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WEIGHTED MEANS AND WEIGHTING FUNCTIONS[1]

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We present some properties of mixture and generalized mixture operators, with special stress on their monotonicity. We introduce new sufficient conditions for weighting functions to ensure the monotonicity of the corresponding operators. However, mixture operators, generalized mixture operators neither quasi-arithmetic means weighted by a weighting function need not be non-decreasing operators, in general.

Keywords: mixture operator, generalized mixture operator, monotonicity of the mixture operator, quasi-arithmetic mean, ordinal approach

AMS Subject Classification: 91B06

1. INTRODUCTION

In multicriteria decision making, alternatives are characterized by score vectors describing the degree of fulfilment of chosen criteria. Without going into details (for them, we recommend monographs [5, 6], for example), suppose that each alternative $\boldsymbol{a}$ is characterized by a score vector $\boldsymbol{a} = (a_1, \ldots, a_n) \in [0,1]^n$, where $n \in N - \{1\}$ is the number of applied criteria and the decision is based on an aggregation operator $A : [0,1]^n \to [0,1]$. Commonly used anonymous (i. e., stable under permutations of score) aggregation operator is the arithmetic mean M,

$$M(a_1, \ldots, a_n) = \frac{1}{n} \sum_{i=1}^{n} a_i.$$

Standard incorporation of (fixed) weights (for details see [6] or [4]) models possibly different criteria importance and leads to the class of weighted arithmetic means. In this case, weights are assigned to single criteria (i. e., coordinates of score vector), independently of the actual score vector. Alternative approaches of introducing weights to the arithmetic mean aggregation link the weights and single observed score values. If this link is based on the ordinal approach, (i. e., fixed weight is assigned to the largest score, another fixed weight is assigned to the second largest score, etc.), we obtain the *OWA* (Ordered Weighted Average) operators introduced by [16]. Observe that *OWA* operators can be viewed as "symmetrization" of weighted arithmetic means, see [3], and that both *OWA*'s and weighted arithmetic means are

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strongly related to the number n of all criteria. Two cardinal approaches generalizing the arithmetic mean are based on transformation and on weighting function. Transformed arithmetic mean $M^{(f)} : [0,1]^n \to [0,1]$ based on a transformation $f : [0,1] \to [-\infty, \infty]$ (f is continuous and strictly monotone) is given by

$$M^{(f)}(a_1, \dots, a_n) = f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(a_i)\right). \tag{1}$$

Observe that M_f is also called a quasi-arithmetic mean ([3, 6]) and that it is continuous up to the case $Ranf = [-\infty, \infty]$. The aim of this paper is a closer look to the second cardinal approach based on a weighting function $g : [0,1] \to]0, \infty]$, which is supposed to be continuous. Arithmetic mean weighted by a weighting function g (also called a mixture operator in [12, 15]), $M_g : [0,1]^n \to [0,1]$ is given by

$$M_g(a_1, \dots, a_n) = \frac{\sum_{i=1}^{n} g(a_i) \cdot a_i}{\sum_{i=1}^{n} g(a_i)}. \tag{2}$$

Observe that due to the continuity of weighting function g, each mixture operator M_g is continuous. Note that sometimes different weighting functions are applied for different criteria score, thus leading to a generalized mixture operator, see [12, 15], $M_{\boldsymbol{g}} : [0,1]^n \to [0,1]$ given by

$$M_{\boldsymbol{g}}(a_1, \dots, a_n) = \frac{\sum_{i=1}^{n} g_i(a_i) \cdot a_i}{\sum_{i=1}^{n} g_i(a_i)}, \tag{3}$$

where $\boldsymbol{g} = (g_1, \dots, g_n)$ is a vector of weighting functions. Obviously generalized mixture operators are continuous. Observe also that quasi-arithmetic means (based on the transformation f) weighted by a weighting function g,
$M_g^{(f)} : [0,1]^n \to [0,1]$, given by

$$M_g^{(f)}(a_1, \dots, a_n) = f^{-1}\left(\frac{\sum_{i=1}^{n} g(a_i) \cdot f(a_i)}{\sum_{i=1}^{n} g(a_i)}\right) \tag{4}$$

were studied by several authors, see, e. g., [1, 11]. Recall that operators
$M_{\boldsymbol{g}}^{(f)} : [0,1]^n \to [0,1]$ given by

$$M_{\boldsymbol{g}}^{(f)}(a_1, \dots, a_n) = f^{-1}\left(\frac{\sum_{i=1}^{n} g_i(a_i) \cdot f(a_i)}{\sum_{i=1}^{n} g_i(a_i)}\right) \tag{5}$$

are often called Losonczi means due to [10] (and if all g_i are equal, i. e., when (4) is applied, then $M_g^{(f)}$ is called a simple Losonczi mean). In different papers, different names for operators we will investigate are used. To avoid any confusion, throughout this paper we will use the following terminology:

- operators $M_{\boldsymbol{g}}^{(f)}$ will be called Losonczi means, in short L-means
- operators $M_g^{(f)}$ will be called simple Losonczi means, SL-means
- operators $M_{\boldsymbol{g}}$ will be called generalized mixture operators
- operators M_g will be called mixture operators.

Interesting are relations of formulas (1), (2) and (4) (which clearly generalizes both (1) and (2)). For example, put $f : [0,1] \to]0, \infty]$, $f(x) = \frac{1}{x}$. Then $M^{(f)} = M_f$. Finally observe that formulas (1), (2), (4) do not depend on the number of criteria n.

2. BASIC PROPERTIES OF WEIGHTED OPERATORS

All operators introduced in the introduction are obviously unanimous (i. e., idempotent) operators, $A(a, \ldots, a) = a$ for each $a \in [0, 1]$. Up to the weighted arithmetic mean (and possibly operators given by (3) and (5)) all of them are also anonymous (i. e., symmetric),

$$A(a_1, \ldots, a_n) = A(a_{\sigma(1)}, \ldots, a_{\sigma(n)}),$$

where $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is any permutation. However, not all of them are aggregation operators [3, 9].

Definition 2.1. A mapping $A : [0, 1]^n \to [0, 1]$ is called an aggregation operator whenever it is non-decreasing in each coordinate and fulfils

$$A(0, \ldots, 0) = 0, \qquad A(1, \ldots, 1) = 1.$$

The non-decreasigness of an aggregation operator A means (in multicriteria decision making) that better score of an alternative $\boldsymbol{a}$ when comparing with the score of an alternative $\boldsymbol{b}$ is compatible with the preference of $\boldsymbol{a}$ over $\boldsymbol{b}$ (i. e., Pareto principle is satisfied). However, mixture operators, generalized mixture operators neither (simple) Losonczi means need not be non-decreasing operators, in general. Evidently, this possible failure is connected with the use of weighting function g. In the next section, we will have a closer look to this problem.

Recall that penalization of bad attribute performances and reward of good attribute performances in case of quasi-arithmetic means can be simply interpreted as $M^{(f)} \geq M$ (i. e., $M^{(f)}(a_1, \ldots, a_n) \geq M(a_1, \ldots, a_n)$ for all $(a_1, \ldots, a_n) \in [0, 1]^n$). Following [8], we have $M^{(f)} \geq M$ if and only if f is increasing and convex or it is decreasing and concave. We will look for a similar characterization in the case of mixture operators.

Proposition 2.1. Let $M_g : [0, 1]^n \to [0, 1]$ be a mixture operator. Then $M_g \geq M$ if and only if the weighting function $g : [0, 1] \to]0, \infty]$ is non-decreasing.

Proof. Let g be non-decreasing and let X be a random variable with uniform distribution on $(a_1, \ldots, a_n)$ (even if $a_i = a_j$ for some $i \neq j$, formally we can distinguish them). Then $g(X)$ and X have non-negative correlation, i. e., non-negative covariance $E\left[(g(X) - E(g(X))\right] \cdot [X - E(X)] \geq 0$. This means that

$$\frac{1}{n} \sum_{i=1}^{n} g(a_i) a_i \geq \left(\frac{1}{n} \sum_{i=1}^{n} g(a_i) \right) \cdot \left(\frac{1}{n} \sum_{i=1}^{n} a_i \right),$$

i. e.,

$$M_g(a_i) \geq M(a_i).$$

Vice-versa, let $M_g \geq M$. For arbitrary $x, y \in [0, 1]$, $x < y$,

$$M_g(x, y, \ldots, y) \geq M(x, y, \ldots, y),$$

i. e.,

$$\frac{g(x)\cdot x+(n-1)\cdot g(y)\cdot y}{g(x)+(n-1)\cdot g(y)}\geq\frac{x+(n-1)\cdot y}{n}.$$

Therefore,

$$ng(x)x+n(n-1)g(y)y\geq g(x)x+(n-1)^2g(y)y+(n-1)\left(g(x)y+g(y)x\right),$$

and, equivalently,

$$(n-1)\left(g(y)-g(x)\right)(y-x)\geq 0,$$

i. e., $g(y)\geq g(x)$. Hence g is non-decreasing. □

Recall that to each operator $A:[0,1]^n\to[0,1]$ its dual $A^d:[0,1]^n\to[0,1]$ is given by $A^d(a_1,\dots,a_n)=1-A(1-a_1,\dots,1-a_n)$. Note that duality preserves monotonicity and boundary conditions of aggregation operators, as well as the anonymity and unanimity. Moreover, weighted means are self-dual ($W=W^d$), while the dual to an *OWA* operator is again an *OWA* operator but with reversed weights (they are taken in the opposite order).

Dual operator to a quasi-arithmetic mean is again a quasi-arithmetic mean, $\left(M^{(f)}\right)^d=M^{(1-f^*)}$, where $f^*:[0,1]\to[-\infty,\infty]$ is given by $f^*(x)=f(1-x)$. Observe that $M^{(f_1)}=M^{(f_2)}$ if and only if $f_2=\alpha f_1+\beta$ for some real constants $\alpha\neq 0$ and β. Therefore, a quasi-arithmetic mean $M^{(f)}$ is self-dual if and only if $f^*=\alpha f+\beta$ for some $\alpha\neq 0$, β. After small computing we can conclude that $M^{(f)}=\left(M^{(f)}\right)^d$ if and only if the graph of the transformation f is symmetric with sort point $\left(\frac{1}{2},f(\frac{1}{2})\right)$, i. e., $f(x)+f(1-x)=2f\left(\frac{1}{2}\right)$ for all $x\in[0,1]$.
Concerning the mixture operators, we have the next results.

Proposition 2.2. Let $M_g:[0,1]^n\to[0,1]$ be a mixture operator. Then

$$(M_g)^d=M_{g^*},$$

where $g^*:[0,1]\to]0,\infty]$ is given by $g^*(x)=g(1-x)$, and $(M_g)^d=M_g$ if and only if $g(x)=g(1-x)$ for all $x\in[0,1]$, i. e., $g=g^\star$.

Proof. For any $(a_1,\dots,a_n)\in[0,1]^n$, we have

$$\begin{aligned}&(M_g)^d(a_1,\dots,a_n)=1-M_g(1-a_1,\dots,1-a_n)\\ =\;&1-\frac{\sum_{i=1}^n g(1-a_i)\cdot(1-a_i)}{\sum_{i=1}^n g(1-a_i)}=\frac{\sum_{i=1}^n g^\star(a_i)\cdot a_i}{\sum_{i=1}^n g^\star(a_i)}=M_{g^*}(a_1,\dots,a_n).\end{aligned}$$

Observe that $M_{g_1}=M_{g_2}$ if and only if $g_1=\alpha g_2$ for some real constant $\alpha>0$. However, then $(M_{\boldsymbol{g}})^d=M_{\boldsymbol{g}}$ if and only if the graph of the weighting function g is symmetric with respect to the axis $x=\frac{1}{2}$, i. e., $g(x)=g(1-x)$ for all $x\in[0,1]$.
Similarly, for a generalized mixture operator $M_{\boldsymbol{g}}$ we have $(M_{\boldsymbol{g}})^d=M_{\boldsymbol{g}^\star}$, where $\boldsymbol{g}^\star=(g_1^\star,\dots,g_n^\star)$. In the case of Losonczi means, we have $(M_g^{(f)})^d=M_{g^\star}^{(1-f^\star)}$ and $(M_{\boldsymbol{g}}^{(f)})^d=M_{\boldsymbol{g}^\star}^{(1-f^\star)}$. □

3. MONOTONICITY OF MIXTURE OPERATORS

As already observed, each mixture operator M_g is unanimous (idempotent), and thus it is monotone if and only if it is non-decreasing (i. e., if it is an aggregation operator). Our main interest is in mixture operators rewarding the good attribute performances, i. e., in mixture operators stronger than arithmetic mean, $M_g \geq M$. Therefore, we will deal with non-decreasing weighting functions g mostly. However, if M_g is an aggregation operator then also $M_g^d = M_{g^\star}$ (see Proposition 2.2) is an aggregation operator. Thus description of non-decreasing weighting functions g yielding monotone mixture operator M_g straightforwardly gives also description of non-increasing g yielding monotone M_g (observe that g is non-decreasing if and only if g^* is non-increasing, and that $(g^\star)^\star = g$). Due to the anonymity (symmetry) of M_g it is enough to investigate its monotonicity in the first coordinate only. For application reasons, we will restrict our considerations to piecewise smooth weighting functions g (i. e., all derivatives of g exist possibly up to countably many points). Due to the continuity of M_g (and of g), the monotonicity of M_g in the first coordinate is equivalent to the non-negativity of the first partial derivative $\frac{\partial M_g}{\partial a_1}$ in all points $a_1 \in [0,1]$ where this derivative exist. Therefore, with no loss of generality we can assume that g is smooth (and thus $\frac{\partial M_g}{\partial a_1}$ exist everywhere on $[0,1]$). Several non-decreasing weighting functions g yielding a monotone mixture operator were introduced in [15], namely linear and special quadratic functions. Note, however, that description of such fitting quadratic functions in [15] is not exhaustive.

Definition 3.1. Let $g : [0,1] \to]0,\infty]$ be a non-decreasing continuous function such that the mixture operator M_g is an aggregation operator. Then g is called a fitting weighting function.

Observe that fitting weighting functions (up to special classes) were not yet characterized. The first sufficient condition for a weighting function g to be fitting was stated in [13].

Proposition 3.1. A smooth weighting function $g : [0,1] \to]0,\infty]$ is a fitting weighting function (independently of n) whenever the next condition is satisfied:

$$0 \leq g'(x) \leq g(x) \qquad \text{for all} \qquad x \in [0,1]. \tag{C1}$$

Remark 3.1. Note that the requirement of (piecewise) smoothness of the weighting function g in the above proposition is substantial. Though for any non-decreasing weighting function g its derivative exists almost everywhere, (C1) fulfilled in points of existence of g' need not guarantee the non-decreasingness of the mixture operator M_g. Indeed, consider the Cantor function $g : [0,1] \to [0,1]$. Then $g(\frac{1}{3}) = \frac{1}{2}$, $g(\frac{1}{9}) = \frac{1}{4}$, $g(\frac{2}{27}) = \frac{1}{8}$, and thus

$$M_g\left(\frac{1}{3}, \frac{2}{27}\right) = \frac{(\frac{1}{8} \cdot \frac{2}{27} + \frac{1}{2} \cdot \frac{1}{3})}{(\frac{1}{8} + \frac{1}{2})} = \frac{38}{135},$$

$$M_g\left(\frac{1}{3},\frac{1}{9}\right) = \frac{\left(\frac{1}{4}\cdot\frac{1}{9}+\frac{1}{2}\cdot\frac{1}{3}\right)}{\left(\frac{1}{4}+\frac{1}{2}\right)} = \frac{7}{27},$$

i. e.,

$$M_g\left(\frac{1}{3},\frac{2}{27}\right) > M_g\left(\frac{1}{3},\frac{1}{9}\right)$$

violates the non-decreasingness of M_g. Observe that the set of points, where the Cantor function has not derivative is the Cantor set (i. e., it is uncountable) and that $g'(x) = 0$ whenever $g'(x)$ exist.

This result can be generalized. Indeed, from (2) we see that

$$\frac{\partial M_g}{\partial a_1} = \frac{(g(a_1)+g'(a_1)\cdot a_1)\cdot\left(\sum_{i=1}^n (g(a_i))\right) - \left(\sum_{i=1}^n g(a_i)\cdot a_i\right)\cdot g'(a_1)}{\left(\sum_{i=1}^n g(a_i)\right)^2} \geq 0$$

if and only if

$$g^2(a_1) + \alpha\left(g(a_1) + g'(a_1)(a_1-\beta)\right) \geq 0, \tag{6}$$

where $\alpha = \sum_{i=2}^n g(a_i)$ and $\alpha\cdot\beta = \sum_{i=2}^n g(a_i)\cdot a_i$ (and thus necessarily $\beta \in [0,1]$ and $\alpha \in [(n-1)\cdot g(0), (n-1)\cdot g(1)]$). Now it is easy to see that (C1) implies (6). However, (6) is satisfied also whenever

$$g(a_1) + g'(a_1)\cdot(a_1-\beta) \geq 0 \tag{7}$$

for each $a_1 \in [0,1]$ and each $\beta \in [0,1]$. Because of $g'(a_1) \geq 0$, (7) is fulfilled whenever

$$0 \leq g'(x)(1-x) \leq g(x) \qquad \text{for all} \qquad x \in [0,1]. \tag{C2}$$

We have just shown a sufficient condition more general than (C1).

Proposition 3.2. If a non-decreasing smooth weighting function g satisfies (C2) then

$$M_g : [0,1]^n \to [0,1]$$

is an aggregation operator (independently of n).

Though (6) is a general characterization of fitting weighting functions g, in general constants α and β are not independent (and they obviously are linked to g). Therefore to find an "if and only if" general condition to characterize fitting weighting functions is a problem with rather high computational complexity even in special classes of functions. However, we are able still to improve sufficient condition (C2), but constraint by n.

Proposition 3.3. Let g be a non-decreasing smooth weighting function such that for a fixed $n \in N$,$n > 1$, it satisfies the next condition:

$$\frac{g^2(x)}{(n-1)g(1)} + g(x) \geq g'(x)(1-x) \qquad \text{for all} \qquad x \in [0,1] \tag{C3}$$

Then g is a fitting weighting function, i. e., $M_g : [0,1]^n \to [0,1]$ is an aggregation operator.

Proof. Minimal value of $g(a_1) + g'(a_1) \cdot (a_1 - \beta)$ for $\beta \in [0,1]$ is attained for $\beta = 1$, i.e., it is $g(a_1) + g'(a_1) \cdot (a_1 - 1)$. Therefore, (6) is surely satisfied whenever

$$\frac{g^2(a_1)}{\alpha} + g(a_1) \geq g'(a_1) \cdot (1 - a_1).$$

Suppose that (C3) holds. Then

$$\frac{g^2(a_1)}{\alpha} + g(a_1) \geq \frac{g^2(a_1)}{(n-1) \cdot g(1)} + g(a_1) \geq g'(a_1) \cdot (1 - a_1),$$

i.e., (6) is satisfied and thus g is a fitting weighting function.

Remark 3.2. Observe that if we require that (C3) holds for all $n \in N$, $n > 1$, then this condition reduces to (C2).

Example 3.1.

1. For linear non-decreasing weighting functions each sufficient condition ((C1)–(C3)) characterizes all increasing linear fitting weighting functions, namely $g(x) = \alpha \cdot x + \beta$, $\alpha \geq 0$, $\beta \geq \alpha$. Note that if $\alpha = 0$, necessarily $\beta > 0$ and then $M_\beta = M$ is the common arithmetic mean. The strongest operator M_g from this class is determined by the weighting function $s(x) = \alpha(x+1)$ (recall that positive multiplicative constant has no influence on the resulting mixture operator

   $$M_s(a_1, \ldots, a_n) = M(a_1, \ldots, a_n) + \frac{\sigma^2(a_1, \ldots, a_n)}{M(a_1, \ldots, a_n) + 1},$$

   where $\sigma^2(a_1, \ldots, a_n)$ is the dispersion of a random variable X uniformly distributed over $(a_1, \ldots, a_n)$ (obviously, then $M(a_1, \ldots, a_n)$ is the corresponding mean value). By duality, for decreasing linear fitting weighting functions given by formula $g(x) = \alpha x + \beta$ we have $\alpha < 0$ and $\beta + 2\alpha \geq 0$. The weakest aggregation operator M_g with linear weighting function g is the dual operator to M_s, $M_w = (M_s)^d$, and it is given by

   $$M_w(a_1, \ldots, a_n) = M(a_1, \ldots, a_n) - \frac{\sigma^2(a_1, \ldots, a_n)}{2 - M(a_1, \ldots, a_n)}.$$

2. Let $g_\gamma : [0,1]^2 \to]0, \infty]$ be given by $g_\gamma(x) = 1 + \gamma x^2$ for $\gamma \in [0, \infty[$. From (C1) we get inequality

   $$\gamma x^2 - 2\gamma x + 1 \geq 0.$$

   This inequality is fulfilled for all $x \in [0,1]$ if and only if $\gamma \in [0,1]$ (this was observed also in [12, 15] with several illustrative examples from multicriteria decision making).

   However, applying (C2), we get the inequality given by

   $$3\gamma x^2 - 2\gamma x + 1 \geq 0.$$

The function on the left side inequality attains minimum for $x = \frac{1}{3}$. We can conclude that g_γ is a fitting weighting function whenever $\gamma \in [0, 3]$. The condition (C3) for $n = 2$ leads to the inequality

$$\gamma^2 x^4 + (3\gamma^2 + 5\gamma)x^2 - (2\gamma^2 + 2\gamma)x + \gamma + 2 \geq 0$$

(observe, that in this case (C3) yields an "if and only if" condition for γ so that g_γ is a fitting weighting function). Thus for $n = 2$ we see that g_γ is a fitting weighting function whenever $\gamma \in [0, 4.081553896]$ (the result was obtained by means of MAPLE system). Similarly, applying (C3) for $n = 3$ we get inequality

$$\gamma^2 x^4 + (6\gamma^2 + 8\gamma)x^2 - (4\gamma^2 + 4\gamma)x + 2\gamma + 3 \geq 0.$$

By system MAPLE we have obtained $\gamma \in [0, 3.581118151]$.

As already observed in Section 2, M_g is an aggregation operator if and only if M_{g^*} is monotone (i.e., an aggregation operator). This fact allows us to introduce sufficient conditions for non-increasing smooth weighting functions g yielding monotone mixture operators M_g:

$$g(x) + g'(x) \geq 0 \qquad \text{for all} \qquad x \in [0,1] \tag{C1'}$$

$$g(x) + g'(x)x \geq 0 \qquad \text{for all} \qquad x \in [0,1] \tag{C2'}$$

$$\frac{g(x)^2}{(n-1)g(0)} + g(x) + g'(x)x \geq 0 \qquad \text{for all} \qquad x \in [0,1]. \tag{C3'}$$

4. MONOTONICITY OF GENERALIZED MIXTURE OPERATORS

Generalized mixture operators are not stable under permutations of score vectors (i.e., they are not anonymous), in general, and thus the monotonicity of $M_{\boldsymbol{g}}$ should be checked in each coordinate separately. However, for each $i \in \{1, \dots, n\}$, the monotonicity of $M_{\boldsymbol{g}}$ in ith coordinate is equivalent with the fulfilment of a version of inequality (6), namely

$$g_i^2(a_i) + \alpha_i \left(g_i(a_i) + g_i'(a_i)(a_i - \beta_i)\right) \geq 0 \tag{8}$$

for all $(a_1, \dots, a_n) \in [0,1]^n$, where $\alpha_i = \sum_{i \neq j} g_j(a_j)$ and $\alpha_i \cdot \beta_i = \sum_{j \neq i} g_j(a_j) \cdot a_j$. Observe that this means that $\beta \in [0, 1]$ and thus we can apply sufficient conditions (C1), (C2) or (C1'), (C2') whenever g_i is monotone.

Proposition 4.1. Let $\boldsymbol{g} = (g_1, \dots, g_n)$ be a vector of monotone smooth weighting functions. Then the generalized mixture operator $M_{\boldsymbol{g}} : [0,1]^n \to [0,1]$ is monotone (i.e., it is an aggregation operator) whenever all non-decreasing g_i fulfil (C1) or (C2) and all non-increasing g_i fulfil (C1'), or (C2').

Example 4.1. For $n = 2$, let $\boldsymbol{g} = (g_1, g_2)$, where $g_1(x) = x + 1$ and $g_2(x) = 2 - x$. It is easy to check that g_1 fulfils (C1) and g_2 fulfils (C1') and thus $M_{\boldsymbol{g}} : [0,1]^2 \to [0,1]$ given by

$$M_{\boldsymbol{g}}(a_1, a_2) = \frac{a_1 + a_1^2 + 2a_2 - a_2^2}{3 + a_1 - a_2}$$

is an aggregation operator. Observe that this operator coincide with the (binary) arithmetic mean M on both diagonals, i. e.,

$$\begin{aligned} M_{\boldsymbol{g}}(a,a) &= a = M(a,a), \\ M_{\boldsymbol{g}}(a,1-a) &= \frac{1}{2} = M(a,1-a). \end{aligned}$$

Observe also that $g_2 = g_1^*$, and that the last property is true for arbitrary $g_2 = g_1^*$, i. e.,

$$M_{(g,g^*)}(a,1-a) = \frac{1}{2}$$

for each $a \in [0,1]$ and any weighting function g.

5. CONCLUDING REMARKS

We have discussed some properties of mixture and generalized mixture operators with special stress on their monotonicity. New sufficient conditions for weighting functions to ensure the monotonicity of the corresponding operators have been introduced. Note that the monotonicity of quasi-arithmetic means weighted by a weighting function g will be subject of our next study. For readers interested in the applications of mixture and generalized mixture operators in multicriteria decision making we recommend recent paper [15], where several illustrative examples can be found. Obtained results can be further generalized for the Losonczi means. For example, if $f : [0,1] \to [0,1]$ is an increasing differentiable bijection, then for smooth and non-decreasing g the simple Losonczi mean $M_g^{(f)}$ is an aggregation operator whenever $g \cdot f' \geq g'$. Similarly, for Losonczi mean $M_{\boldsymbol{g}}^{(f)}$ with $\boldsymbol{g} = (g_1, \ldots, g_n)$, where each weighting function g_i, $i = 1, 2, \ldots, n$, is smooth and non-decreasing, the validity of g_i, $f' \geq g'$, $i = 1, \ldots, n$ is sufficient to guarantee the monotonicity of $M_{\boldsymbol{g}}^{(f)}$. Note that each strictly monotone quasi-arithmetic mean can be expressed as $M^{(f)}$ with $f : [0,1] \to [0,1]$ an increasing bijection. This is not the case of non-strictly monotone quasi-arithmetic means (which possess necessarily an annihilator), such as the geometric mean (with $f(x) = \log x$) or harmonic mean (with $f(x) = \frac{1}{x}$). In such case, also the investigation of the monotonicity of the related (simple) Losonczi means is more complicated. Admitting 0 in the range of g, let $f(x) = \frac{1}{x}$ (i. e., $M^{(f)}$ is the harmonic mean) and let $g(x) = \sqrt{x}$. Then the simple Losonczi mean $M_g^{(f)}$ is an aggregation operator which coincide with the geometric mean in the binary case, but differs from the geometric mean whenever $n > 2$. A deeper study of monotonicity of Losonczi means and of relationships between Losonczi means $M_{\boldsymbol{g}_1}^{(f_1)}$ and $M_{\boldsymbol{g}_2}^{(f_2)}$ will be subject of our further investigations.

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