 \in C$ and $d \in D$.

1b) $d_1 \leq d_2$ implies $c \bullet d_1 \leq c \bullet d_2$ for all $c \in C$ and $d_1, d_2 \in D$.

2a) If $c \bullet d \leq p$ holds for $d \in D$, $p \in P$ and for all $c \in X \subseteq C$, then $\sup X \bullet d \leq p$.

2b) If $c \bullet d \leq p$ holds for $c \in C$, $p \in P$ and for all $d \in Y \subseteq D$, then $c \bullet \sup Y \leq p$.

Let A and B be non-empty sets and let R be P-fuzzy relation on their Cartesian product, i. e. $R : A \times B \to P$.

Define the following mapping $\uparrow : D^B \to C^A$:
If $g : B \to D$ then $\uparrow(g) : A \to C$ is defined in the following way:

$$\uparrow(g)(a) = \sup\{c \in C : (\forall\, b \in B) c \bullet g(b) \leq R(a, b)\}.$$

Symmetrically we define the mapping $\downarrow : C^A \to D^B$:
If $f : A \to C$ then $\downarrow(f) : B \to D$ is defined in the following way:

$$\downarrow(f)(b) = \sup\{d \in D : (\forall\, a \in A) f(a) \bullet d \leq R(a,b)\}.$$

Mappings $\downarrow$ and $\uparrow$ form a Galois connection, namely

1a) $g_1 \leq g_2$ implies $\uparrow(g_1) \geq \uparrow(g_2)$. 1b) $f_1 \leq f_2$ implies $\downarrow(f_1) \geq \downarrow(f_2)$.

2a) $g \leq \downarrow(\uparrow(g))$. 2b) $f \leq \uparrow(\downarrow(f))$.

Then a pair of functions $\langle g, f\rangle$ from $D^B \times C^A$ such that $g^\uparrow = f$ and $f^\downarrow = g$, is called a *(generalized) concept*. If $\langle g_1, f_1\rangle$ and $\langle g_2, f_2\rangle$ are concepts, we write $\langle g_1, f_1\rangle \leq \langle g_2, f_2\rangle$ iff $g_1 \leq g_2$ (or equivalently $f_1 \geq f_2$). The set of all such concepts with the order $\leq$ is called a *(generalized) concept lattice* and it is denoted by $\mathrm{GCL}(A, B, R, C, D, P, \bullet)$.

The following analogy of the Basic Theorem on Concept Lattice can be formulated (it is proven in [7] and [8]):

Theorem 1.

1) A generalized concept lattice $\mathrm{GCL}(A, B, R, C, D, P, \bullet)$ is a complete lattice in which
$$\bigwedge_{i\in I} \langle g_i, f_i\rangle = \left\langle \bigwedge_{i\in I} g_i, \uparrow\left(\downarrow\left(\bigvee_{i\in I} f_i\right)\right)\right\rangle$$
and
$$\bigvee_{i\in I} \langle g_i, f_i\rangle = \left\langle \downarrow\left(\uparrow\left(\bigvee_{i\in I} g_i\right)\right), \bigwedge_{i\in I} f_i\right\rangle.$$

2) Let moreover P have the least element 0_P and $0_C \bullet d = 0_P$ and $c \bullet 0_D = 0_P$ for every $c \in C$ and $d \in D$. Then a complete lattice V is isomorphic to $\mathrm{GCL}(A, B, R, C, D, P, \bullet)$ if and only if there are mappings $\alpha : A \times C \to V$ and $\beta : B \times D \to V$ s. t.

   1a) α is non-increasing in the second argument.

   1b) β is non-decreasing in the second argument.

   2a) $\alpha[A \times C]$ is infimum-dense.

   2b) $\beta[B \times D]$ is supremum-dense.

   3) For every $a \in A$, $b \in B$, $c \in C$, $d \in D$

$$\alpha(a,c) \geq \beta(b,d) \qquad \text{if and only if} \qquad c \bullet d \leq R(a,b).$$

This construction is really a generalization of known fuzzifications of concept lattice. For Pollandt's ([11, 12]) and Bělohlávek's ([1, 2]) fuzzy concept lattice we have $C = D = P = L$ and $\bullet$ is the product, for an one-sided fuzzy concept lattice ([3, 5, 10]) $C = P = [0,1]$ but $D = \{0,1\}$ and $\bullet$ is the minimum (or the product again).

Note that this notion is not the only common platform for notions of an one-sided fuzzy concept lattice and of a fuzzy concept lattice. The alternative answer is the approach given by Bělohlávek et al. [4] again. They define a so-called concept lattice with hedges. In the paper [9] we show that this construction can be understood as a special case of a generalized concept lattice.

5. A GENERALIZED CHU SPACE

Now we are going to try to express these ideas by means of the category theory. We define a notion of generalized Chu space what will be a pendant of a generalized concept lattice.

Let A and B be non-empty sets, P be (the support of) a poset, $R : A \times B \to P$, C and D be (the supports of) complete lattices and $\bullet : C \times D \to P$ be isotone and left-continuous in both its arguments. By a *generalized Chu space* it will be understood the tuple $(A, B, R, P, C, D, \bullet)$.

It is easy to see that an ordinary Chu space can be seen as a special case of a generalized Chu space (A, B, R') for $P = \{0, 1\}$ where R is the characteristic function of the relation R', i. e.

$$R(a, b) = \begin{cases} 1, & \text{if } \langle a, b\rangle \in R', \\ 0, & \text{if } \langle a, b\rangle \notin R', \end{cases}$$

$C = D = \{0, 1\}$ (a set is identified with its characteristic function) and $\bullet$ is the product.

For a generalized Chu space $(A, B, R, P, C, D, \bullet)$ we can again define functions $\uparrow : D^B \to C^A$ and $\downarrow : C^A \to D^B$ in the same way as before. Hence it is easy to see that the generalized Chu space $(A, B, R, P, C, D, \bullet)$ naturally leads to the corresponding generalized concept lattice $\mathrm{GCL}(A, B, R, P, C, D, \bullet)$.

If we want to speak about some category, we should say something about its morphisms. In our case of generalized Chu space which will be objects of our category we can imagine more types of such morphisms. For simplicity (as the beginning of this approach) we will focus to morphisms which work only with a fixed C, D, P and $\bullet$ of generalized Chu spaces $(A, B, R, P, C, D, \bullet)$:

Let $(A_1, B_1, R_1, P, C, D, \bullet)$ and $(A_2, B_2, R_2, P, C, D, \bullet)$ be generalized Chu spaces. Let $p : A_2 \to A_1$ and $q : B_1 \to B_2$ (note the mutually inverse directions again) are such that

$$R_2(a_2, q(b_1)) = R_1(p(a_2), b_1)$$

holds for all $a_2 \in A_2$ and $b_1 \in B_1$. Then $\langle p, q\rangle$ will be called an *object-attribute (OA-) morphism* of these generalized Chu spaces.

It is easy to see the following:

Lemma 5. The system of all generalized Chu spaces and their OA-morphisms is a category.

P r o o f. $\langle \mathrm{id}_A, \mathrm{id}_B\rangle$ is the unit morphism of a space $(A, B, R, P, C, D, \bullet)$ on itself. If $\langle p_{12}, q_{12}\rangle$ is a morphism of a space $(A_1, B_1, R_1, P, C, D, \bullet)$ to a space

$(A_2, B_2, R_2, P, C, D, \bullet)$, i.e. $R_2(a_2, q_{12}(b_1)) = R_1(p_{12}(a_2), b_1)$, and $\langle p_{23}, q_{23}\rangle$ is a morphism of a space $(A_2, B_2, R_2, P, C, D, \bullet)$ to a space $(A_3, B_3, R_3, P, C, D, \bullet)$, i.e. $R_3(a_3, q_{23}(b_2)) = R_2(p_{12}(a_3), b_2)$, then $\langle p_{23} \circ p_{12}, q_{12} \circ q_{23}\rangle$ is a morphism of $(A_1, B_1, R_1, P, C, D, \bullet)$ to $(A_3, B_3, R_3, P, C, D, \bullet)$, because:

$$R_3(a_3, (q_{12} \circ q_{23})(b_1)) = R_3(a_3, q_{23}(q_{12}(b_1))) = R_2(p_{23}(a_3), q_{12}(b_1))$$
$$= R_1(p_{12}(p_{23}(a_3)), b_1) = R_1((p_{23} \circ p_{12})(a_3), b_1).$$

6. RELATIONSHIP BETWEEN LIFTINGS OF GENERALIZED CHU MORPHISM AND MAPPINGS DEFINING A GENERALIZED CONCEPT LATTICE

In this chapter we assume that $(A_1, B_1, R_1, P, C, D, \bullet)$ and $(A_2, B_2, R_2, P, C, D, \bullet)$ are generalized Chu spaces, $\langle p, q\rangle$ is some their object-attribute (OA-) morphism, and the mappings $\downarrow_i$ and $\uparrow_i$ correspond to the space $(A_i, B_i, R_i, P, C, D, \bullet)$ and they are defined as before. We will show the following two commutativity diagrams. The first is a fuzzification of Lemma 1:

Lemma 6. $\uparrow_1 \circ p^-_C = q^+_D \circ \uparrow_2$, i.e. the diagram

$$\begin{array}{ccc} D^{B_1} & \xrightarrow{\uparrow_1} & C^{A_1} \\ \big\downarrow q^+_D & \# & \big\downarrow p^-_C \\ D^{B_2} & \xrightarrow{\uparrow_2} & C^{A_2} \end{array}$$

commutes.

Proof. Let $g : B_1 \to D$, then really $(\uparrow_1 \circ p^-_C)(g) : A_2 \to C$ and $(q^+_D \circ \uparrow_2)(g) : A_2 \to C$. Then for arbitrary $a_2 \in A_2$ the following holds:

$(\uparrow_1 \circ p^-_C(g))(a_2) = p^+_C(\uparrow_2(g))(a_2) = (\uparrow_2(g))(p(a_2))$
$= \sup\{c \in C : (\forall\, b_1 \in B_1) c \bullet g(b_1) \le R_1(p(a_2), b_1)\}$
$= \sup\{c \in C : (\forall\, b_1 \in B_1) c \bullet g(b_1) \le R_2(a_2, q(b_1))\}$
(because $\langle p, q\rangle$ is an OA-morphism),
$= \sup\{c \in C : (\forall\, b_1 \in B_1)(\forall\, b_2 \in B_2 : q(b_1) = b_2) c \bullet g(b_1) \le R_2(a_2, q(b_1))\}$
$= \sup\{c \in C : (\forall\, b_1 \in B_1)(\forall\, b_2 \in B_2 : q(b_1) = b_2) c \bullet g(b_1) \le R_2(a_2, b_2)\}$
$= \sup\{c \in C : (\forall\, b_2 \in B_2)(\forall\, b_1 \in B_1 : q(b_1) = b_2) c \bullet g(b_1) \le R_2(a_2, b_2)\}$
(the exchange of quantifiers),
$= \sup\{c \in C : (\forall\, b_2 \in B_2) c \bullet \sup\{g(b_1) : q(b_1) = b_2\} \le R_2(a_2, b_2)\}$
(a property of $\bullet$),
$= \sup\{c \in C : (\forall\, b_2 \in B_2) c \bullet (q^+_D(g))(b_2)$
$\le R_2(a_2, b_2)\} = \uparrow_2(q^+_D(g))(a_2) = (q^+_D \circ \uparrow_2(g))(a_2).$ $\square$

The second diagram is analogous, but under necessary assumption of surjectivity of mappings p and q:

Lemma 7. If p and q are surjective then $\uparrow_2 \circ p_C^+ = q_D^- \circ \uparrow_1$, i.e. the diagram

$$\begin{array}{ccc} D^{B_2} & \xrightarrow{\uparrow_2} & C^{A_2} \\ \downarrow q_D^- & \# & \downarrow p_C^+ \\ D^{B_1} & \xrightarrow{\uparrow_1} & C^{A_1} \end{array}$$

commutes.

Proof. Let $g : B_2 \to D$, then really $(\uparrow_2 \circ p_C^+)(g) : A_1 \to C$ and $(q_D^- \circ \uparrow_1)(g) : A_1 \to C$. Then for arbitrary $a_1 \in A_1$ the following holds:

$(\uparrow_2 \circ p_C^+(g))(a_1) = p_C^+(\uparrow_2(g))(a_1) = \sup\{(\uparrow_2(g))(a_2) : p(a_2) = a_1\}$
$= \sup\{\sup\{c \in C : (\forall b_2 \in B_2) c \bullet g(b_2) \le R_2(a_2, b_2)\} : p(a_2) = a_1\}$
$= \sup\{\sup\{c \in C : (\forall b_1 \in B_1) c \bullet g(q(b_1)) \le R_2(a_2, q(b_1))\} : p(a_2) = a_1\}$

(because q is surjective),

$= \sup\{\sup\{c \in C : (\forall b_1 \in B_1) c \bullet g(q(b_1)) \le R_1(p(a_2), b_1)\} : p(a_2) = a_1\}$

(because $\langle p, q\rangle$ is an OA-morphism),

$= \sup\{\sup\{c \in C : (\forall b_1 \in B_1) c \bullet g(q(b_1)) \le R_1(a_1, b_1)\} : p(a_2) = a_1\}$
$= \sup\{c \in C : (\forall b_1 \in B_1) c \bullet g(q(b_1)) \le R_1(a_1, b_1)\}$

(the set $\{c \in C : (\forall b_1 \in B_1) c \bullet g(q(b_1)) \le R_1(a_1, b_1)\}$ does not depend on a_2, but from the surjectivity of p there exists at least one such a_2),

$= \sup\{c \in C : (\forall b_1 \in B_1) c \bullet (q_D^-(g))(b_1)$
$\le R_1(a_1, b_1)\} = \uparrow_1(q_D^-(g))(a_1) = (q_D^- \circ \uparrow_1(g))(a_1)$.

7. CONCLUSIONS AND FUTURE WORK

In this paper we try to continue in ideas of the work of Zhang how to look at a concept lattice from the categorical point of view. We define very naturally an appropriate modification of a Chu space corresponding to our generalized concept lattice. It seems that there are more ways to define morphisms between such generalized Chu spaces, but for the beginning we started with mappings which transform the sets of objects and attributes only. Moreover we have defined a fuzzy version of liftings of a function, show their basic properties and discuss some commutative relationships to the mappings defining a generalized concept lattice.

Our future plan in this field of research is to precise relationships between these generalized Chu mappings (and/or their liftings) and mappings defining generalized concept lattice (in the direction stated by Zhang's paper [14]), and use them for a better understanding of transformation of one (generalized) concept lattice to another.

Another interesting challenge is to study Shostak's notion of a fuzzy category and its relationship to a category of fuzzily defined structures (e.g. in [13]) in this (generalized) concept lattice context.

We have said that a generalized concept lattice is a generalization of till known fuzzifications of concept lattice. But what does mean the word generalization here? It is rather intuitive, because all these constructions lead to (all) complete lattices.

We hope that the more precise answer will be given by the category theory, maybe it will be the existence of some canonical mapping, i. e. some functor between corresponding Chu spaces.

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