Kybernetika

Lubomír Kubáček

Outliers in models with constraints

Kybernetika, Vol. 42 (2006), No. 6, 673--698

Persistent URL: http://dml.cz/dmlcz/135744

KYBERNETIKA — VOLUME *42* (2006), NUMBER 6, PAGES 673 – 698

OUTLIERS IN MODELS WITH CONSTRAINTS

Lubomír Kubáček

Outliers in univariate and multivariate regression models with constraints are under consideration. The covariance matrix is assumed either to be known or to be known only partially.

Keywords: univariate regression model, multivariate regression model, constraints, outlier, variance components

AMS Subject Classification: 62J05

1. INTRODUCTION

The problem is how to test suspicious measurement whether it is a rough error or a mistake (outlier) in an observation when parameters of a regression model satisfy some constraints. The covariance matrix need not be known; some unknown parameters can occur in it. The solution of the mentioned problem or a contribution to it is the aim of the paper. Although this problem is intensively studied, cf. e.g. [2], many problems are not yet solved. Some comments to several of them are presented in the paper.

2. NOTATION AND SYMBOLS

The following notation will be used:

$\mathbf{Y}$ ... n-dimensional random vector (observation vector),

$\underline{\mathbf{Y}}$ ... $n \times m$ random matrix (observation matrix),

$\boldsymbol{\beta}$ ... k-dimensional unknown vector parameter,

$\underline{\boldsymbol{\beta}}$ ... $k \times m$ matrix of unknown parameters,

$\mathbf{X}$ ... $n \times k$ given matrix (design matrix),

$\boldsymbol{\Sigma}$ ... $n \times n$ covariance matrix of the observation vector $\mathbf{Y}$
(it is assumed to be positive definite),

$\mathbf{b}$ ... given q-dimensional vector,

$\mathbf{B}$ ... $q \times k$ given matrix,

$\mathbf{G}$ ... $q \times k$ given matrix,

$\mathbf{H}$ ... $m \times r$ given matrix,

$\mathbf{G}_0$ ... $q \times r$ given matrix,

$\mathcal{M}(\mathbf{A})$ ... column subspace of the matrix $\mathbf{A}$,

$\mathbf{P}_{\mathbf{A}}$ ... projection matrix (in the Euclidean norm) on the column subspace of the matrix $\mathbf{A}$,

$\mathbf{I}$ ... identity matrix,

$\mathrm{vec}(\mathbf{A})$ ... vector consisted of the columns of the matrix $\mathbf{A}$,

$\mathbf{M}_{\mathbf{A}}$ ... projection matrix on the orthogonal complement of the subspace $\mathcal{M}(\mathbf{A})$, i.e. $\mathbf{M}_{\mathbf{A}} = \mathbf{I} - \mathbf{P}_{\mathbf{A}}$,

$\mathbf{A}^-$ ... generalized inverse of the matrix $\mathbf{A}$, i.e. $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$ (in more detail cf. [10]),

$\mathbf{A}^+$ ... the Moore–Penrose inverse of the matrix $\mathbf{A}$, i.e. $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}, \mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+, \mathbf{A}\mathbf{A}^+ = (\mathbf{A}\mathbf{A}^+)', \mathbf{A}^+\mathbf{A} = (\mathbf{A}^+\mathbf{A})'$ (in more detail cf. [10]),

$\mathbf{P}_{\mathbf{A}}^{\Sigma^{-1}}$ ... projection matrix in the norm $\|\mathbf{x}\| = \sqrt{\mathbf{x}'\boldsymbol{\Sigma}^{-1}\mathbf{x}}, \mathbf{x} \in R^n$, on the column subspace of the matrix $\mathbf{A}$,

$\mathbf{M}_{\mathbf{A}}^{\Sigma^{-1}} = \mathbf{I} - \mathbf{P}_{\mathbf{A}}^{\Sigma^{-1}}$,

$\xi \sim \chi^2_q(0)$ ... a random variable ξ has the central chi-square distribution with q degrees of freedom,

$\overset{H_0}{\sim}$ ... the random variable is distributed under the true null hypothesis,

$\chi^2_q(0; 1-\alpha)$ ... $(1-\alpha)$-quantile of the central chi-square distribution with q degrees of freedom,

$u(1-\alpha/2)$ ... $(1-\alpha/2)$-quantile of the normal distribution $N(0,1)$,

$\mathbf{F} = (\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_r}), \mathbf{e}_{i_j} \in R^n, j = 1, \ldots, r, \{\mathbf{e}_{i_j}\}_k = \begin{cases} 0, & k \neq i_j, \\ 1, & k = i_j. \end{cases}$

$\chi^2_r(\delta)$ ... random variable with noncenteral chi-squared distribution with r degrees of freedom and with the parameter noncentrality equal to δ,

$F_{r,f}(\delta)$ ... random variable with nonceneral Fisher–Snedecor distribution with r and f degrees of freedom and with the parameter of noncentrality equal to δ,

$F_{r,f}(0; 1-\alpha)$ ... $(1-\alpha)$-quantile of the central Fisher–Snedecor distribution with r and f degrees of freedom,

$\mathbf{E} = \left(\mathbf{e}_{i_1}^{(m)} \otimes \mathbf{e}_{j_1}^{(n)}, \ldots, \mathbf{e}_{i_s}^{(m)} \otimes \mathbf{e}_{j_s}^{(n)}\right).$

An univariate regression model with normally distributed observation vector and with constraints will be denoted as

$$\mathbf{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}), \quad \mathbf{b} + \mathbf{B}\boldsymbol{\beta} = \mathbf{0}. \tag{1}$$

A multivariate regression model [1] with normally distributed observation matrix and with constraints will be considered in the form

$$\underline{\mathbf{Y}} \sim N_{nm}(\mathbf{X}\underline{\boldsymbol{\beta}}, \boldsymbol{\Sigma} \otimes \mathbf{I}), \tag{2}$$

where $\boldsymbol{\Sigma} \otimes \mathbf{I}$ is the covariance matrix of the vector $\mathrm{vec}(\underline{\mathbf{Y}})$. Constraints can be given in different forms, e. g. $\mathbf{G}\underline{\boldsymbol{\beta}}\mathbf{H} + \mathbf{G}_0 = \mathbf{0}, \mathbf{G}\underline{\boldsymbol{\beta}} + \mathbf{G}_0 = \mathbf{0}, \underline{\boldsymbol{\beta}}\mathbf{H} + \mathbf{G}_0 = \mathbf{0}$, etc.

The univariate model is regular if the rank of the matrix $\mathbf{X}$ is $r(\mathbf{X}) = k < n$, $\boldsymbol{\Sigma}$ is positive definite (p.d.) and $r(\mathbf{B}) = q < k$.

The multivariate model considered is regular if $r(\mathbf{X}) = k < n, r(\mathbf{G}) = q < k, r(\mathbf{H}) = r < m$ and $\boldsymbol{\Sigma}$ is p.d.

3. MODELS WITH OUTLIERS

3.1. Univariate models

Lemma 3.1.1. In the regular univariate model with constraints the best linear unbiased estimator (BLUE) is

$$\begin{aligned}
\widehat{\widehat{\boldsymbol{\beta}}} &= \widehat{\boldsymbol{\beta}} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}(\mathbf{B}\widehat{\boldsymbol{\beta}} + \mathbf{b}) \\
&= (\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{b}, \\
\widehat{\boldsymbol{\beta}} &= \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}, \quad \mathbf{C} = \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}, \\
\mathrm{Var}(\widehat{\widehat{\boldsymbol{\beta}}}) &= \mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1} = (\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}.
\end{aligned}$$

P r o o f is given, e. g. in [5], p. 80. □

Corollary 3.1.2. The residual vector $\mathbf{v}_I = \mathbf{Y} - \mathbf{X}\widehat{\widehat{\boldsymbol{\beta}}}$ is distributed as

$$\begin{aligned}
\mathbf{v}_I &\sim N_n[\mathbf{0}, \mathrm{Var}(\mathbf{v}_I)], \\
\mathrm{Var}(\mathbf{v}_I) &= \boldsymbol{\Sigma} - \mathbf{X}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}' \\
&= \mathrm{Var}(\mathbf{Y} - \mathbf{X}\widehat{\boldsymbol{\beta}}) + \mathbf{X}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\mathbf{X},
\end{aligned}$$

where $\mathrm{Var}(\mathbf{Y} - \mathbf{X}\widehat{\boldsymbol{\beta}}) = \boldsymbol{\Sigma} - \mathbf{X}\mathbf{C}^{-1}\mathbf{X}'$.

If $\mathbf{v}_I'\boldsymbol{\Sigma}^{-1}\mathbf{v}_I \geq \chi^2_{n+q-k}(0; 1-\alpha)$ for sufficiently small α, then the measured data are not compatible with the model. Thus outliers could occur. A thorough inspection of data, mainly their genesis, must be realized and on this basis it is sometimes possible to decide which of data are suspicious.

It is not the only way how to detect outlier (cf. [2, 3]. In the following text also the way given in [13] pp. 92–94 is followed.

Let the measurements $\{\mathbf{Y}\}_{i_1},\dots,\{\mathbf{Y}\}_{i_r}$ be suspicious. In such a case the model (1) is rewritten in the form

$$\mathbf{Y} \sim N_n\left[(\mathbf{X},\mathbf{F})\begin{pmatrix}\boldsymbol{\beta}\\ \boldsymbol{\Delta}\end{pmatrix},\boldsymbol{\Sigma}\right],\quad \mathbf{b}+\mathbf{B}\boldsymbol{\beta}=\mathbf{0}. \tag{3}$$

In the model (3) the hypothesis $H_0:\boldsymbol{\Delta}=\mathbf{0}$ versus $H_0:\boldsymbol{\Delta}\neq\mathbf{0}$, can be tested if and only if the vector $\boldsymbol{\Delta}$ is unbiasedly estimable. It can be formulated as follows.

Lemma 3.1.3. The hypothesis $H_0:\boldsymbol{\Delta}=\mathbf{0}$ versus $H_0:\boldsymbol{\Delta}\neq\mathbf{0}$ can be tested in the model (3) iff $\mathcal{M}(\mathbf{X}\mathbf{M}_{\mathbf{B}'})\cap\mathcal{M}(\mathbf{F})=\{\mathbf{0}\}$ (intersection both subspaces is the set with a single point, i. e. null vector $\mathbf{0}$ only), what is equivalent to $\mathcal{M}\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}\cap\mathcal{M}\begin{pmatrix}\mathbf{F}\\ \mathbf{0}\end{pmatrix}=\{\mathbf{0}\}$.

Proof. The hypothesis can be tested iff (cf. [13])

$$\mathcal{M}\begin{pmatrix}\mathbf{0}\\ \mathbf{I}\end{pmatrix}\subset\mathcal{M}\begin{pmatrix}\mathbf{X}', & \mathbf{B}'\\ \mathbf{F}', & \mathbf{0}\end{pmatrix}\Leftrightarrow\exists\{\mathbf{U},\mathbf{V}\}\mathbf{X}'\mathbf{U}+\mathbf{B}'\mathbf{V}=\mathbf{0}\quad\&\quad\mathbf{F}'\mathbf{U}=\mathbf{I}.$$

The equality $\mathbf{X}'\mathbf{U}+\mathbf{B}'\mathbf{V}=\mathbf{0}$ implies $\mathcal{M}(\mathbf{U})=\mathcal{M}(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}})$ & $\mathcal{M}(\mathbf{V})=\mathcal{M}(\mathbf{M}_{\mathbf{B}\mathbf{M}_{\mathbf{X}'}})$. Further $\mathcal{M}(\mathbf{F}'\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}})$ must be equal to $\mathcal{M}(\mathbf{I})=R^r$. Since ([10], p. 137)

$$r\begin{pmatrix}\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\\ \mathbf{F}'\end{pmatrix}=r\left(\mathbf{F}'\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)+r(\mathbf{X}\mathbf{M}_{\mathbf{B}'}),$$

the equality $r(\mathbf{F}'\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}})=r$ can be valid iff $r\begin{pmatrix}\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\\ \mathbf{F}'\end{pmatrix}=r+r(\mathbf{X}\mathbf{M}_{\mathbf{B}'})$ $(r(\mathbf{F}')=r)$, what is equivalent to $\mathcal{M}(\mathbf{X}\mathbf{M}_{\mathbf{B}'})\cap\mathcal{M}(\mathbf{F})=\{\mathbf{0}\}$.

The equivalence

$$\mathcal{M}(\mathbf{X}\mathbf{M}_{\mathbf{B}'})\cap\mathcal{M}(\mathbf{F})=\{\mathbf{0}\}\Leftrightarrow\mathcal{M}\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}\cap\mathcal{M}\begin{pmatrix}\mathbf{F}\\ \mathbf{0}\end{pmatrix}=\{\mathbf{0}\}$$

is the consequence of the following consideration

$$\mathcal{M}(\mathbf{X}\mathbf{M}_{\mathbf{B}'})\cap\mathcal{M}(\mathbf{F})=\{\mathbf{0}\}\Leftrightarrow r\begin{pmatrix}\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\\ \mathbf{F}'\end{pmatrix}=r(\mathbf{F}')+r(\mathbf{M}_{\mathbf{B}'}\mathbf{X}').$$

However in general

$$r\begin{pmatrix}\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\\ \mathbf{F}'\end{pmatrix}=r(\mathbf{F}'\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}})+r(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')$$

and therefore $r(\mathbf{F}')=r(\mathbf{F}'\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}})$. Analogously

$$\mathcal{M}\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}\cap\mathcal{M}\begin{pmatrix}\mathbf{F}\\ \mathbf{0}\end{pmatrix}=\{\mathbf{0}\}\Leftrightarrow r\begin{pmatrix}\mathbf{X}, & \mathbf{F}\\ \mathbf{B}, & \mathbf{0}\end{pmatrix}=r\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}+r(\mathbf{F}).$$

Further

$$r\begin{pmatrix}\mathbf{X}, & \mathbf{F}\\ \mathbf{B}, & \mathbf{0}\end{pmatrix} = r\left[(\mathbf{X},\mathbf{F})\mathbf{M}_{(\mathbf{B},\mathbf{0})'}\right] + r(\mathbf{B})$$
$$= r\left[(\mathbf{X},\mathbf{F})\begin{pmatrix}\mathbf{M}_{\mathbf{B}'}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{I}\end{pmatrix}\right] + r(\mathbf{B})$$
$$= r(\mathbf{X}\mathbf{M}_{\mathbf{B}'},\mathbf{F}) + r(\mathbf{B}) = r\left(\mathbf{F}'\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right) + r(\mathbf{X}\mathbf{M}_{\mathbf{B}'}) + r(\mathbf{B})$$
$$= r(\mathbf{F}'\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}) + r\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}.$$

In both cases the equality $r(\mathbf{F}'\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}) = r(\mathbf{F}')$ is necessary and sufficient condition for equivalence

$$\mathcal{M}(\mathbf{X}\mathbf{M}_{\mathbf{B}'}) \cap \mathcal{M}(\mathbf{F}) = \{\mathbf{0}\} \Leftrightarrow \mathcal{M}\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix} \cap \mathcal{M}\begin{pmatrix}\mathbf{F}\\ \mathbf{0}\end{pmatrix} = \{\mathbf{0}\}. \qquad \square$$

Lemma 3.1.4. In the regular model (3) the BLUE of the vector $\begin{pmatrix}\boldsymbol{\beta}\\ \boldsymbol{\Delta}\end{pmatrix}$ is $\begin{pmatrix}\widehat{\widehat{\boldsymbol{\beta}}}_{\text{out}}\\ \widehat{\widehat{\boldsymbol{\Delta}}}\end{pmatrix}$, where

$$\widehat{\widehat{\boldsymbol{\beta}}}_{\text{out}} = \widehat{\widehat{\boldsymbol{\beta}}} - (\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\widehat{\widehat{\boldsymbol{\Delta}}},$$
$$\widehat{\widehat{\boldsymbol{\beta}}} = \widehat{\boldsymbol{\beta}} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}(\mathbf{B}\widehat{\boldsymbol{\beta}} + \mathbf{b}),$$
$$\widehat{\boldsymbol{\beta}} = \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}, \quad \mathbf{C} = \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}$$

(the estimator $\widehat{\boldsymbol{\beta}}$ is the BLUE in the regular model $\mathbf{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta},\boldsymbol{\Sigma})$, $\widehat{\widehat{\boldsymbol{\beta}}}$ is the BLUE of $\boldsymbol{\beta}$ in (1)) and

$$\widehat{\widehat{\boldsymbol{\Delta}}} = \left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+}\mathbf{F}\right]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \mathbf{X}\widehat{\widehat{\boldsymbol{\beta}}}).$$

Further

$$\operatorname{Var}(\widehat{\widehat{\boldsymbol{\beta}}}_{\text{out}}) = \operatorname{Var}(\widehat{\widehat{\boldsymbol{\beta}}}) + (\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{F}$$
$$\times\left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+}\mathbf{F}\right]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\left(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'}\right)^{+},$$
$$\operatorname{Var}(\widehat{\widehat{\boldsymbol{\beta}}}) = \operatorname{Var}(\widehat{\boldsymbol{\beta}}) - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1} = (\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+},$$
$$\operatorname{Var}(\widehat{\boldsymbol{\beta}}) = \mathbf{C}^{-1},$$
$$\operatorname{Var}(\widehat{\widehat{\boldsymbol{\Delta}}}) = \left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+}\mathbf{F}\right]^{-1},$$
$$\operatorname{cov}(\widehat{\widehat{\boldsymbol{\beta}}}_{\text{out}},\widehat{\widehat{\boldsymbol{\Delta}}}) = -(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+}\mathbf{F}\right]^{-1}.$$

P r o o f. At first it is to be remarked that the matrix

$$\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+}\mathbf{F}$$

is regular, what is implied by the assumptions $\mathcal{M}(\mathbf{X}\mathbf{M}_{\mathbf{B}'}) \cap \mathcal{M}(\mathbf{F}) = \{\mathbf{0}\}$ and $r(\mathbf{F}_{n,r}) = r < n$.

Let $\boldsymbol{\beta}_0$ be any solution of the equation $\mathbf{B}\boldsymbol{\beta} + \mathbf{b} = \mathbf{0}$, i.e. $\boldsymbol{\beta} = \boldsymbol{\beta}_0 + \mathbf{M}_{\mathbf{B}'}\boldsymbol{\gamma}$. Thus we obtain the model without constraints

$$\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0 \sim N_n\left[(\mathbf{X}\mathbf{M}_{\mathbf{B}'}, \mathbf{F})\begin{pmatrix}\boldsymbol{\gamma}\\ \boldsymbol{\Delta}\end{pmatrix}, \boldsymbol{\Sigma}\right], \quad \boldsymbol{\gamma} \in R^k, \boldsymbol{\Delta} \in R^s,$$

which is not regular, however the assumption $\mathcal{M}(\mathbf{X}\mathbf{M}_{\mathbf{B}'}) \cap \mathcal{M}(\mathbf{F}) = \{\mathbf{0}\}$ ensures the estimability of the vectors $\mathbf{M}_{\mathbf{B}'}\boldsymbol{\gamma}$ and $\boldsymbol{\Delta}$. Thus the BLUE of the vector $\begin{pmatrix}\mathbf{M}_{\mathbf{B}'}\boldsymbol{\gamma}\\ \boldsymbol{\Delta}\end{pmatrix}$ is

$$\begin{pmatrix}\widehat{\widehat{\mathbf{M}_{\mathbf{B}'}\boldsymbol{\gamma}}}\\ \widehat{\widehat{\boldsymbol{\Delta}}}\end{pmatrix} = \begin{pmatrix}\mathbf{M}_{\mathbf{B}'}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'}, & \mathbf{M}_{\mathbf{B}'}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\\ \mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{M}_{\mathbf{B}'}, & \mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\end{pmatrix}^{+} \times \begin{pmatrix}\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0)\\ \mathbf{F}'\boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0)\end{pmatrix}.$$

Since

$$\begin{pmatrix}\mathbf{M}_{\mathbf{B}'}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'}, & \mathbf{M}_{\mathbf{B}'}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\\ \mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{M}_{\mathbf{B}'}, & \mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\end{pmatrix}^{+} = \begin{pmatrix}\mathbf{A}_{1,1}, & \mathbf{A}_{1,2}\\ \mathbf{A}_{2,1}, & \mathbf{A}_{2,2}\end{pmatrix},$$

$$\begin{aligned}
\mathbf{A}_{1,1} &= (\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+} + (\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{F} \times\\
&\quad \left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+}\mathbf{F}\right]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{X}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+},\\
\mathbf{A}_{1,2} &= -(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+}\mathbf{F}\right]^{-1}\\
&= \mathbf{A}'_{2,1},\\
\mathbf{A}_{2,2} &= \left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+}\mathbf{F}\right]^{-1},
\end{aligned}$$

the expressions for the estimators can be easily obtained.

The covariance matrix of the estimator $\begin{pmatrix}\widehat{\widehat{\boldsymbol{\beta}}}_{\text{out}}\\ \widehat{\widehat{\boldsymbol{\Delta}}}\end{pmatrix}$ is

$$\operatorname{Var}\begin{pmatrix}\widehat{\widehat{\boldsymbol{\beta}}}_{\text{out}}\\ \widehat{\widehat{\boldsymbol{\Delta}}}\end{pmatrix} = \left[\begin{pmatrix}\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\\ \mathbf{F}'\end{pmatrix}\boldsymbol{\Sigma}^{-1}(\mathbf{X}\mathbf{M}_{\mathbf{B}'}, \mathbf{F})\right]^{+}.$$

Since

$$\begin{aligned}
&\begin{pmatrix}\mathbf{M}_{\mathbf{B}'}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'}, & \mathbf{M}_{\mathbf{B}'}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\\ \mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{M}_{\mathbf{B}'}, & \mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\end{pmatrix}^{+}\\
&\quad = \begin{pmatrix}\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'}, & \mathbf{M}_{\mathbf{B}'}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\\ \mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{M}_{\mathbf{B}'}, & \mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\end{pmatrix}^{+},
\end{aligned}$$

the equality

$$\begin{pmatrix} \mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'}, & \mathbf{M}_{\mathbf{B}'}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{F} \\ \mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{M}_{\mathbf{B}'}, & \mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F} \end{pmatrix}^{+} = \begin{pmatrix} \mathbf{A}_{1,1}, & \mathbf{A}_{1,2} \\ \mathbf{A}_{2,1}, & \mathbf{A}_{2,2} \end{pmatrix}$$

is obvious and the proof can be finished. □

The following theorem is implied by the preceding lemmas.

Theorem 3.1.5. In regular model (3) the hypothesis

$$H_0 : \boldsymbol{\Delta} = \mathbf{0} \quad \text{versus} \quad H_a : \boldsymbol{\Delta} \neq \mathbf{0}$$

can be tested by the help of the statistic

$$\begin{aligned} \hat{\hat{\boldsymbol{\Delta}}}{}' [\mathrm{Var}(\hat{\hat{\boldsymbol{\Delta}}})]^{-1} \hat{\hat{\boldsymbol{\Delta}}} &\sim \chi_r^2(\delta), \quad \delta = \boldsymbol{\Delta}'[\mathrm{Var}(\hat{\hat{\boldsymbol{\Delta}}})]^{-1}\boldsymbol{\Delta}, \\ \hat{\hat{\boldsymbol{\Delta}}} &= \left[\mathbf{F}' \left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}} \boldsymbol{\Sigma} \mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+} \mathbf{F}\right]^{-1} \mathbf{F}'\boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \mathbf{X}\hat{\hat{\boldsymbol{\beta}}}), \\ \mathrm{Var}(\hat{\hat{\boldsymbol{\Delta}}}) &= \left[\mathbf{F}' \left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}} \boldsymbol{\Sigma} \mathbf{M}_{X M_{\mathbf{B}'}}\right)^{+} \mathbf{F}\right]^{-1}, \\ \hat{\hat{\boldsymbol{\beta}}} &= (\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{b}. \end{aligned}$$

If for some i^*

$$|\hat{\hat{\Delta}}_{i^*}| \geq \sqrt{\chi_r^2(0; 1-\alpha)}\sqrt{\mathrm{Var}(\hat{\hat{\Delta}}_{i^*})},$$

then the null-hypothesis $\boldsymbol{\Delta} = \mathbf{0}$ is rejected because of the i^*th measurement $\{\mathbf{Y}\}_{i^*}$, i. e. it is outlier.

Until now the covariance matrix $\boldsymbol{\Sigma}$ is assumed to be known. Let $\boldsymbol{\Sigma} = \sigma^2\mathbf{V}$, where σ^2 is an unknown parameter and $\mathbf{V}$ be an $n \times n$ p.d. given matrix.

Lemma 3.1.6. In the regular model (3) with the covariance matrix $\boldsymbol{\Sigma} = \sigma^2\mathbf{V}$ the residual vector $\mathbf{v}_{I,\mathrm{out}} = \mathbf{Y} - \mathbf{X}\hat{\hat{\boldsymbol{\beta}}}_{\mathrm{out}} - \mathbf{F}\hat{\hat{\boldsymbol{\Delta}}}$ can be expressed as

$$\mathbf{v}_{I,\mathrm{out}} = \mathbf{v}_I - \mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}^{\mathbf{V}^{-1}}\mathbf{F}\hat{\hat{\boldsymbol{\Delta}}}.$$

The expression for $\mathbf{v}_I$ is given by Lemma 3.1.1, however the matrix $\mathbf{C}$ must be substituted by $\mathbf{C}_0 = \mathbf{X}'\mathbf{V}^{-1}\mathbf{X}$. Thus

$$\begin{aligned} \mathbf{v}_I &= \mathbf{Y} - \mathbf{X}\hat{\hat{\boldsymbol{\beta}}} = \mathbf{Y} - \mathbf{X}(\mathbf{M}_{\mathbf{B}}\mathbf{C}_0\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\mathbf{V}^{-1}\mathbf{Y} + \mathbf{X}\mathbf{C}_0^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}_0^{-1}\mathbf{B}')^{-1}\mathbf{b} \\ &= \mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}} + \mathbf{X}\mathbf{C}_0^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}_0^{-1}\mathbf{B}')^{-1}(\mathbf{B}\hat{\boldsymbol{\beta}} + \mathbf{b}). \end{aligned}$$

Another expression for $\mathbf{v}_{I,\mathrm{out}}$ is

$$\mathbf{v}_{I,\mathrm{out}} = \left\{\mathbf{I} - \mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}^{V^{-1}}\mathbf{F}\left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{V}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+}\mathbf{F}\right]^{-1}\mathbf{F}'\mathbf{V}^{-1}\right\}\mathbf{v}_I.$$

P r o o f. It is a direct consequence of Lemma 3.1.4. □

Corollary 3.1.7. In the regular model (3) with covariance matrix $\mathbf{\Sigma} = \sigma^2\mathbf{V}$ the best estimator of σ^2 is

$$\widehat{\sigma}^2_{I,\text{out}} = \mathbf{v}'_{I,\text{out}}\mathbf{V}^{-1}\mathbf{v}_{I,\text{out}}/[n+q-(k+r)] \sim \sigma^2\chi^2_{n+q-(k+r)}(0)/[n+q-(k+r)].$$

Analogously as in Theorem 3.1.5 the test statistic is now

$$\widehat{\widehat{\mathbf{\Delta}}}'\mathbf{F}'(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{V}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}})^+\mathbf{F}\widehat{\widehat{\mathbf{\Delta}}}/(r\widehat{\sigma}^2_{I,\text{out}}) \sim F_{r,n+q-(k+r)}(\delta),$$

$$\delta = \frac{1}{\sigma^2}\mathbf{\Delta}'\mathbf{F}'(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{V}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}})^+\mathbf{\Delta},$$

$$\widehat{\widehat{\mathbf{\Delta}}} = \left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{V}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{F}\right]^{-1}\mathbf{F}'\mathbf{V}^{-1}\mathbf{v}_I.$$

Remark 3.1.8. The procedure for testing suspicious data can be described by the following steps.

Let $\{\mathbf{Y}\}_{i_1}, \dots, \{\mathbf{Y}\}_{i_r}$ be denoted as possible outliers.

The BLUEs of $\boldsymbol{\beta}$ and $\mathbf{\Delta}$ in the model (3) are

$$\widehat{\widehat{\boldsymbol{\beta}}}_{\text{out}} = \widehat{\widehat{\boldsymbol{\beta}}} - (\mathbf{M}_{\mathbf{B}'}\mathbf{C}_0\mathbf{M}_{\mathbf{B}'})^+\mathbf{X}'\mathbf{V}^{-1}\widehat{\widehat{\mathbf{\Delta}}},$$

$$\widehat{\widehat{\mathbf{\Delta}}} = \left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{V}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{F}\right]^{-1}\mathbf{F}'\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{X}\widehat{\widehat{\boldsymbol{\beta}}}),$$

$$\widehat{\widehat{\mathbf{\Delta}}} \overset{H_0}{\sim} N_s\left\{\mathbf{0}, \sigma^2\left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{V}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{F}\right]^{-1}\right\}.$$

The residual vector is

$$\mathbf{v}_{I,\text{out}} = \mathbf{v}_I - \mathbf{M}^{\mathbf{V}^{-1}}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{F}\widehat{\widehat{\mathbf{\Delta}}} = \mathbf{Y} - \mathbf{X}\widehat{\widehat{\boldsymbol{\beta}}} - \mathbf{M}^{\mathbf{V}^{-1}}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{F}\widehat{\widehat{\mathbf{\Delta}}}$$

$$\sim N_n\left[\mathbf{0}, \text{Var}(\mathbf{v}_I) - \mathbf{M}^{\mathbf{V}^{-1}}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{F}\text{Var}(\widehat{\widehat{\mathbf{\Delta}}})\mathbf{F}'\left(\mathbf{M}^{\mathbf{V}^{-1}}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)'\right]$$

and the best estimator of σ^2 is

$$\widehat{\sigma}^2_{I,\text{out}} = \frac{\mathbf{v}'_{I,\text{out}}\mathbf{V}^{-1}\mathbf{v}_{I,\text{out}}}{n+q-(k+r)} \sim \sigma^2\frac{\chi^2_{n+q-(k+r)}(0)}{n+q-(k+r)}.$$

The test statistic of the hypothesis $\mathbf{\Delta} = \mathbf{0}$ versus $\mathbf{\Delta} \neq \mathbf{0}$ is

$$T = \frac{\widehat{\widehat{\mathbf{\Delta}}}'\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{V}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{F}\widehat{\widehat{\mathbf{\Delta}}}}{r\widehat{\sigma}^2_{I,\text{out}}} \sim F_{r,n+q-(k+r)}(\delta),$$

$$\delta = \frac{\mathbf{\Delta}'\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{V}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{F}\mathbf{\Delta}}{\sigma^2}.$$

If $T > F_{r,n+q-(k+r)}(0; 1-\alpha)$, and for some i^*

$$|\{\widehat{\widehat{\mathbf{\Delta}}}\}_{i^*}| \geq \sqrt{r\widehat{\sigma}^2_{I,\text{out}}F_{r,n+q-(k+r)}(0;1-\alpha)}\sqrt{\left\{\left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{V}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{F}\right]^{-1}\right\}_{i^*,i^*}},$$

then the i^*th measurement contributes to the rejection of the null-hypothesis H_0, thus it is outlier.

3.2. Multivariate model

Lemma 3.2.1. In the regular multivariate model

$$\underline{\mathbf{Y}} \sim N_{nm}(\mathbf{X}_{n,k}\underline{\boldsymbol{\beta}}_{k,m}, \boldsymbol{\Sigma} \otimes \mathbf{I}) \tag{4}$$

(i. e. $r(\mathbf{X}) = k < n, \boldsymbol{\Sigma}$ is p.d.) with regular constraints

$$\mathbf{G}\underline{\boldsymbol{\beta}}\mathbf{H} + \mathbf{G}_0 = \mathbf{0} \tag{5}$$

(i. e. $r(\mathbf{G}) = q < k, r(\mathbf{H}) = r < m$) the BLUE of the matrix $\underline{\boldsymbol{\beta}}$ is

$$\underline{\widehat{\widehat{\boldsymbol{\beta}}}} = \underline{\widehat{\boldsymbol{\beta}}} - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}(\mathbf{G}\underline{\widehat{\boldsymbol{\beta}}}\mathbf{H} + \mathbf{G}_0)(\mathbf{H}'\boldsymbol{\Sigma}\mathbf{H})^{-1}\mathbf{H}'\boldsymbol{\Sigma},$$

where $\underline{\widehat{\boldsymbol{\beta}}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{\mathbf{Y}}$ (the BLUE in the model (4) without constraints (5)). The covariance matrix of the vector $\mathrm{vec}(\underline{\widehat{\widehat{\boldsymbol{\beta}}}})$ is

$$\mathrm{Var}[\mathrm{vec}(\underline{\widehat{\widehat{\boldsymbol{\beta}}}})] = \mathrm{Var}[\mathrm{vec}(\underline{\widehat{\boldsymbol{\beta}}})]$$
$$-[\boldsymbol{\Sigma}\mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}\mathbf{H})^{-1}\mathbf{H}'\boldsymbol{\Sigma}] \otimes \{(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\},$$

where $\mathrm{Var}[\mathrm{vec}(\underline{\widehat{\boldsymbol{\beta}}})] = \boldsymbol{\Sigma} \otimes (\mathbf{X}'\mathbf{X})^{-1}$.

P r o o f . It is implied by Lemma 3.1.1. It suffices to rewrite the model in the form

$$\mathrm{vec}(\underline{\mathbf{Y}}) \sim N_{nm}[(\mathbf{I} \otimes \mathbf{X})\mathrm{vec}(\underline{\boldsymbol{\beta}}), \boldsymbol{\Sigma} \otimes \mathbf{I}], \quad (\mathbf{H}' \otimes \mathbf{G})\mathrm{vec}(\underline{\boldsymbol{\beta}}) + \mathrm{vec}(\mathbf{G}_0) = \mathbf{0}. \quad \square$$

Corollary 3.2.2. The residual matrix $\underline{\mathbf{v}}_I = \underline{\mathbf{Y}} - \mathbf{X}\underline{\widehat{\widehat{\boldsymbol{\beta}}}}$ is distributed as

$$\mathrm{vec}(\underline{\mathbf{v}}_I) \sim N_{nm}\left\{\mathbf{0}, \mathrm{Var}[\mathrm{vec}(\underline{\mathbf{v}})] + \mathbf{K}\right\}.$$

The matrix $\underline{\mathbf{v}}_I$ can be written as

$$\begin{aligned}
\underline{\mathbf{v}}_I &= \underline{\mathbf{Y}} - \mathbf{X}\underline{\widehat{\widehat{\boldsymbol{\beta}}}} = \underline{\mathbf{Y}} - \mathbf{X}\underline{\widehat{\boldsymbol{\beta}}} + \underline{\mathbf{k}}_I = \underline{\mathbf{v}} + \underline{\mathbf{k}}_I,\\
\underline{\mathbf{k}}_I &= \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}(\mathbf{G}\underline{\widehat{\boldsymbol{\beta}}}\mathbf{H} + \mathbf{G}_0)(\mathbf{H}'\boldsymbol{\Sigma}\mathbf{H})^{-1}\mathbf{H}'\boldsymbol{\Sigma}.
\end{aligned}$$

The matrices $\underline{\mathbf{v}}$ and $\mathbf{k}_I$ are stochastically independent and thus

$$\begin{aligned}
\mathrm{Var}[\mathrm{vec}(\underline{\mathbf{v}}_I)] &= \mathrm{Var}[\mathrm{vec}(\underline{\mathbf{v}})] + \mathrm{Var}(\mathrm{vec}(\underline{\mathbf{k}}_I),\\
\mathrm{Var}[\mathrm{vec}(\underline{\mathbf{v}})] &= \boldsymbol{\Sigma} \otimes \mathbf{M}_{\mathbf{X}},\\
\mathrm{Var}(\mathrm{vec}(\underline{\mathbf{k}}_I) &= \mathbf{K} = [\boldsymbol{\Sigma}\mathbf{H}[\mathbf{H}'\boldsymbol{\Sigma}\mathbf{H})^{-1}\mathbf{H}'\boldsymbol{\Sigma}] \otimes \{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\\
&\quad \times \mathbf{G}']^{-1}\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\} = (\boldsymbol{\Sigma}\mathbf{P}_{\mathbf{H}}^{\Sigma}) \otimes \mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}.
\end{aligned}$$

If $\mathrm{Tr}(\underline{\mathbf{v}}_I'\underline{\mathbf{v}}_I\boldsymbol{\Sigma}^{-1}) \geq \chi^2_{m(n-k)+qs}(0; 1-\alpha)$ for sufficiently small α, then the measured data are not compatible with the model. (It is to be remarked that $\boldsymbol{\Sigma}^{-1}$ is a generalized inverse of the matrix $\mathrm{Var}[\mathrm{vec}(\underline{\mathbf{v}}_I)]$.) On the basis of thorough inspection of the data genesis it is sometimes possible to decide which of data are suspicious. Let it be made. Then the model (4) and (5) is rewritten as

$$\mathrm{vec}(\underline{\mathbf{Y}}) \sim N_{nm}\left[(\mathbf{I}\otimes\mathbf{X},\mathbf{E})\begin{pmatrix}\mathrm{vec}(\underline{\boldsymbol{\beta}})\\ \boldsymbol{\Delta}\end{pmatrix},\boldsymbol{\Sigma}\otimes\mathbf{I}\right],\quad \mathbf{G}\underline{\boldsymbol{\beta}}\mathbf{H}+\mathbf{G}_0=\mathbf{0}. \tag{6}$$

The indices i_r, j_r in the matrix $\mathbf{E}$ are chosen such that

$$\{\underline{\mathbf{Y}}\}_{i_r,j_r},\quad r=1,\dots,s,$$

are suspicious observations.

Lemma 3.2.3. The hypothesis $H_0:\boldsymbol{\Delta}=\mathbf{0}$ versus $H_a:\boldsymbol{\Delta}\neq\mathbf{0}$ in the model (6) can be tested iff

$$\mathcal{M}\begin{pmatrix}\mathbf{I}\otimes\mathbf{X}\\ \mathbf{H}'\otimes\mathbf{G}\end{pmatrix}\cap\mathcal{M}\begin{pmatrix}\mathbf{E}\\ \mathbf{0}\end{pmatrix}=\{\mathbf{0}\},$$

what is equivalent to $\mathcal{M}\left[(\mathbf{I}\otimes\mathbf{X})\mathbf{M}_{(\mathbf{H}\otimes\mathbf{G}')}\right]\cap\mathcal{M}(\mathbf{E})=\{\mathbf{0}\}$. The last equality can be rewritten as

$$(\mathbf{I}\otimes\mathbf{X})\mathbf{M}_{(\mathbf{H}\otimes\mathbf{G}')}=\mathbf{M}_{\mathbf{H}\otimes\mathbf{X}}+\mathbf{P}_{\mathbf{H}\otimes(\mathbf{X}\mathbf{M}_{\mathbf{G}'})}.$$

P r o o f. It is a consequence of Lemma 3.1.3. □

Theorem 3.2.4. The BLUE of the vector $\begin{pmatrix}\mathrm{vec}(\underline{\boldsymbol{\beta}})\\ \boldsymbol{\Delta}\end{pmatrix}$ in the regular model (6) is $\begin{pmatrix}\mathrm{vec}(\widehat{\widehat{\underline{\boldsymbol{\beta}}}}_{\mathrm{out}})\\ \widehat{\widehat{\boldsymbol{\Delta}}}\end{pmatrix}$,

$$\begin{aligned}\mathrm{vec}(\widehat{\widehat{\underline{\boldsymbol{\beta}}}}_{\mathrm{out}}) &= \mathrm{vec}(\widehat{\widehat{\underline{\boldsymbol{\beta}}}})-\Big(\mathbf{I}\otimes[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']-[\boldsymbol{\Sigma}\mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}\mathbf{H})^{-1}\mathbf{H}']\\ &\quad\otimes\{(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\}\Big)\mathbf{E}\widehat{\widehat{\boldsymbol{\Delta}}},\\ \mathrm{vec}(\widehat{\widehat{\underline{\boldsymbol{\beta}}}}) &= \mathrm{vec}(\widehat{\underline{\boldsymbol{\beta}}})-\Big([\boldsymbol{\Sigma}\mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}\mathbf{H})^{-1}]\otimes\{(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\}\Big)\\ &\quad\times[(\mathbf{H}'\otimes\mathbf{G})\mathrm{vec}(\widehat{\underline{\boldsymbol{\beta}}})+\mathrm{vec}(\mathbf{G}_0)]\end{aligned}$$

(the BLUE of $\mathrm{vec}(\underline{\boldsymbol{\beta}})$ in the model (4) with constraints (5)),

$$\mathrm{vec}(\widehat{\underline{\boldsymbol{\beta}}})=\{\mathbf{I}\otimes[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\}\mathrm{vec}(\underline{\mathbf{Y}})$$

(the BLUE of $\mathrm{vec}(\underline{\boldsymbol{\beta}})$ in the model (4) without constraints) and

$$\begin{aligned}\widehat{\widehat{\boldsymbol{\Delta}}} &= \left[\mathbf{E}'\left(\boldsymbol{\Sigma}^{-1}\otimes\mathbf{M}_{\mathbf{X}}+[\mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}\mathbf{H})^{-1}\mathbf{H}']\otimes\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\right)\mathbf{E}\right]^{-1}\\ &\quad\times\mathbf{E}'(\boldsymbol{\Sigma}^{-1}\otimes\mathbf{I})[\mathrm{vec}(\underline{\mathbf{Y}})-(\mathbf{I}\otimes\mathbf{X})\mathrm{vec}(\widehat{\widehat{\underline{\boldsymbol{\beta}}}})].\end{aligned}$$

Further

$$\begin{aligned}
\mathrm{Var}[\mathrm{vec}(\underline{\hat{\hat{\boldsymbol{\beta}}}}_{\mathrm{out}})] &= \mathrm{Var}[\mathrm{vec}(\underline{\hat{\hat{\boldsymbol{\beta}}}}])] + \mathbf{A}'\mathrm{Var}(\hat{\hat{\boldsymbol{\Delta}}})\mathbf{A},\\
\mathrm{Var}[\mathrm{vec}(\underline{\hat{\hat{\boldsymbol{\beta}}}})] &= \mathrm{Var}[\mathrm{vec}(\underline{\hat{\boldsymbol{\beta}}})] - [\boldsymbol{\Sigma}\mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}\mathbf{H})^{-1}\mathbf{H}'\boldsymbol{\Sigma}]\\
&\quad \otimes\{(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\},\\
\mathrm{Var}[\mathrm{vec}(\underline{\hat{\boldsymbol{\beta}}})] &= \boldsymbol{\Sigma}\otimes(\mathbf{X}'\mathbf{X})^{-1},\\
\mathbf{A} &= \mathbf{E}'(\boldsymbol{\Sigma}^{-1}\otimes\mathbf{X})\left\{\mathbf{M}_{\mathbf{H}\otimes\mathbf{G}'}[\boldsymbol{\Sigma}^{-1}\otimes(\mathbf{X}'\mathbf{X})]\mathbf{M}_{\mathbf{H}\otimes\mathbf{G}'}\right\}^{+}\\
&= \mathbf{E}'\Big(\mathbf{I}\otimes[\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}] - \mathbf{P}_{\mathbf{H}}^{\Sigma}\otimes\{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'\\
&\quad \times[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\}\Big),\\
\mathrm{Var}(\hat{\hat{\boldsymbol{\Delta}}}) &= \Big[\mathbf{E}'\Big(\boldsymbol{\Sigma}^{-1}\otimes\mathbf{M}_{\mathbf{X}} + (\mathbf{P}_{\mathbf{H}}^{\Sigma}\boldsymbol{\Sigma}^{-1})\otimes\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\Big)\mathbf{E}\Big]^{-1},\\
\mathrm{cov}[\mathrm{vec}(\underline{\hat{\hat{\boldsymbol{\beta}}}}_{\mathrm{out}}),\hat{\hat{\boldsymbol{\Delta}}}] &= -\mathbf{A}'\mathrm{Var}(\hat{\hat{\boldsymbol{\Delta}}}).
\end{aligned}$$

P r o o f. With respect to Lemma 3.1.4 it is valid

$$\mathrm{vec}(\underline{\hat{\hat{\boldsymbol{\beta}}}}_{\mathrm{out}}) = \mathrm{vec}(\underline{\hat{\hat{\boldsymbol{\beta}}}}) - \left\{\mathbf{M}_{\mathbf{H}\otimes\mathbf{G}'}[\boldsymbol{\Sigma}^{-1}\otimes(\mathbf{X}'\mathbf{X})]\mathbf{M}_{\mathbf{H}\otimes\mathbf{G}'}\right\}^{+}(\boldsymbol{\Sigma}^{-1}\otimes\mathbf{X}')\mathbf{E}\hat{\hat{\boldsymbol{\Delta}}}.$$

Since

$$\begin{aligned}
&\left\{\mathbf{M}_{\mathbf{H}\otimes\mathbf{G}'}[\boldsymbol{\Sigma}^{-1}\otimes(\mathbf{X}'\mathbf{X})]\mathbf{M}_{\mathbf{H}\otimes\mathbf{G}'}\right\}^{+}\\
&\quad= \boldsymbol{\Sigma}\otimes(\mathbf{X}'\mathbf{X})^{-1} - [\boldsymbol{\Sigma}\otimes(\mathbf{X}'\mathbf{X})^{-1}](\mathbf{H}\otimes\mathbf{G}')\\
&\quad\quad \times\Big\{(\mathbf{H}'\otimes\mathbf{G})[\boldsymbol{\Sigma}\otimes(\mathbf{X}'\mathbf{X})^{-1}](\mathbf{H}\otimes\mathbf{G}')\Big\}^{-1}(\mathbf{H}'\otimes\mathbf{G})[\boldsymbol{\Sigma}\otimes(\mathbf{X}'\mathbf{X})^{-1}]\\
&\quad= \boldsymbol{\Sigma}\otimes(\mathbf{X}'\mathbf{X})^{-1} - (\boldsymbol{\Sigma}\mathbf{P}_{\mathbf{H}}^{\Sigma})\otimes\{(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\},
\end{aligned}$$

the expression for $\mathrm{vec}(\underline{\hat{\hat{\boldsymbol{\beta}}}}_{\mathrm{out}})$ can be easily obtained. Analogously the expression

$$\begin{aligned}
\hat{\hat{\boldsymbol{\Delta}}} = \Big\{\mathbf{E}'\Big[\mathbf{M}_{(\mathbf{I}\otimes\mathbf{X})\mathbf{M}_{\mathbf{H}\otimes\mathbf{G}'}}(\boldsymbol{\Sigma}\otimes\mathbf{I})\mathbf{M}_{(\mathbf{I}\otimes\mathbf{X})\mathbf{M}_{\mathbf{H}\otimes\mathbf{G}'}}\Big]^{+}\mathbf{E}\Big\}^{-1}\mathbf{E}'(\boldsymbol{\Sigma}^{-1}\otimes\mathbf{I})\\
\times[\mathrm{vec}(\underline{\mathbf{Y}}) - (\mathbf{I}\otimes\mathbf{X})\mathrm{vec}(\underline{\hat{\hat{\boldsymbol{\beta}}}})]
\end{aligned}$$

can be easily reestablished into expression given in the statement. Further, again with respect to Lemma 3.1.4,

$$\begin{aligned}
\mathrm{Var}[\mathrm{vec}(\underline{\hat{\hat{\boldsymbol{\beta}}}}_{\mathrm{out}})] = \mathrm{Var}[\mathrm{vec}(\underline{\hat{\hat{\boldsymbol{\beta}}}})] + \Big\{\mathbf{M}_{\mathbf{H}\otimes\mathbf{G}'}[\boldsymbol{\Sigma}^{-1}\otimes(\mathbf{X}'\mathbf{X})]\mathbf{M}_{\mathbf{H}\otimes\mathbf{G}'}\Big\}^{+}(\boldsymbol{\Sigma}^{-1}\otimes\mathbf{X}')\\
\times\mathbf{E}\mathrm{Var}(\hat{\hat{\boldsymbol{\Delta}}})\mathbf{E}'(\boldsymbol{\Sigma}^{-1}\otimes\mathbf{X})\Big\{\mathbf{M}_{\mathbf{H}\otimes\mathbf{G}'}[\boldsymbol{\Sigma}^{-1}\otimes(\mathbf{X}'\mathbf{X})]\mathbf{M}_{\mathbf{H}\otimes\mathbf{G}'}\Big\}^{+}
\end{aligned}$$

and Corollary 3.2.2, the proof can be easily finished. □

Corollary 3.2.5. The hypothesis $H_0 : \mathbf{\Delta} = \mathbf{0}$ versus $H_a : \mathbf{\Delta} \neq \mathbf{0}$, can be tested on the base of Theorem 3.2.4. The test statistic is

$$\tau = \widehat{\widehat{\mathbf{\Delta}}}'[\mathrm{Var}(\widehat{\widehat{\mathbf{\Delta}}})]^{-1}\widehat{\widehat{\mathbf{\Delta}}} \sim \chi^2_s(\delta), \quad \delta = \mathbf{\Delta}'[\mathrm{Var}(\widehat{\widehat{\mathbf{\Delta}}})]^{-1}\mathbf{\Delta}.$$

If the hypothesis $\mathbf{\Delta} = \mathbf{0}$ is rejected and it is valid

$$|\{\widehat{\widehat{\Delta}}\}_i| > \sqrt{\chi^2_s(0, 1-\alpha)}\sqrt{\{\mathrm{Var}(\widehat{\widehat{\mathbf{\Delta}}}\}_{i,i}},$$

then the measurement $\{\mathrm{vec}(\underline{\mathbf{Y}})\}_i$ can be declared to be outlier.

If $\mathbf{\Sigma} = \sigma^2\mathbf{V}$, where σ^2 is unknown parameter and $\mathbf{V}$ is a known p.d. matrix, then σ^2 must be estimated and the test must be a little modified.

Lemma 3.2.6. Let

$$\begin{aligned}
\underline{\mathbf{v}} &= \underline{\mathbf{Y}} - \mathbf{X}\widehat{\underline{\boldsymbol{\beta}}}, \quad \widehat{\underline{\boldsymbol{\beta}}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{\mathbf{Y}},\\
\underline{\mathbf{v}}_I &= \underline{\mathbf{Y}} - \mathbf{X}\widehat{\widehat{\underline{\boldsymbol{\beta}}}}, \quad \underline{\mathbf{v}}_{I,\mathrm{out}} = \underline{\mathbf{Y}} - \mathbf{X}\widehat{\widehat{\underline{\boldsymbol{\beta}}}}_{\mathrm{out}} - \mathbf{E}\widehat{\widehat{\mathbf{\Delta}}}.
\end{aligned}$$

Then

$$\mathrm{vec}(\mathbf{v}_{I,\mathrm{out}}) = \mathrm{vec}(\mathbf{v}_I) - \Big[\mathbf{I} \otimes \mathbf{M}_{\mathbf{X}} + \big(\mathbf{P}^{\mathbf{V}}_{\mathbf{H}}\big)' \otimes \mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\Big]\mathbf{E}\widehat{\widehat{\mathbf{\Delta}}}$$

and

$$\begin{aligned}
\mathrm{vec}(\underline{\mathbf{v}}_{I,\mathrm{out}}) &\sim N_{nm}\Big\{\mathbf{0}, \mathrm{Var}[\mathrm{vec}(\underline{\mathbf{v}}_I)] - \Big[\mathbf{I} \otimes \mathbf{M}_{\mathbf{X}} + \big(\mathbf{P}^{\mathbf{V}}_{\mathbf{H}}\big)' \otimes \mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\Big]\\
&\quad \times\mathbf{E}\mathrm{Var}(\widehat{\widehat{\mathbf{\Delta}}})\mathbf{E}'\left[\mathbf{I} \otimes \mathbf{M}_{\mathbf{X}} + \mathbf{P}^{\mathbf{V}}_{\mathbf{H}} \otimes \mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\right]\Big\},\\
\mathrm{vec}(\underline{\mathbf{v}}_I) &= \mathrm{vec}(\underline{\mathbf{v}}) + \mathrm{vec}(\underline{\mathbf{k}}_I),\\
\underline{\mathbf{k}}_I &= \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}(\mathbf{G}\widehat{\underline{\boldsymbol{\beta}}}\mathbf{H} + \mathbf{G}_0)(\mathbf{H}'\mathbf{V}\mathbf{H})^{-1}\mathbf{H}'\mathbf{V},
\end{aligned}$$

$\underline{\mathbf{v}}$ and $\underline{\mathbf{k}}_I$ are stochastically independent,

$$\mathrm{vec}(\underline{\mathbf{v}}) \sim N_{n,m}[\mathbf{0}, \sigma^2(\mathbf{V} \otimes \mathbf{M}_{\mathbf{X}})], \quad \mathrm{vec}(\underline{\mathbf{k}}_I) \sim N_{n,m}[\mathbf{0}, \sigma^2(\mathbf{V}\mathbf{P}^{\mathbf{V}}_{\mathbf{H}}) \otimes \mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}],$$

and

$$\mathrm{vec}(\underline{\mathbf{v}}_I) \sim N_{nm}\Big\{\mathbf{0}, \sigma^2\Big[\mathbf{V} \otimes \mathbf{M}_{\mathbf{X}} + (\mathbf{V}\mathbf{P}^{\mathbf{V}}_{\mathbf{H}}) \otimes \mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\Big]\Big\}.$$

Proof. With respect to Theorem 3.2.4

$$\begin{aligned}
&\mathrm{vec}(\underline{\mathbf{Y}}) - (\mathbf{I} \otimes \mathbf{X})\mathrm{vec}(\widehat{\widehat{\underline{\boldsymbol{\beta}}}}_{\mathrm{out}}) - \mathbf{E}\widehat{\widehat{\mathbf{\Delta}}} = \mathrm{vec}(\underline{\mathbf{Y}}) - (\mathbf{I} \otimes \mathbf{X})\mathrm{vec}(\widehat{\widehat{\underline{\boldsymbol{\beta}}}}) + \Big(\mathbf{I} \otimes \mathbf{P}_{\mathbf{X}} - (\mathbf{P}^{\mathbf{V}}_{\mathbf{H}})'\\
&\otimes \mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\Big)\mathbf{E}\widehat{\widehat{\mathbf{\Delta}}} - \mathbf{E}\widehat{\widehat{\mathbf{\Delta}}} = \mathrm{vec}(\underline{\mathbf{v}}_I) - \Big[\mathbf{I} \otimes \mathbf{M}_{\mathbf{X}} + (\mathbf{P}^{\mathbf{V}}_{\mathbf{H}})' \otimes \mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\Big]\mathbf{E}\widehat{\widehat{\mathbf{\Delta}}}.
\end{aligned}$$

Since $\underline{\mathbf{v}} = \mathbf{M}_{\mathbf{X}}\underline{\mathbf{Y}}$ and $\underline{\mathbf{k}}_I$ is a function of $\widehat{\underline{\boldsymbol{\beta}}}$, they are stochastically independent. Obviously

$$\mathrm{vec}(\underline{\mathbf{k}}_I) \sim N_{nm}\Big[\mathbf{0}, \sigma^2\Big(\mathbf{V}\mathbf{P}^{\mathbf{V}}_{\mathbf{H}}\Big) \otimes \mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\Big],$$

and thus

$$\mathrm{vec}(\underline{\mathbf{v}}+\underline{\mathbf{k}}_I)=\mathrm{vec}(\underline{\mathbf{v}}_I)\sim N_{nm}\Big\{\mathbf{0},\sigma^2\Big[\mathbf{V}\otimes\mathbf{M_X}+\Big(\mathbf{VP^V_H}\Big)\otimes\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\Big]\Big\}.$$

Since $\underline{\mathbf{v}}_{I,\mathrm{out}}$ can be expressed as

$$\mathrm{vec}(\underline{\mathbf{v}}_{I,\mathrm{out}})=\Bigg\{\mathbf{I}-\Big[\mathbf{I}\otimes\mathbf{M_X}+\Big(\mathbf{P^V_H}\Big)'\otimes\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\Big]\mathbf{E}\Bigg(\mathbf{E}'\Big\{\mathbf{V}^{-1}\otimes\mathbf{M_X}$$
$$+[\mathbf{H}(\mathbf{H}'\mathbf{VH})^{-1}\mathbf{H}']\otimes\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\Big\}\mathbf{E}\Bigg)^{-1}\mathbf{E}'(\mathbf{V}^{-1}\otimes\mathbf{I})\Bigg\}\mathrm{vec}(\underline{\mathbf{v}}_I),$$

we have

$$\mathrm{Var}[\mathrm{vec}(\underline{\mathbf{v}}_{I,\mathrm{out}})]$$
$$=\Bigg\{\mathbf{I}-\Big[\mathbf{I}\otimes\mathbf{M_X}+\Big(\mathbf{P^V_H}\Big)'\otimes\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\Big]\mathbf{E}\frac{1}{\sigma^2}\mathrm{Var}(\widehat{\widehat{\boldsymbol{\Delta}}})\mathbf{E}'(\mathbf{V}^{-1}\otimes\mathbf{I})\Bigg\}\mathrm{Var}(\underline{\mathbf{v}}_I)$$
$$\times\Bigg\{\mathbf{I}-(\mathbf{V}^{-1}\otimes\mathbf{I})\mathbf{E}\frac{1}{\sigma^2}\mathrm{Var}(\widehat{\widehat{\boldsymbol{\Delta}}})\mathbf{E}'(\mathbf{I}\otimes\mathbf{M_X}+\mathbf{P^V_H}\otimes\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'})\Bigg\}.$$

Now the equalities

$$\mathbf{E}'(\mathbf{V}^{-1}\otimes\mathbf{I})\mathrm{Var}[\mathrm{vec}(\underline{\mathbf{v}}_I)](\mathbf{V}^{-1}\otimes\mathbf{I})\mathbf{E}=\sigma^2\mathbf{E}'\Big(\mathbf{V}^{-1}$$
$$\otimes\mathbf{M_X}+[\mathbf{H}(\mathbf{H}'\mathbf{VH})^{-1}\mathbf{H}']\otimes\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\Big)\mathbf{E}=\sigma^4[\mathrm{Var}(\widehat{\widehat{\boldsymbol{\Delta}}})]^{-1}$$

and

$$[\mathbf{I}\otimes\mathbf{M_X}+(\mathbf{P^V_H})'\otimes\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}]\mathbf{E}\frac{1}{\sigma^2}\mathrm{Var}(\widehat{\widehat{\boldsymbol{\Delta}}})\mathbf{E}'(\mathbf{V}^{-1}\otimes\mathbf{I})\mathrm{Var}[\mathrm{vec}(\underline{\mathbf{v}}_I)]$$
$$=[\mathbf{I}\otimes\mathbf{M_X}+(\mathbf{P^V_H})'\otimes\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}]\mathbf{E}\frac{1}{\sigma^2}\mathrm{Var}(\widehat{\widehat{\boldsymbol{\Delta}}})\mathbf{E}'[\mathbf{I}\otimes\mathbf{M_X}+\mathbf{P^V_H}\otimes\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}]$$

must be taken into account in order to obtain the expression for $\mathrm{Var}[\mathrm{vec}(\underline{\mathbf{v}}_{I,\mathrm{out}})]$. □

Corollary 3.2.7. The best estimator of σ^2 in the model (4), (5) is

$$\begin{aligned}\widehat{\sigma}^2_{I,\mathrm{out}} &= \frac{[\mathrm{vec}(\underline{\mathbf{v}}_{I,\mathrm{out}})]'(\mathbf{V}^{-1}\otimes\mathbf{I})\mathrm{vec}(\underline{\mathbf{v}}_{I,\mathrm{out}})}{nm+qr-(km+s)}=\frac{\mathrm{Tr}(\underline{\mathbf{v}}'_{I,\mathrm{out}}\underline{\mathbf{v}}_{I,\mathrm{out}}\mathbf{V}^{-1})}{m(n-k)+qr-s}\\ &\sim \sigma^2\frac{\chi^2_{m(n-k)+qr-s}}{m(n-k)+qr-s}\end{aligned}$$

and the test statistic is

$$\frac{\widehat{\widehat{\boldsymbol{\Delta}}}'\mathbf{E}'\Big\{\mathbf{V}^{-1}\otimes\mathbf{M_X}+[\mathbf{H}(\mathbf{H}'\mathbf{VH})^{-1}\mathbf{H}']\otimes\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\Big\}\mathbf{E}\widehat{\widehat{\boldsymbol{\Delta}}}}{s\widehat{\sigma}^2_{I,\mathrm{out}}}\sim F_{s,m(n-k)-s}(\delta),$$

$$\delta=\frac{\boldsymbol{\Delta}'\mathbf{E}'\Big\{\mathbf{V}^{-1}\otimes\mathbf{M_X}+[\mathbf{H}(\mathbf{H}'\mathbf{VH})^{-1}\mathbf{H}']\otimes\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\Big\}\mathbf{E}\boldsymbol{\Delta}}{\sigma^2}.$$

Remark 3.2.8. The hypothesis $\boldsymbol{\Delta} = \mathbf{0}$ is rejected due to those measurements $\{\underline{\mathbf{Y}}\}_{i_r,j_r}$ for which

$$|\widehat{\widehat{\Delta}}_{i_r,j_r}| \geq \widehat{\sigma}_{I,\text{out}}\sqrt{sF_{s,m(n-k)+qr-s}(0;1-\alpha)}\sqrt{(\mathbf{e}_{i_r}^{(m)} \otimes \mathbf{e}_{j_r}^{(n)})'\mathbf{U}(\mathbf{e}_{i_r}^{(m)} \otimes \mathbf{e}_{j_r}^{(n)})},$$
$$\mathbf{U} = \left[\mathbf{E}'\Big(\mathbf{V}^{-1} \otimes \mathbf{M}_{\mathbf{X}} + [\mathbf{H}(\mathbf{H}'\mathbf{V}\mathbf{H})^{-1}\mathbf{H}'] \otimes \mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\Big)\mathbf{E}\right]^{-1}.$$

4. PROBLEM OF VARIANCE COMPONENTS

4.1. Univariate models

Let a regular univariate linear model

$$\mathbf{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \textstyle\sum_{i=1}^p \vartheta_i \mathbf{V}_i), \quad \boldsymbol{\beta} \in \mathcal{V}_I = \{\mathbf{u} : \mathbf{b} + \mathbf{B}\mathbf{u} = \mathbf{0}\}, \qquad (7)$$
$$\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_p)' \in \underline{\vartheta} \subset R^p,$$

be under consideration. Here except $\boldsymbol{\beta}$ also the vector parameter $\boldsymbol{\vartheta}$ is unknown. The parameter space $\underline{\vartheta}$ is an open set in the p-dimensional Euclidean space, $\vartheta_i > 0, i = 1, \dots, p$, and symmetric matrices $\mathbf{V}_1, \dots, \mathbf{V}_p$, are p.s.d. and known. An estimator of the variance components $\vartheta_1, \dots, \vartheta_p$, is calculated often in an iterative way. An arbitrary value $\boldsymbol{\vartheta}_0$ of the vector is chosen and the $\boldsymbol{\vartheta}_0$-MINQUE (minimum norm quadratic unbiased estimator; in more detail cf. [11] and [5]) $\widehat{\boldsymbol{\vartheta}}$ is determined. In the next step this estimator is chosen instead of $\boldsymbol{\vartheta}_0$ and the procedure continues. For the sake of simplicity in the following text it is assumed that $\boldsymbol{\vartheta}_0$ is such good starting point of this procedure that only one step of iteration is necessary.

Lemma 4.1.1. The MINQUE of the vector $\boldsymbol{\vartheta}$ in the model (7) is

$$\widehat{\boldsymbol{\vartheta}} = \mathbf{S}^{-1}_{(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}_0\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}})^+} \begin{pmatrix} \mathbf{v}_I'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_1\boldsymbol{\Sigma}_0^{-1}\mathbf{v}_I \\ \vdots \\ \mathbf{v}_I'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_1\boldsymbol{\Sigma}_0^{-1}\mathbf{v}_I \end{pmatrix},$$

$$\left\{\mathbf{S}_{(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}_0\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}})^+}\right\}_{i,j}$$
$$= \operatorname{Tr}\left[\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}_0\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{V}_i\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}_0\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{V}_j\right], \quad i,j = 1,\dots,p,$$

$\boldsymbol{\Sigma}_0 = \sum_{i=1}^p \vartheta_i^{(0)}\mathbf{V}_i, \mathbf{v}_I = \mathbf{Y} - \mathbf{X}\widehat{\widehat{\boldsymbol{\beta}}}$, and $\widehat{\widehat{\boldsymbol{\beta}}}$ is the $\boldsymbol{\vartheta}^{(0)}$-LBLUE (locally best linear unbiased estimator) of the vector $\boldsymbol{\beta}$ given by Lemma 3.1.1 for $\mathbf{C} = \mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X}$. If $\mathbf{Y}$ is normally distributed, then $\operatorname{Var}_{\vartheta^{(0)}}(\widehat{\boldsymbol{\vartheta}}) = 2\mathbf{S}^{-1}_{(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}_0\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}})^+}$.

Proof. Cf. [5]. □

The problem is whether $\widehat{\boldsymbol{\vartheta}}$ can be used instead of the actual value $\boldsymbol{\vartheta}^*$ of the vector $\boldsymbol{\vartheta}$.

One approach to the problem is given in the following text.

Let

$$\begin{aligned}
\phi(\boldsymbol{\vartheta}) &= \mathbf{v}_I'(\boldsymbol{\vartheta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\mathbf{v}_I(\boldsymbol{\vartheta}),\\
\mathbf{v}_I(\boldsymbol{\vartheta}) &= \mathbf{Y}-\mathbf{X}\widehat{\widehat{\boldsymbol{\beta}}}(\boldsymbol{\vartheta}) = \mathbf{Y}-\mathbf{X}\widehat{\boldsymbol{\beta}}(\boldsymbol{\vartheta})+\mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta})\mathbf{B}'\\
&\quad \times[\mathbf{B}\mathbf{C}^{-1}(\boldsymbol{\vartheta})\mathbf{B}']^{-1}[\mathbf{B}\widehat{\boldsymbol{\beta}}(\boldsymbol{\vartheta})+\mathbf{b}],\\
\widehat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}) &= \mathbf{C}^{-1}(\boldsymbol{\vartheta})\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\mathbf{Y},\\
\mathbf{C}(\boldsymbol{\vartheta}) &= \mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\mathbf{X}.
\end{aligned}$$

Lemma 4.1.2. Under the given notation the following relationships are valid.

$$\begin{aligned}
\phi(\boldsymbol{\vartheta}_0+\delta\boldsymbol{\vartheta}) &= \phi(\boldsymbol{\vartheta}_0)+\sum_{i=1}^{p}\mathbf{v}_I'(\boldsymbol{\vartheta}_0)\Big\{2\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}_0)\mathbf{M}_{\mathbf{B}'}]^+\\
&\quad \times\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)-\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\Big\}\delta\vartheta_i\\
&\quad + \ \text{terms of higher orders}\\
&= \phi(\boldsymbol{\vartheta}_0)+\boldsymbol{\eta}'(\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta}+\ \text{terms of higher orders},\\
E_{\vartheta_0}[\boldsymbol{\eta}'(\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta}] &= -\mathbf{a}'(\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta}+\ \text{terms of higher orders},\\
\mathbf{a}'(\boldsymbol{\vartheta}_0) &= [a_1(\boldsymbol{\vartheta}_0),\dots,a_p(\boldsymbol{\vartheta}_0)],\\
a_i(\boldsymbol{\vartheta}_0) &= \mathrm{Tr}\Big\{\Big[\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\Big]^+\mathbf{V}_i\Big\},\quad i=1,\dots,p,\\
\mathrm{Var}_{\vartheta_0}[\boldsymbol{\eta}'(\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta}] &= 2\delta\boldsymbol{\vartheta}'\mathbf{S}_{[\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}(\vartheta_0)\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}]^+}\delta\boldsymbol{\vartheta},
\end{aligned}$$

$$\begin{gathered}
\Big\{\mathbf{S}_{[\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}(\vartheta_0)\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}]^+}\Big\}_{i,j}\\
=\mathrm{Tr}\Big\{\Big[\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\Big]^+\mathbf{V}_i\Big[\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\Big]^+\mathbf{V}_j\Big\},\\
i,j=1,\dots,p.
\end{gathered}$$

P r o o f. If the relationship

$$\frac{\partial\mathbf{A}(\boldsymbol{\vartheta})}{\partial\vartheta_i}=-\mathbf{A}^{-1}(\boldsymbol{\vartheta})\frac{\partial\mathbf{A}(\boldsymbol{\vartheta})}{\partial\vartheta_i}\mathbf{A}^{-1}(\boldsymbol{\vartheta})$$

which is valid for any matrix regular in a neighbourhood of the vector $\boldsymbol{\vartheta}$, is taken into account, we obtain the following relationships (for the sake of simplicity the dependence on $\boldsymbol{\vartheta}$ is not written).

$$\begin{aligned}
\frac{\partial\phi}{\partial\vartheta_i} &= 2\mathbf{v}_I'\boldsymbol{\Sigma}^{-1}\frac{\partial\mathbf{v}_I}{\partial\vartheta_i}-\mathbf{v}_I'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\mathbf{v}_I,\\
\frac{\partial\mathbf{v}_I}{\partial\vartheta_i} &= -\mathbf{X}\frac{\partial\widehat{\boldsymbol{\beta}}}{\partial\vartheta_i}+\mathbf{X}\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}(\mathbf{B}\widehat{\boldsymbol{\beta}}+\mathbf{b})\\
&\quad -\mathbf{X}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}
\end{aligned}$$

$$\times(\mathbf{B}\widehat{\boldsymbol{\beta}}+\mathbf{b})+\mathbf{X}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\frac{\partial\widehat{\boldsymbol{\beta}}}{\partial\vartheta_i},$$

$$\begin{aligned}
\frac{\partial\widehat{\boldsymbol{\beta}}}{\partial\vartheta_i} &= \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}-\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\mathbf{Y}\\
&= -\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\mathbf{Y}-\mathbf{X}\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y})\\
&= -\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\mathbf{Y}-\mathbf{X}\widehat{\boldsymbol{\beta}}).
\end{aligned}$$

Let $\mathbf{v}=\mathbf{Y}-\mathbf{X}\widehat{\boldsymbol{\beta}}$. Thus $\mathbf{v}_I=\mathbf{v}+\mathbf{X}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}(\mathbf{B}\widehat{\boldsymbol{\beta}}+\mathbf{b})$ and

$$\begin{aligned}
\frac{\partial\mathbf{v}_I}{\partial\vartheta_i} &= +\mathbf{X}\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\mathbf{v}\\
&\quad +\mathbf{X}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}(\mathbf{B}\widehat{\boldsymbol{\beta}}+\mathbf{b})\\
&\quad +\mathbf{X}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\Big[-\mathbf{B}\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\mathbf{v}\Big]\\
&= \mathbf{X}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}(\mathbf{B}\widehat{\boldsymbol{\beta}}+\mathbf{b})\\
&\quad +\mathbf{X}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\mathbf{v}\\
&= \mathbf{X}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\mathbf{v}_I.
\end{aligned}$$

Thus we have

$$\frac{\partial\phi}{\partial\vartheta_i}=\mathbf{v}_I'\Big[2\boldsymbol{\Sigma}^{-1}\mathbf{X}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}-\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\Big]\mathbf{v}_I.$$

Let

$$\mathbf{A}_i=2\boldsymbol{\Sigma}^{-1}\mathbf{X}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}-\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}.$$

Thus

$$E\left(\frac{\partial\phi}{\partial\vartheta_i}\right)=\mathrm{Tr}[\mathbf{A}_i\mathrm{Var}(\mathbf{v}_I)].$$

Since $\mathrm{Var}(\mathbf{v}_I)=\boldsymbol{\Sigma}-\mathbf{X}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'$, we have

$$\begin{aligned}
\mathrm{Tr}[\mathbf{A}_i\mathrm{Var}(\mathbf{v}_I)] &= \mathrm{Tr}\Big[2\boldsymbol{\Sigma}^{-1}\mathbf{X}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i-\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\\
&\quad -2\boldsymbol{\Sigma}^{-1}\mathbf{X}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\mathbf{X}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\\
&\quad +\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\mathbf{X}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\Big]\\
&= -\mathrm{Tr}\Big[(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}})^{+}\mathbf{V}_i\Big].
\end{aligned}$$

Further

$$\begin{aligned}
\mathrm{cov}\left(\frac{\partial\phi}{\partial\vartheta_i},\frac{\partial\phi}{\partial\vartheta_j}\right) &= \mathrm{cov}(\mathbf{v}_I'\mathbf{A}_i\mathbf{v}_I,\mathbf{v}_I'\mathbf{A}_j\mathbf{v}_I)=2\mathrm{Tr}\Big[\mathbf{A}_i\mathrm{Var}(\mathbf{v}_I)\mathbf{A}_j\mathrm{Var}(\mathbf{v}_I)\Big]\\
&= 2\mathrm{Tr}\Big[(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}})^{+}\mathbf{V}_i(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}})^{+}\mathbf{V}_j\Big]\\
&= 2\left\{\mathbf{S}_{(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}})^{+}}\right\}_{i,j}.
\end{aligned}$$

Thus the statement is proved. □

Theorem 4.1.3. Let $\delta_{\max}$ be a solution of the equation

$$P\{\chi^2_{n+q-k}(0) + \delta_{\max} \geq \chi^2_{n+q-k}(0; 1-\alpha)\} = \alpha + \varepsilon,$$

i. e. $\delta_{\max} = \chi^2_{n+q-k}(0; 1-\alpha) - \chi^2_{n+q-k}(0; 1-\alpha-\varepsilon)$ and let $t > 0$ be such real number that $P\{\boldsymbol{\eta}'\delta\boldsymbol{\vartheta} < \delta_{\max}\} \geq 1 - \frac{1}{t^2}$, i. e.

$$-\mathbf{a}'\delta\boldsymbol{\vartheta} + t\sqrt{\delta\boldsymbol{\vartheta}' 2\mathbf{S}_{(\mathbf{M_{XM_{B'}}}\boldsymbol{\Sigma}\mathbf{M_{XM_{B'}}})^+}\delta\boldsymbol{\vartheta}} \leq \delta_{\max},$$

where t is sufficiently large. Let

$$\mathbf{A} = 2t^2\mathbf{S}_{(\mathbf{M_{XM_{B'}}}\boldsymbol{\Sigma}\mathbf{M_{XM_{B'}}})^+} - \mathbf{a}\mathbf{a}'.$$

Then

$$\delta\boldsymbol{\vartheta} \in \mathcal{N} = \Big\{\delta\boldsymbol{\vartheta} : (\delta\boldsymbol{\vartheta} - \delta_{\max}\mathbf{A}^+\mathbf{a})'\mathbf{A}(\delta\boldsymbol{\vartheta} - \delta_{\max}\mathbf{A}^+\mathbf{a}) \leq \delta^2_{\max}(1 + \mathbf{a}'\mathbf{A}^+\mathbf{a})\Big\}$$
$$\Rightarrow P_{\vartheta_0}\Big\{\mathbf{v}'_I(\boldsymbol{\vartheta}_0)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{v}_I(\boldsymbol{\vartheta}_0) + \boldsymbol{\eta}'(\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta} \geq \chi^2_{n+q-k}(0; 1-\alpha)\Big\} \leq \alpha + \varepsilon.$$

Proof. With respect to Lemma 4.1.2, when the terms of higher orders are neglected,

$$P_{\vartheta_0}\Big\{\mathbf{v}'_I(\boldsymbol{\vartheta}_0)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{v}_I(\boldsymbol{\vartheta}_0) + \boldsymbol{\eta}'(\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta} \geq \chi^2_{n+q-k}(0; 1-\alpha)\Big\} \leq \alpha + \varepsilon$$
$$\Leftrightarrow P\{\boldsymbol{\eta}'(\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta} \leq \delta_{\max}\} = 1,$$

i. e. $E_{\vartheta_0}(\boldsymbol{\eta}'(\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta}) + t\sqrt{\mathrm{Var}_{\vartheta_0}[\boldsymbol{\eta}'(\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta}]} \leq \delta_{\max}$ for sufficiently large t. Let

$$t^2\mathrm{Var}_{\vartheta_0}[\boldsymbol{\eta}'(\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta}] \leq (\delta_{\max} + \mathbf{a}'\delta\boldsymbol{\vartheta})^2.$$

From this inequality we obtain

$$\delta\boldsymbol{\vartheta}'\left(2t^2\mathbf{S}_{(\mathbf{M_{XM_{B'}}}\boldsymbol{\Sigma}\mathbf{M_{XM_{B'}}})^+} - \mathbf{a}\mathbf{a}'\right)\delta\boldsymbol{\vartheta} - 2\delta_{\max}\mathbf{a}'\delta\boldsymbol{\vartheta} \leq \delta^2_{\max}.$$

If $\mathbf{a} \in \mathcal{M}(\mathbf{A})$, then it can be written as

$$(\delta\boldsymbol{\vartheta} - \delta_{\max}\mathbf{A}^+\mathbf{a})'\mathbf{A}(\delta\boldsymbol{\vartheta} - \delta_{\max}\mathbf{A}^+\mathbf{a}) \leq \delta^2_{\max}(1 + \mathbf{a}'\mathbf{A}^+\mathbf{a}).$$

The relationship $\mathbf{a} \in \mathcal{M}(\mathbf{A})$ can be proved as follows.

The matrix $(\mathbf{M_{XM_{B'}}}\boldsymbol{\Sigma}\mathbf{M_{XM_{B'}}})^+$ is p.s.d., thus it can be written as $\mathbf{J}\mathbf{J}'$. Therefore

$$\mathbf{a}' = [\mathrm{Tr}(\mathbf{J}'\mathbf{V}_1\mathbf{J}), \ldots, \mathrm{Tr}(\mathbf{J}'\mathbf{V}_p\mathbf{J})]$$

and

$$\Big\{\mathbf{S}_{(\mathbf{M_{XM_{B'}}}\boldsymbol{\Sigma}\mathbf{M_{XM_{B'}}})^+}\Big\}_{i,j} = \mathrm{Tr}(\mathbf{J}'\mathbf{V}_i\mathbf{J}\mathbf{J}'\mathbf{V}_j\mathbf{J}), \quad i,j = 1, \ldots, p,$$

i. e. the matrix $\mathbf{S}_{(\mathbf{M_{XM_{B'}}}, \boldsymbol{\Sigma}\mathbf{M_{XM_{B'}}})^+}$ is the Gramm matrix of the p-tuple $\{\mathbf{J}'\mathbf{V}_i\mathbf{J}\}_{i=1}^p$ in the Hilbert space $\mathcal{H}$ of the symmetric $r(\mathbf{J}) \times r(\mathbf{J})$ matrices with the inner product $\langle \mathbf{A}, \mathbf{B} \rangle = \mathrm{Tr}(\mathbf{AB})$, $\mathbf{A}, \mathbf{B} \in \mathcal{H}$. Since

$$\mathrm{Tr}(\mathbf{J}'\mathbf{V}_i\mathbf{J}) = \mathrm{Tr}(\mathbf{J}'\mathbf{V}_i\mathbf{J}\mathbf{I}) = \mathrm{Tr}\Big(\mathbf{J}'\mathbf{V}_i\mathbf{J}\sum_{j=1}^p \alpha_j \mathbf{J}'\mathbf{V}_j\mathbf{J}\Big),$$

where $\sum_{j=1}^p \alpha_j \mathbf{J}'\mathbf{V}_j\mathbf{J}$ is the Euclidean projection of the matrix $\mathbf{I}$ on the subspace generated by the p-tuple $\{\mathbf{J}'\mathbf{V}_i\mathbf{J}\}_{i=1}^p$, the vector $\mathbf{a}$ can be expressed as $\mathbf{S}_{(\mathbf{M_{XM_{B'}}}, \boldsymbol{\Sigma}\mathbf{M_{XM_{B'}}})^+} \boldsymbol{\alpha}$. Thus

$$\mathbf{a} \in \mathcal{M}(\mathbf{S}_{(\mathbf{M_{XM_{B'}}}, \boldsymbol{\Sigma}\mathbf{M_{XM_{B'}}})^+}) \Rightarrow \mathbf{a} \in \mathcal{M}(2t^2\mathbf{S}_{(\mathbf{M_{XM_{B'}}}, \boldsymbol{\Sigma}\mathbf{M_{XM_{B'}}})^+} - \mathbf{aa}'),$$

since t^2 can be chosen more or less arbitrarily. □

More on the nonsensitivity regions and their optimization cf. [6, 7, 8, 9]. With respect to these references it seems that in practice the value t need not be larger than 5; in some cases it is sufficient to use the value 3.

Corollary 4.1.4. The random variable $\mathbf{v}_I'(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})\mathbf{v}_I(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})$ can be expressed as $\chi^2_{n+q-k}(0) + \boldsymbol{\eta}'(\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta}$ (cf. Lemma 4.1.2). If $\delta\boldsymbol{\vartheta} \in \mathcal{N}$ (Theorem 4.1.3) and

$$\mathbf{v}_I'(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})\mathbf{v}_I(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta}) \geq \chi^2_{n+q-k}(0; 1-\alpha), \tag{8}$$

then we can conclude that outliers occur in measurement results.

The problem is how to recognize whether $\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta} = \boldsymbol{\vartheta}^*$ (actual value of the parameter $\boldsymbol{\vartheta}$) satisfies the relationship $\boldsymbol{\vartheta}^* - \boldsymbol{\vartheta}_0 \in \mathcal{N}$. Some information can be obtained by a comparison of the set $\mathcal{N}$ and the set

$$\mathcal{C} = \left\{\delta\boldsymbol{\vartheta} : (\delta\boldsymbol{\vartheta} - \widehat{\delta\boldsymbol{\vartheta}})' \left[2\mathbf{S}^{-1}_{(\mathbf{M_{XM_{B'}}}, \boldsymbol{\Sigma_0}\mathbf{M_{XM_{B'}}})^+}\right]^{-1} (\delta\boldsymbol{\vartheta} - \widehat{\delta\boldsymbol{\vartheta}}) \leq \frac{p}{\alpha}\right\}.$$

It is valid (the Scheffé theorem; cf. [12])

$$\delta\boldsymbol{\vartheta} \in \mathcal{C} \Leftrightarrow \forall\{\mathbf{h} \in R^p\} |\mathbf{h}'(\delta\boldsymbol{\vartheta} - \widehat{\delta\boldsymbol{\vartheta}})| \leq \sqrt{\frac{p}{\alpha}}\sqrt{\mathrm{Var}_{\vartheta^{(0)}}(\mathbf{h}'\widehat{\vartheta})}$$
$$\Rightarrow \forall\{i = 1, \dots, p\} |\delta\vartheta_i - \widehat{\delta\vartheta}_i| \leq \sqrt{\frac{p}{\alpha}}\sqrt{\mathrm{Var}_{\vartheta^{(0)}}(\widehat{\vartheta}_i)}.$$

Let

$$\mathcal{C}_i = \left\{\delta\vartheta_i : |\delta\vartheta_i - \widehat{\delta\vartheta}_i| \leq \sqrt{\frac{p}{\alpha}}\sqrt{\mathrm{Var}(\widehat{\vartheta}_i)}\right\}.$$

Then regarding the Chebyshev inequality

$$P\{\delta\vartheta_i \notin \mathcal{C}_i\} \leq \frac{\alpha}{p}, \quad i = 1, \dots, p.$$

With respect to the Bonferroni theorem (cf. [4], p. 492)

$$P\Big\{\delta\boldsymbol{\vartheta}\in\cap_{i=1}^{p}(\mathcal{C}_i\times R^{p-1}\Big\} = 1-P\Big\{\delta\boldsymbol{\vartheta}\notin\cup_{i=1}^{p}(\mathcal{C}_i\times R^{p-1})\Big\} \geq 1-\sum_{i=1}^{p}P\{\delta\vartheta_i\notin\mathcal{C}_i\}\geq 1-\alpha.$$

If the difference

$$P\Big\{\delta\boldsymbol{\vartheta}\in\cap_{i=1}^{p}(\mathcal{C}_i\times R^{p-1})\Big\}-P\{\delta\boldsymbol{\vartheta}\in\mathcal{C}\}$$

is neglected, then $\mathcal{C}\subset\mathcal{N}$ enables us to use $\widehat{\boldsymbol{\vartheta}}$ instead of the actual however unknown value $\boldsymbol{\vartheta}^*$.

If (8) is valid and $\boldsymbol{\vartheta}^*-\boldsymbol{\vartheta}_0\in\mathcal{N}$, then by the inspection of data, it is sometimes possible to indicate suspicious of them. In this case the model

$$\mathbf{Y}\sim N_n\left[(\mathbf{X},\mathbf{F})\begin{pmatrix}\boldsymbol{\beta}\\ \boldsymbol{\Delta}\end{pmatrix},\sum_{i=1}^{p}\vartheta_i\mathbf{V}_i\right],\quad \mathbf{b}+\mathbf{B}\boldsymbol{\beta}=\mathbf{0}, \tag{9}$$

will be considered.

Lemma 4.1.5. Let in the regular mixed linear model (9) the statistic $T(\boldsymbol{\vartheta})=\widehat{\widehat{\boldsymbol{\Delta}}}'[\mathrm{Var}(\widehat{\widehat{\boldsymbol{\Delta}}})]^{-1}\widehat{\widehat{\boldsymbol{\Delta}}}$, where (cf. Lemma 3.1.4)

$$\begin{aligned}\widehat{\widehat{\boldsymbol{\Delta}}} &= \left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{XM_{B'}}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{XM_{B'}}}\right)^{+}\mathbf{F}\right]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\mathbf{Y}-\mathbf{X}\widehat{\widehat{\boldsymbol{\beta}}})\\ &= \left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{XM_{B'}}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{XM_{B'}}}\right)^{+}\mathbf{F}\right]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}[\mathbf{M}_{\mathbf{XM_{B'}}}^{\Sigma^{-1}}\mathbf{Y}+\\ &\quad +\mathbf{XC}^{-1}\mathbf{B}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{b}]\\ &= \left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{XM_{B'}}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{XM_{B'}}}\right)^{+}\mathbf{F}\right]^{-1}\mathbf{F}'\left(\mathbf{M}_{\mathbf{XM_{B'}}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{XM_{B'}}}\right)^{+}\mathbf{Y}\\ &\quad +\left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{XM_{B'}}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{XM_{B'}}}\right)^{+}\mathbf{F}\right]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{XC}^{-1}\mathbf{B}'(\mathbf{BC}^{-1}\mathbf{B}')^{-}\mathbf{b}\end{aligned}$$

be considered. Then

$$\begin{aligned}\frac{\partial T}{\partial\vartheta_i} &= -\widehat{\widehat{\boldsymbol{\Delta}}}'\mathbf{F}'(\mathbf{M}_{\mathbf{XM_{B'}}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{XM_{B'}}})^{+}\mathbf{V}_i(\mathbf{M}_{\mathbf{XM_{B'}}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{XM_{B'}}})^{+}\mathbf{F}\widehat{\widehat{\boldsymbol{\Delta}}}\\ &\quad -2\widehat{\widehat{\boldsymbol{\Delta}}}'\mathbf{F}'(\mathbf{M}_{\mathbf{XM_{B'}}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{XM_{B'}}})^{+}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\mathbf{v}_{I,\mathrm{out}},\\ \mathbf{v}_{I,\mathrm{out}} &= \mathbf{Y}-\mathbf{X}\widehat{\widehat{\boldsymbol{\beta}}}_{\mathrm{out}}-\mathbf{F}\widehat{\widehat{\boldsymbol{\Delta}}}.\end{aligned}$$

Proof. We have

$$\frac{\partial T}{\partial\vartheta_i}=2\widehat{\widehat{\boldsymbol{\Delta}}}'[\mathrm{Var}(\widehat{\widehat{\boldsymbol{\Delta}}})]^{-1}\frac{\partial\widehat{\widehat{\boldsymbol{\Delta}}}}{\partial\vartheta_i}-\widehat{\widehat{\boldsymbol{\Delta}}}'[\mathrm{Var}(\widehat{\widehat{\boldsymbol{\Delta}}})]^{-1}\frac{\partial\mathrm{Var}(\widehat{\widehat{\boldsymbol{\Delta}}})}{\partial\vartheta_i}[\mathrm{Var}(\widehat{\widehat{\boldsymbol{\Delta}}})]^{-1}\widehat{\widehat{\boldsymbol{\Delta}}},$$

and

$$\begin{aligned}
\frac{\partial \widehat{\widehat{\boldsymbol{\Delta}}}}{\partial \vartheta_i} &= \left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+}\mathbf{F}\right]^{-1}\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+}\mathbf{V}_i \\
&\quad \times\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+}\mathbf{F}\left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+}\mathbf{F}\right]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1} \\
&\quad \times(\mathbf{Y}-\mathbf{X}\widehat{\widehat{\boldsymbol{\beta}}})-\left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+}\mathbf{F}\right]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1} \\
&\quad \times(\mathbf{Y}-\mathbf{X}\widehat{\widehat{\boldsymbol{\beta}}})-\left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+}\mathbf{F}\right]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\frac{\partial\widehat{\widehat{\boldsymbol{\beta}}}}{\partial\vartheta_i}, \\
\frac{\partial \widehat{\widehat{\boldsymbol{\beta}}}}{\partial \vartheta_i} &= \frac{\partial}{\partial\vartheta_i}\left[\widehat{\boldsymbol{\beta}}-\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}(\mathbf{B}\widehat{\boldsymbol{\beta}}+\mathbf{b})\right] \\
&= \frac{\partial}{\partial\vartheta_i}\left[(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}-\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{b}\right] \\
&= (\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\mathbf{X}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \\
&\quad -(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\mathbf{Y}-\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1}\mathbf{B}' \\
&\quad \times(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{b}+\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1} \\
&\quad \times\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{b} \\
&= -(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\left[\mathbf{Y}-\mathbf{X}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}\right. \\
&\quad \left.+\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{b}\right] \\
&= -(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\mathbf{Y}-\mathbf{X}\widehat{\widehat{\boldsymbol{\beta}}}).
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\partial \widehat{\widehat{\boldsymbol{\Delta}}}}{\partial \vartheta_i} &= \left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+}\mathbf{F}\right]^{-1}\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+} \\
&\quad \times\mathbf{V}_i\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+}\mathbf{F}\widehat{\widehat{\boldsymbol{\Delta}}} \\
&\quad -\left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+}\mathbf{F}\right]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\mathbf{Y}-\mathbf{X}\widehat{\widehat{\boldsymbol{\beta}}}) \\
&\quad +\left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+}\mathbf{F}\right]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\left(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'}\right)^{+} \\
&\quad \times\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\mathbf{Y}-\mathbf{X}\widehat{\widehat{\boldsymbol{\beta}}}).
\end{aligned}$$

Now the equality

$$\boldsymbol{\Sigma}^{-1}\mathbf{X}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}-\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+}=\boldsymbol{\Sigma}^{-1}$$

and the relationships

$$\begin{aligned}
\mathbf{v}_{I,\mathrm{out}} &= \mathbf{Y}-\mathbf{X}\widehat{\widehat{\boldsymbol{\beta}}}-\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}^{\Sigma^{-1}}\mathbf{F}\widehat{\widehat{\boldsymbol{\Delta}}}, \\
\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^{+} &= \boldsymbol{\Sigma}^{-1}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}^{\Sigma^{-1}}
\end{aligned}$$

can be utilized and thus

$$\frac{\partial \hat{\hat{\mathbf{\Delta}}}}{\partial \vartheta_i} = -\left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{F}\right]^{-1}\mathbf{F}'\mathbf{\Sigma}^{-1}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}^{\Sigma^{-1}}\mathbf{V}_i\mathbf{\Sigma}^{-1}\mathbf{v}_{I,\mathrm{out}}.$$

The rest of the proof is elementary. □

Let in the following text the notation

$$\mathbf{A}_i = \mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{V}_i\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{F}$$

and

$$\mathbf{B}_i = \mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{V}_i\mathbf{\Sigma}^{-1}$$

be used. Then

$$\frac{\partial T}{\partial \vartheta_i} = -\hat{\hat{\mathbf{\Delta}}}'\mathbf{A}_i\hat{\hat{\mathbf{\Delta}}} - 2\hat{\hat{\mathbf{\Delta}}}'\mathbf{B}_i\mathbf{v}_{I,\mathrm{out}}.$$

It is to be remarked that $\hat{\hat{\mathbf{\Delta}}}$ and $\mathbf{v}_{I,\mathrm{out}}$ are stochastically independent.

Lemma 4.1.6. Let $\boldsymbol{\eta}' = \frac{\partial T}{\partial \boldsymbol{\vartheta}'}$. Then

$$\begin{aligned}
E(\boldsymbol{\eta}'\delta\boldsymbol{\vartheta}) &= -\mathbf{a}'\delta\boldsymbol{\vartheta}, \mathbf{a}' = (a_1,\ldots,a_p), a_i = \mathrm{Tr}(\mathbf{Z}\mathbf{V}_i), i = 1,\ldots,p,\\
\mathbf{Z} &= \left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{F}\left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{F}\right]^{-1}\\
&\quad \times\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+,\\
\mathrm{Var}(\boldsymbol{\eta}'\delta\boldsymbol{\vartheta}) &= \delta\boldsymbol{\vartheta}'(4\mathbf{C}_{(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}})^+,\mathbf{Z}} - 2\mathbf{S}_{\mathbf{Z}})\delta\boldsymbol{\vartheta},
\end{aligned}$$

where

$$\begin{aligned}
\left\{\mathbf{C}_{(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}})^+,\mathbf{Z}}\right\}_{i,j} &= \mathrm{Tr}\left[\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{V}_i\mathbf{Z}\mathbf{V}_j\right],\\
&\quad i,j = 1,\ldots,p,\\
\left\{\mathbf{S}_{\mathbf{Z}}\right\}_{i,j} &= \mathrm{Tr}(\mathbf{Z}\mathbf{V}_i\mathbf{Z}\mathbf{V}_j), \quad i,j = 1,\ldots,p.
\end{aligned}$$

P r o o f. If the null hypothesis on outliers is valid, i. e. $\mathbf{\Delta} = \mathbf{0}$, then

$$\begin{aligned}
E\left(\frac{\partial T}{\partial \vartheta_i}\right) &=\\
&= -\mathrm{Tr}[\mathbf{A}_i\mathrm{Var}(\hat{\hat{\mathbf{\Delta}}})]\\
&= -\mathrm{Tr}\Bigg\{\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{V}_i\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{F}\\
&\quad \times\left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{F}\right]^{-1}\Bigg\}\\
&= -\mathrm{Tr}\Bigg\{\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{F}\left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{F}\right]^{-1}\\
&\quad \times\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\mathbf{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{V}_i\Bigg\} = -\mathrm{Tr}(\mathbf{Z}\mathbf{V}_i).
\end{aligned}$$

Thus $E(\boldsymbol{\eta}'\delta\boldsymbol{\vartheta}) = -\mathbf{a}'\delta\boldsymbol{\vartheta}$.

Further

$$\begin{aligned}
&\operatorname{cov}\left(\frac{\partial T}{\partial \vartheta_i}, \frac{\partial T}{\partial \vartheta_j}\right)\\
&= \operatorname{cov}(-\widehat{\widehat{\boldsymbol{\Delta}}}'\mathbf{A}_i\widehat{\widehat{\boldsymbol{\Delta}}} - 2\widehat{\widehat{\boldsymbol{\Delta}}}'\mathbf{B}_i\mathbf{v}_{I,\text{out}}, -\widehat{\widehat{\boldsymbol{\Delta}}}'\mathbf{A}_j\widehat{\widehat{\boldsymbol{\Delta}}} - 2\widehat{\widehat{\boldsymbol{\Delta}}}'\mathbf{B}_j\mathbf{v}_{I,\text{out}})\\
&= 2\operatorname{Tr}[\mathbf{A}_i \operatorname{Var}(\widehat{\widehat{\boldsymbol{\Delta}}})\mathbf{A}_j \operatorname{Var}(\widehat{\widehat{\boldsymbol{\Delta}}})] + 4\operatorname{Tr}[\operatorname{Var}(\widehat{\widehat{\boldsymbol{\Delta}}})\mathbf{B}_j' \operatorname{Var}(\widehat{\widehat{\boldsymbol{\Delta}}})\mathbf{B}_i].
\end{aligned}$$

Since

$$\operatorname{Var}(\widehat{\widehat{\boldsymbol{\Delta}}}) = \left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{F}\right]^{-1}$$

and

$$\operatorname{Var}(\mathbf{v}_{I,\text{out}}) = \boldsymbol{\Sigma}\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbf{Z}\boldsymbol{\Sigma},$$

we have

$$\begin{aligned}
&\operatorname{cov}\left(\frac{\partial T}{\partial \vartheta_i}, \frac{\partial T}{\partial \vartheta_j}\right)\\
&= 2\operatorname{Tr}\Big\{\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{V}_i\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\\
&\quad \times\mathbf{F}\left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{F}\right]^{-1}\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\\
&\quad \times\mathbf{V}_j\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{F}\left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{F}\right]^{-1}\Big\}\\
&\quad +4\operatorname{Tr}\Big\{\left[\boldsymbol{\Sigma}\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbf{Z}\boldsymbol{\Sigma}\right]\boldsymbol{\Sigma}^{-1}\mathbf{V}_j\\
&\quad \times\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{F}\left[\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{F}\right]^{-1}\\
&\quad \times\mathbf{F}'\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\Big\}\\
&= 2\operatorname{Tr}(\mathbf{Z}\mathbf{V}_i\mathbf{Z}\mathbf{V}_j) + 4\operatorname{Tr}\Big\{\left[\boldsymbol{\Sigma}\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbf{Z}\boldsymbol{\Sigma}\right]\\
&\quad \times\boldsymbol{\Sigma}^{-1}\mathbf{V}_j\mathbf{Z}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\Big\}\\
&= 2\operatorname{Tr}(\mathbf{Z}\mathbf{V}_i\mathbf{Z}\mathbf{V}_j) + 4\operatorname{Tr}\left[\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{V}_j\mathbf{Z}\mathbf{V}_i\right] - 4\operatorname{Tr}(\mathbf{Z}\mathbf{V}_j\mathbf{Z}\mathbf{V}_i)\\
&= -2\operatorname{Tr}(\mathbf{Z}\mathbf{V}_i\mathbf{Z}\mathbf{V}_j) + 4\operatorname{Tr}\left[\left(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\right)^+\mathbf{V}_j\mathbf{Z}\mathbf{V}_i\right].
\end{aligned}$$

The rest of the proof is elementary. □

Now, analogously as Theorem 4.1.3, the following theorem can be stated.

Theorem 4.1.7. Let $\delta_{\max}$ be a solution of the equation $P\{\chi^2_{n+q-(k+s)} + \delta_{\max} \geq \chi^2_{n+q-(k+s)}(0; 1-\alpha)\} = \alpha + \varepsilon$ and let $t > 0$ be such real number that $P\{\boldsymbol{\eta}'\delta\boldsymbol{\vartheta} < \delta_{\max}\} \geq 1 - \frac{1}{t^2}$, i. e.

$$-\mathbf{a}'\delta\boldsymbol{\vartheta} + \sqrt{\delta\boldsymbol{\vartheta}'(4\mathbf{C}_{(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}})^+,\mathbf{Z}} - 2\mathbf{S}_{\mathbf{Z}})\delta\boldsymbol{\vartheta}} \leq \delta_{\max}$$

for sufficiently large t. Let

$$\mathbf{A} = t^2(4\mathbf{C}_{(\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}})^+,\mathbf{Z}} - 2\mathbf{S}_{\mathbf{Z}}) - \mathbf{a}\mathbf{a}'.$$

Then

$$\begin{aligned}\delta\boldsymbol{\vartheta} \in \mathcal{N}_{\text{out}} = \{\delta\boldsymbol{\vartheta} : (\delta\boldsymbol{\vartheta} - \delta_{\max}\mathbf{A}^+\mathbf{a})'\mathbf{A}(\delta\boldsymbol{\vartheta} - \delta_{\max}\mathbf{A}^+\mathbf{a}) \leq \delta^2_{\max}(1 + \mathbf{a}'\mathbf{A}^+\mathbf{a})\} \\ \Rightarrow P_{H_0}\{\hat{\hat{\boldsymbol{\Delta}}}'(\delta\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})\mathbf{F}'\Big[\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})\mathbf{M}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\Big]^+\mathbf{F}\hat{\hat{\boldsymbol{\Delta}}}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta}) \\ \geq \chi^2_{n+q-(k+s)}(0; 1-\alpha)\} \leq \alpha + \varepsilon.\end{aligned}$$

4.2. Multivariate model

The problem to find a nonsensitive region in the regular multivariate models with constraints

$$\text{vec}(\underline{\mathbf{Y}}) \sim N_{nm}\left[(\mathbf{I} \otimes \mathbf{X})\text{vec}(\underline{\boldsymbol{\beta}}), \sum_{i=1}^{p}\vartheta_i(\mathbf{V}_i \otimes \mathbf{I})\right], (\mathbf{H}' \otimes \mathbf{G})\text{vec}(\underline{\boldsymbol{\beta}}) + \text{vec}(\mathbf{G}_0) = \mathbf{0}$$

and

$$\begin{aligned}\text{vec}(\underline{\mathbf{Y}}) \sim N_{nm}\left[(\mathbf{I} \otimes \mathbf{X}, \mathbf{E})\begin{pmatrix}\text{vec}(\underline{\boldsymbol{\beta}}) \\ \boldsymbol{\Delta}\end{pmatrix}, \sum_{i=1}^{p}\vartheta_i(\mathbf{V}_i \otimes \mathbf{I})\right], \\ (\mathbf{H}' \otimes \mathbf{G})\text{vec}(\underline{\boldsymbol{\beta}}) + \text{vec}(\mathbf{G}_0) = \mathbf{0},\end{aligned}$$

respectively, is quite similar as in the preceding section. That is why only statements with short comments are given as follows.

Theorem 4.2.1. Let $\delta_{\max}$ be a solution of the equation $P\{\chi^2_{nm+qr-km} + \delta_{\max} \geq \chi^2_{nm+qr-km}(0; 1-\alpha)\} = \alpha + \varepsilon$ and let $t > 0$ be such real number that $P\{\boldsymbol{\eta}'\delta\boldsymbol{\vartheta} < \delta_{\max}\} \geq 1 - \frac{1}{t^2}$,

$$\begin{aligned}\boldsymbol{\eta}'\delta\boldsymbol{\vartheta} &= \sum_{i=1}^{p}\Big\{2\text{Tr}\Big[\underline{\mathbf{v}}_I'\mathbf{P}_{\mathbf{X}}\underline{\mathbf{v}}_I\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1} - \underline{\mathbf{v}}_I'\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\underline{\mathbf{v}}_I\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1} \\ &\quad \times\mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}\mathbf{H})^{-1}\mathbf{H}'\Big] - \text{Tr}(\underline{\mathbf{v}}_I'\underline{\mathbf{v}}_I\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1})\Big\}\delta\vartheta_i,\end{aligned}$$

i. e.

$$\begin{aligned}-\mathbf{a}'\delta\boldsymbol{\vartheta} + t\sqrt{\delta\boldsymbol{\vartheta}'2\Big[(n-k)\mathbf{S}_{\Sigma^{-1}} + q\mathbf{S}_{\mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}\mathbf{H})^{-1}\mathbf{H}'}\Big]\delta\boldsymbol{\vartheta}} \leq \delta_{\max}, \\ \mathbf{a}' = (a_1, \ldots, a_p),\end{aligned}$$

$$\begin{aligned}
\left\{\mathbf{S}_{\Sigma^{-1}}\right\}_{i,j} &= \mathrm{Tr}(\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\mathbf{V}_j), \quad i,j=1,\dots,p,\\
\left\{\mathbf{S}_{\mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}\mathbf{H})^{-1}\mathbf{H}'}\right\}_{i,j} &= \mathrm{Tr}\Big[\mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}\mathbf{H})^{-1}\mathbf{H}'\mathbf{V}_i\mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}\mathbf{H})^{-1}\mathbf{H}'\mathbf{V}_j\Big],\\
&\quad i,j=1,\dots,p,\\
a_i &= \mathrm{Tr}\Bigg(\Bigg\{\boldsymbol{\Sigma}^{-1}\otimes\mathbf{M}_{\mathbf{X}} + [\mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}\mathbf{H})^{-1}\otimes\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\Bigg\}\\
&\quad \times(\mathbf{V}_i\otimes\mathbf{I})\Bigg)\\
&= (n-k)\mathrm{Tr}(\boldsymbol{\Sigma}^{-1}\mathbf{V}_i) + q\mathrm{Tr}[\mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}\mathbf{H})^{-1}\mathbf{H}'],\\
&\quad i=1,\dots,p,
\end{aligned}$$

where t is sufficiently large. Let

$$\mathbf{A} = 2t^2\Big[(n-k)\mathbf{S}_{\Sigma^{-1}} + q\mathbf{S}_{\mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}\mathbf{H})^{-1}\mathbf{H}'}\Big] - \mathbf{a}\mathbf{a}'.$$

Then

$$\delta\boldsymbol{\vartheta}\in\mathcal{N} = \Big\{\delta\boldsymbol{\vartheta} : (\delta\boldsymbol{\vartheta}-\delta_{\max}\mathbf{A}^+\mathbf{a})'\mathbf{A}(\delta\boldsymbol{\vartheta}-\delta_{\max}\mathbf{A}^+\mathbf{a}) \le \delta^2_{\max}(1+\mathbf{a}'\mathbf{A}^+\mathbf{a})\Big\}$$
$$\Rightarrow$$
$$P\Big\{\mathrm{Tr}\Big[\underline{\mathbf{v}}_I'(\boldsymbol{\vartheta}_0+\delta\boldsymbol{\vartheta})\underline{\mathbf{v}}_I(\boldsymbol{\vartheta}_0+\delta\boldsymbol{\vartheta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0+\delta\boldsymbol{\vartheta})\Big] \ge \chi^2_{nm+qr-km}(0;1-\alpha)\Big\} \le \alpha+\varepsilon.$$

P r o o f. It is sufficient to take into account Theorem 4.1.2, the following relationships

$$\begin{aligned}
\boldsymbol{\eta}'\delta\boldsymbol{\vartheta} &= \sum_{i=1}^p[\mathrm{vec}(\underline{\mathbf{v}}_I)]'\Bigg(2(\boldsymbol{\Sigma}^{-1}\otimes\mathbf{I})(\mathbf{I}\otimes\mathbf{X})\Big\{\mathbf{M}_{\mathbf{H}\otimes\mathbf{G}'}[\boldsymbol{\Sigma}^{-1}\otimes(\mathbf{X}'\mathbf{X})]\mathbf{M}_{\mathbf{H}\otimes\mathbf{G}'}\Big\}^+\\
&\quad \times(\mathbf{I}\otimes\mathbf{X}')(\boldsymbol{\Sigma}^{-1}\otimes\mathbf{I})(\mathbf{V}_i\otimes\mathbf{I})(\boldsymbol{\Sigma}^{-1}\otimes\mathbf{I}) - (\boldsymbol{\Sigma}^{-1}\otimes\mathbf{I})(\mathbf{V}_i\otimes\mathbf{I})\\
&\quad \times(\boldsymbol{\Sigma}^{-1}\otimes\mathbf{I})\Bigg)\mathrm{vec}(\underline{\mathbf{v}}_I)\delta\vartheta_i\\
&= \sum_{i=1}^p\Bigg\{2\mathrm{Tr}\Big[\underline{\mathbf{v}}_I'\mathbf{P}_{\mathbf{X}}\underline{\mathbf{v}}_I\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1} - \underline{\mathbf{v}}_I'\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\underline{\mathbf{v}}_I\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\mathbf{H}\\
&\quad \times(\mathbf{H}'\boldsymbol{\Sigma}\mathbf{H})^{-1}\mathbf{H}'\Big] - \mathrm{Tr}(\underline{\mathbf{v}}_I'\underline{\mathbf{v}}_I\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}\Bigg\}\delta\vartheta_i,\\
a_i &= \mathrm{Tr}\Bigg\{\Bigg[\mathbf{M}_{(\mathbf{I}\otimes\mathbf{X})\mathbf{M}_{\mathbf{H}\otimes\mathbf{G}'}}(\boldsymbol{\Sigma}\otimes\mathbf{I})\mathbf{M}_{(\mathbf{I}\otimes\mathbf{X})\mathbf{M}_{\mathbf{H}\otimes\mathbf{G}'}}\Bigg]^+(\mathbf{V}_i\otimes\mathbf{I})\Bigg\}\\
&= \mathrm{Tr}\Big(\Big\{\boldsymbol{\Sigma}^{-1}\otimes\mathbf{M}_{\mathbf{X}} + [\mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}\mathbf{H})^{-1}\mathbf{H}']\otimes\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\Big\}(\mathbf{V}_i\otimes\mathbf{I})\\
&= (n-k)\mathrm{Tr}(\boldsymbol{\Sigma}^{-1}\mathbf{V}_i) + q\mathrm{Tr}[\mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}\mathbf{H})^{-1}\mathbf{H}'\mathbf{V}_i],
\end{aligned}$$

and

$$\left\{\mathbf{S}_{\{\mathbf{M}_{(\mathbf{I}\otimes\mathbf{X})\mathbf{M}_{\mathbf{H}\otimes\mathbf{G}'}}(\Sigma\otimes I)M_{\mathbf{H}\otimes\mathbf{X})\mathbf{M}_{\mathbf{H}\otimes\mathbf{G}'}}\}^+}\right\}_{i,j}$$

$$\begin{aligned}
&= \mathrm{Tr}\Bigg(\Big\{\mathbf{\Sigma}^{-1}\otimes\mathbf{M}_{\mathbf{X}} + \big[\mathbf{H}(\mathbf{H}'\mathbf{\Sigma}\mathbf{H})^{-1}\mathbf{H}'\big]\otimes\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\Big\}(\mathbf{V}_i\otimes\mathbf{I}) \\
&\quad \times\Big\{\mathbf{\Sigma}^{-1}\otimes\mathbf{M}_{\mathbf{X}} + \big[\mathbf{H}(\mathbf{H}'\mathbf{\Sigma}\mathbf{H})^{-1}\mathbf{H}'\big]\otimes\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\Big\}(\mathbf{V}_j\otimes\mathbf{I})\Bigg) \\
&= \mathrm{Tr}\Big\{(\mathbf{\Sigma}^{-1}\mathbf{V}_i\mathbf{\Sigma}^{-1}\mathbf{V}_j)\otimes\mathbf{M}_{\mathbf{X}} + \big[\mathbf{H}(\mathbf{H}'\mathbf{\Sigma}\mathbf{H})^{-1}\mathbf{H}'\mathbf{V}_i\mathbf{H}(\mathbf{H}'\mathbf{\Sigma}\mathbf{H})^{-1}\mathbf{H}'\big]\mathbf{V}_j\big] \\
&\quad \otimes\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}\Big\} \\
&= (n-k)\mathrm{Tr}(\mathbf{\Sigma}^{-1}\mathbf{V}_i\mathbf{\Sigma}^{-1}\mathbf{V}_j) + q\mathrm{Tr}\big[\mathbf{H}(\mathbf{H}'\mathbf{\Sigma}\mathbf{H})^{-1}\mathbf{H}'\mathbf{V}_i\mathbf{H}(\mathbf{H}'\mathbf{\Sigma}\mathbf{H})^{-1}\mathbf{H}'\big]\mathbf{V}_j\big] \\
&= \left\{(n-k)\mathbf{S}_{\Sigma^{-1}} + q\mathbf{S}_{\mathbf{H}(\mathbf{H}'\mathbf{\Sigma}\mathbf{H})^{-1}\mathbf{H}'}\right\}_{i,j}. \qquad \square
\end{aligned}$$

Corollary 4.2.2. If $\delta\boldsymbol{\vartheta}\in\mathcal{N}$ and $\mathrm{Tr}(\underline{\mathbf{v}}'_I\underline{\mathbf{v}}_I\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}))\geq\chi^2_{mr+qr-km}(0;1-\alpha)$, then some outliers can occur in the measurement. Analogously as in preceding sections the model

$$\mathrm{vec}(\underline{\mathbf{Y}})\sim N_{nm}\left[(\mathbf{I}\otimes\mathbf{X},\mathbf{E})\begin{pmatrix}\mathrm{vec}(\underline{\boldsymbol{\beta}})\\ \mathbf{\Delta}\end{pmatrix},\sum_{i=1}^{p}\vartheta_i(\mathbf{V}_i\otimes\mathbf{I})\right],$$
$$(\mathbf{H}'\otimes\mathbf{G})\mathrm{vec}(\underline{\mathbf{B}})+\mathrm{vec}(\mathbf{G}_0)=\mathbf{0}$$

will be considered. Let

$$\begin{aligned}
\mathbf{U} &= \Big[\mathbf{M}_{(\mathbf{I}\otimes\mathbf{X})\mathbf{M}_{\mathbf{H}\otimes\mathbf{G}'}}(\mathbf{\Sigma}\otimes\mathbf{I})\mathbf{M}_{(\mathbf{I}\otimes\mathbf{X})\mathbf{M}_{\mathbf{H}\otimes\mathbf{G}'}}\Big]^+ \\
&= \mathbf{\Sigma}^{-1}\otimes\mathbf{M}_{\mathbf{X}} + [\mathbf{H}(\mathbf{H}'\mathbf{\Sigma}\mathbf{H})^{-1}\mathbf{H}']\otimes\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'}
\end{aligned}$$

and $\mathbf{Z}=\mathbf{U}\mathbf{E}(\mathbf{E}'\mathbf{U}\mathbf{E})^{-1}\mathbf{E}'\mathbf{U}$. Then the following theorem can be proved analogously as Theorem 4.2.1.

Theorem 4.2.3. Let $\delta_{\max}$ be a solution of the equation $P\{\chi^2_{nm+qr-mk-s}(0)+\delta_{\max}\geq\chi^2_{nm+qr-mk-s}(0;1-\alpha)\}=\alpha+\varepsilon$ and let $t>0$ be such real number that $P\{\boldsymbol{\eta}'\delta\boldsymbol{\vartheta}<\delta_{\max}\}\approx 1$, where $\boldsymbol{\eta}'=\left(\frac{\partial T(\boldsymbol{\vartheta})}{\partial\vartheta_1},\ldots,\frac{\partial T(\boldsymbol{\vartheta})}{\partial\vartheta_p}\right)$, $T=\hat{\hat{\mathbf{\Delta}}}'\left[\mathrm{Var}(\hat{\hat{\mathbf{\Delta}}})\right]^{-1}\hat{\hat{\mathbf{\Delta}}}$ and

$$\frac{\partial T(\boldsymbol{\vartheta})}{\partial\vartheta_i}=-\hat{\hat{\mathbf{\Delta}}}'\mathbf{E}'\mathbf{U}(\mathbf{V}_i\otimes\mathbf{I})\mathbf{U}\mathbf{E}\hat{\hat{\mathbf{\Delta}}}-2\hat{\hat{\mathbf{\Delta}}}'\mathbf{E}'\mathbf{U}[(\mathbf{V}_i\mathbf{\Sigma}^{-1})\otimes\mathbf{I}]\mathrm{vec}(\underline{\mathbf{v}}_{I,\mathrm{out}}).$$

It means $E(\boldsymbol{\eta}'\delta\boldsymbol{\vartheta})+t\sqrt{\mathrm{Var}(\boldsymbol{\eta}'\delta\boldsymbol{\vartheta})}\leq\delta_{\max}$ for sufficiently large t,

$$\begin{aligned}
E(\boldsymbol{\eta}'\delta\boldsymbol{\vartheta}) &= -\mathbf{a}'\delta\boldsymbol{\vartheta},\quad \mathbf{a}'=(a_1,\ldots,a_p),\\
a_i &= \mathrm{Tr}[\mathbf{Z}(\mathbf{V}_i\otimes\mathbf{I})],\quad i=1,\ldots,p,\\
\mathrm{Var}(\boldsymbol{\eta}'\delta\boldsymbol{\vartheta}) &= \delta\boldsymbol{\vartheta}(6\mathbf{S}_{\mathbf{Z}}+4\mathbf{C}_{\mathbf{U},\mathbf{Z}})\delta\boldsymbol{\vartheta}.
\end{aligned}$$

Let

$$\mathbf{A} = t^2(4\mathbf{C}_{\mathbf{U},\mathbf{Z}} - 2\mathbf{S}_{\mathbf{Z}}) - \mathbf{a}\mathbf{a}'.$$

Then

$$\delta\boldsymbol{\vartheta} \in \mathcal{N}_{\text{out}} = \left\{\delta\boldsymbol{\vartheta} : (\delta\boldsymbol{\vartheta} - \delta_{\max}\mathbf{A}^{+}\mathbf{a})'\mathbf{A}(\delta\boldsymbol{\vartheta} - \delta_{\max}\mathbf{A}^{+}\mathbf{a}) \le \delta^2_{\max}(1 + \mathbf{a}'\mathbf{A}^{+}\mathbf{a})\right\}$$

$$\Rightarrow P_{H_0}\left\{\hat{\hat{\boldsymbol{\Delta}}}(\delta_0 + \delta\boldsymbol{\vartheta})(\mathbf{E}'\mathbf{U}\mathbf{E})^{-1}_{\vartheta_0+\delta\vartheta}\hat{\hat{\boldsymbol{\Delta}}}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta}) \ge \chi^2_{nm+qr-km-s}(0; 1-\alpha)\right\} \le \alpha + \varepsilon.$$

ACKNOWLEDGEMENT

This work was supported by the grant of the Council of Czech Government MSM 6 198 959 214.

(Received April 26, 2005.)

REFERENCES

[1] T. W. Anderson: Introduction to Multivariate Statistical Analysis. Wiley, New York 1958.

[2] V. Barnett and T. Lewis: Outliers in Statistical Data. Wiley, New York 1994.

[3] R. Gnanadesikan: Methods for Statistical Data Analysis of Multivariate Observations. Wiley, New York – Chichester – Weinheim – Brisbane – Singapore – Toronto 1997.

[4] K. M. S. Humak: Statistische Methoden der Modellbildung, Band I. Akademie Verlag, Berlin 1977.

[5] L. Kubáček, L. Kubáčková, and J. Volaufová: Statistical Models with Linear Structures. Veda, Bratislava 1995.

[6] L. Kubáček and L. Kubáčková: Nonsensitiveness regions in universal models. Math. Slovaca *50* (2000), 219–240.

[7] E. Lešanská: Insensitivity regions in mixed models (in Czech). Ph.D. Thesis. Faculty of Science, Palacký University, Olomouc 2001.

[8] E. Lešanská: Insensitivity regions for testing hypotheses in mixed models with constraints. Tatra Mt. Math. Publ. *22* (2001), 209–222.

[9] E. Lešanská: Optimization of the size of nonsensitivity regions. Appl. Math. *47* (2002), 9–23.

[10] C. R. Rao and S. K. Mitra: Generalized Inverse of Matrices and Its Applications. Wiley, New York 1971.

[11] C. R. Rao and J. Kleffe: Estimation of Variance Components and Applications. North–Holland, Amsterdam 1988.

[12] H. Scheffé: The Analysis of Variance. Wiley, New York 1959.

[13] K. Zvára: Regression Analysis (in Czech). Academia, Praha 1989.

Lubomír Kubáček, Department of Mathematical Analysis and Applied Mathematics, Faculty of Science, Palacký University, Tomkova 40, 779 00 Olomouc. Czech Republic.
e-mail: kubacekl@aix.upol.cz.