Kybernetika

Sylvia Pulmannová

A spectral theorem for σ MV-algebras

Kybernetika, Vol. 41 (2005), No. 3, [361]--374

Persistent URL: `http://dml.cz/dmlcz/135661`

KYBERNETIKA — VOLUME 41 (2005), NUMBER 3, PAGES 361 – 374

A SPECTRAL THEOREM FOR SIGMA MV–ALGEBRAS

SYLVIA PULMANNOVÁ

MV-algebras were introduced by Chang, 1958 as algebraic bases for multi-valued logic. MV stands for "multi-valued" and MV algebras have already occupied an important place in the realm of nonstandard (mathematical) logic applied in several fields including cybernetics. In the present paper, using the Loomis–Sikorski theorem for σ-MV-algebras, we prove that, with every element a in a σ-MV algebra M, a spectral measure (i. e. an observable) $\Lambda_a : \mathcal{B}([0,1]) \longrightarrow \mathcal{B}(M)$ can be associated, where $\mathcal{B}(M)$ denotes the Boolean σ-algebra of idempotent elements in M. This result is similar to the spectral theorem for self-adjoint operators on a Hilbert space. We also prove that MV-algebra operations are reflected by the functional calculus of observables.

Keywords: MV-algebras, Loomis–Sikorski theorem, tribe, spectral decomposition, lattice effect algebras, compatibility, block

AMS Subject Classification: 81P10, 03G12

1. INTRODUCTION

MV-algebras were introduced in [6] as the Lindenbaum–Tarski algebras for multi-valued Łukaszievicz calculus. Since then MV-algebras have found increasing interest and become an important tool applied in several fields. For a systematic treatment of MV-algebras see e. g. [9]. Relations of MV-algebras and the quantum logic approach to quantum mechanics can be found in [13]. Measure theoretical and probabilistic aspects of MV-algebras have been developed in [4] and [21].

Effect algebras [14], equivalently D-posets [16] were introduced as algebraic generalizations of the set of Hilbert space effects, i. e. self-adjoint operators between the zero and the identity operators on a Hilbert space. Hilbert space effects play an important role in the theory of unsharp quantum measurements, which take into account the indeterministic nature of quantum mechanics (see [2] for the theory of quantum measurement and [11] for logical aspects in quantum theory). It was shown that MV-algebras can be described as a special subclass of effect algebras [16]. It was also shown in [5] that Hilbert space effect algebras can be covered by MV-algebras, consisting of maximal sets of pairwise commuting effects.

The aim of the present paper is to show that to every element a in a σ-MV-algebra M there is a σ-homomorphism Λ_a from Borel subsets of the unit interval $[0,1]$ of reals to the Boolean σ-algebra $\mathcal{B}(M)$ of the idempotent elements of M. Borrowing a language from physics, Λ_a is called an observable associated with $\mathcal{B}(M)$. In analogy with the spectral theory of self-adjoint operators, we call Λ_a the spectral measure of the element a. We introduce a notion of a regular representation of a σ-MV-algebra M, and show that every regular representation gives rise to an injective mapping $a \mapsto \beta_a$, where β_a is a spectral measure of a.

In order to show that a regular representation always exists, we use the recently proven generalization of the classical Loomis–Sikorski theorem to σ-MV-algebras, [4, 12, 18]. In general, there can be several regular representations, and hence the spectral measures need not be uniquely defined. Uniqueness of spectral measures for elements of σ-MV-algebras is shown in a subsequent paper [20].

The spectral measures associated with elements of σ-MV-algebras enable us to introduce a functional calculus in the sense of Varadarajan [23]. We show that MV-algebra operations are reflected by the functional calculus. The Butnariu and Klement theorem [3] enables us to show that every sigma additive state on $\mathcal{B}(M)$ can be uniquely extended to a σ-additive state on the whole M.

2. DEFINITIONS AND KNOWN RESULTS

An *MV-algebra* is an algebraic structure $(M; \oplus, {}^*, 0, 1)$ consisting of a nonempty set M, a binary operation $\oplus$, a unary operation * and two constants 0 and 1 satisfying the following axioms:

(M1) $a \oplus b = b \oplus a$;

(M2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$;

(M3) $a \oplus a^* = 1$;

(M4) $a \oplus 0 = a$;

(M5) $a^{**} = a$;

(M6) $0^* = 1$;

(M7) $a \oplus 1 = 1$;

(M8) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.

Boolean algebras coincide with MV-algebras satisfying the additional condition $x \oplus x = x$. A routine computation [9] shows that the axiomatization is equivalent to the original one due to Chang [6]. A prototypical MV-algebra is given by the real unit interval $[0,1] = \{x \in R : 0 \leq x \leq 1\}$ equipped with the operations $x^* = 1 - x$, $x \oplus y = \min\{1, x + y\}$. Chang's completeness theorem [7] states that if an equation holds in $[0,1]$ then the equation holds in every MV-algebra. An MV-algebra is ordered by the relation $x \leq y$ iff $x^* \oplus y = 1$. This ordering makes M a distributive lattice with smallest element 0 and largest element 1. Suprema and infima in M are given by

$$x \vee y = (x^* \oplus y)^* \oplus y, \; x \wedge y = (x^* \vee y^*)^*.$$

An additional binary relation $\odot$ is defined by $x \odot y = (x^* \oplus y^*)^*$.

Let (G, u) be an Abelian ℓ-group (additively written) with strong unit u. Let

$$\Gamma(G, u) := \{x \in G : 0 \le x \le u\} = [0, u]$$

be the unit interval of G equipped with the operations $x^* = u - x$, $x \oplus y = u \wedge (x + y)$, $x \odot y = 0 \vee (x + y - 1)$. For any morphism $\lambda : (G, u) \to (G', u')$, let $\Gamma(\lambda)$ be the restriction of λ to $[0, u]$. Then Γ is a categorical equivalence between Abelian ℓ-groups with strong unit and MV-algebras [17].

An MV-algebra is a *σ-MV-algebra* if it is a σ-lattice. On a σ-MV-algebra the following equations hold [13, Prop. 7.1.4]:

1. $b \wedge (\bigvee a_i) = \bigvee (b \wedge a_i)$;
2. $(\bigvee_i a_i) \odot b = \bigvee (a_i \odot b)$;
3. $b \odot (\bigvee a_i)^* = \bigwedge (b \odot a_i^*)$.

A mapping $h : M \to M'$ between two MV-algebras is a *homomorphism* of MV-algebras iff it preserves the operations $\oplus$, *, 0 and 1. An MV-algebra homomorphism of two σ-MV-algebras is a *σ-homomorphism* if it preserves countable joins (and meets).

A *state* (finitely additive) on M is a mapping $m : M \to [0, 1]$ such that for any $a, b \in M$ such that $a \le b^*$ we have $m(a \oplus b) = m(a) + m(b)$. A state which is a homomorphism is called a *state morphism.* State morphisms can be identified with extremal points in the convex set of states of M [13, Th. 6.1.30]. A state m is *σ-additive* if for every sequence $(a_n)_n$, $a_n \nearrow a$ implies $m(a_n) \to m(a)$.

By definition, ideals of MV-algebras are kernels of homomorphisms. An ideal J of M is *prime* iff the quotient M/J is totally ordered. An ideal J is *maximal* if it is not properly contained in any ideal of M. Every maximal ideal is prime, the converse need not hold. Let $\mathcal{P}(M)$ denote the set of all prime ideals of M and $\mathcal{M}(M)$ the set of all maximal ideals of M. Chang's sub-direct representation theorem [7] states that every MV-algebra is embeddable into the direct product $\Pi\{M/I : I \in \mathcal{P}(M)\}$. We say that the MV-algebra M is *semisimple* if

$$\mathcal{R}(M) := \bigcap \mathcal{M}(M) = \{0\}.$$

The set $\mathcal{R}(M)$ is called the *radical* of M. According to [1], every semisimple MV-algebra is isomorphic to some Bold algebra of fuzzy sets on some set $\Omega \neq \emptyset$, where a family $\mathcal{F} \subset [0, 1]^\Omega$ is a *Bold algebra* of fuzzy sets iff

(Bd1) $0_\Omega \in \mathcal{F}$;

(Bd2) $f \in \mathcal{F} \Rightarrow 1_\Omega - f \in \mathcal{F}$;

(Bd3) $f \oplus g(\omega) = \min\{f(\omega) + g(\omega), 1\}$.

An MV-algebra is *Archimedean* (in Belluce sense) if for every $a, b \in M$, $na \le b$ for all $n \in N$ implies $a \odot b = a$, where $na := a \oplus a \oplus \cdots \oplus a$ (n-times). It was proved in [1] that M is Archimedean iff it is semisimple. Moreover, every σ-MV-algebra is Archimedean.

A relation between MV-algebras and self-adjoint operators on a Hilbert space can be seen as follows. If we restrict the total operation $\oplus$ in M to pairs $\{(a, b) : a \le b^*\}$

we obtain a partially defined operation $\dot{+}$ on M, and $(M; \dot{+}, 0, 1)$ admits a structure of an effect algebra (equivalently, D-poset) [16]. An *effect algebra* introduced in [14] is an algebraic structure $(E; \dot{+}, 0, 1)$, where $\dot{+}$ is a partially defined binary operation and 0 and 1 are constants, such that the following axioms hold:

(E1) $a\dot{+}b = b\dot{+}a$;

(E2) $a\dot{+}(b\dot{+}c) = (a\dot{+}b)\dot{+}c$;

(E3) for every $a \in E$ there is a unique $a' \in E$ such that $a\dot{+}a' = 1$;

(E4) $a\dot{+}1$ is defined iff $a = 0$.

The equalities in (E1) and (E2) mean that if one side is defined so is the other and the equality holds. An effect algebra is partially ordered by the relation $a \leq b$ iff there is c such that $a\dot{+}c = b$. The element c is then uniquely defined. This enables us to introduce another partial binary operation $\dot{-}$ by $b\dot{-}a = c$ iff $a\dot{+}c = b$, so that $b\dot{-}a$ is defined iff $a \leq b$. In particular $a' = 1\dot{-}a$. In the ordering $\leq$, 1 is the largest and 0 is the smallest element in E. We also have that $a\dot{+}b$ exists iff $a \leq b'$. We say that a and b are *orthogonal* if $a \leq b'$, i.e., iff $a\dot{+}b$ is defined. In [8] it is proved that an effect algebra E can be organized into an MV-algebra (with the same ordering) iff E is a lattice and for every $a, b \in E$ there holds

$$(a \vee b)\dot{-}a = b\dot{-}(a \wedge b).$$

An important example of effect algebras is obtained in the following way. Let (G, u) be an Abelian group with strong unit u. The unit interval $\{g \in G : 0 \leq g \leq u\} = [0, u]$ endowed with the operation $\dot{+}$ such that $a\dot{+}b$ is defined iff $a + b \leq u$, and then $a\dot{+}b = a + b$, and $a' = u - a$ becomes an effect algebra. Effect algebras arising this way are called *interval effect algebras.* In particular, if we take (G, u) as the group of all self-adjoint operators on a Hilbert space H and u as the identity operator we obtain the effect algebra of Hilbert space effects. In this context, we may consider MV-algebras as interval effect algebras of lattice ordered groups.

3. LOOMIS–SIKORSKI THEOREM

In this paragraph, we briefly recall some basic facts that are used in the proof of the Loomis–Sikorski theorem for σ-MV-algebras [12, 18], see also [4] for a different proof.

The following notion is a direct generalization of a σ-algebra of sets. A *tribe of fuzzy sets* on a set $\Omega \neq \emptyset$ is a nonempty system $\mathcal{T} \subseteq [0, 1]^\Omega$ such that

(T1) $1_\Omega \in \mathcal{T}$;

(T2) if $a \in \mathcal{T}$ then $1_\Omega - a \in \mathcal{T}$;

(T3) $(a_n)_{n=1}^\infty \subseteq \mathcal{T}$ entails

$$\min\left(\sum_{n=1}^\infty a_n, 1\right) \in \mathcal{T}.$$

Elements of $\mathcal{T}$ are called *fuzzy subsets* of Ω. Elements of $\mathcal{T}$ which are characteristic functions are called *crisp subsets* of Ω.

The basic properties of tribes are [13, Prop. 7.16]:

Proposition 3.1. Let $\mathcal{T}$ be a tribe of fuzzy subsets of Ω. Then

(i) $a \vee b = \max\{a, b\} \in \mathcal{T}$, $a \wedge b = \min\{a, b\} \in \mathcal{T}$;

(ii) $b - a \in \mathcal{T}$ if $a \leq b$, i.e. $a(\omega) \leq b(\omega)$ for all $\omega \in \Omega$;

(iii) if $a_n \in \mathcal{T}$, $n \geq 1$, and $a_n \nearrow a$ (point-wise) then $a = \lim_n a_n \in \mathcal{T}$;

(iv) $\mathcal{T}$ is a Bold algebra, in addition a σ-MV-algebra closed under point suprema of sequences of its elements.

Denote by

$$\mathcal{B}(\mathcal{T}) = \{A \subset \Omega : \chi_A \in \mathcal{T}\},$$

i.e., $\mathcal{B}(\mathcal{T})$ is the system of all crisp subsets in $\mathcal{T}$. According to [13, Th. 7.1.7], $\mathcal{B}(\mathcal{T})$ is a σ-algebra of crisp subsets of Ω, and if $f \in \mathcal{T}$, then f is $\mathcal{B}(\mathcal{T})$-measurable. That is, for every $f \in \mathcal{T}$ and every $E \in \mathcal{B}([0,1])$ (where $\mathcal{B}([0,1])$ denotes the Borel subsets of $[0,1]$), the pre-image $f^{-1}(E)$ belongs to $\mathcal{B}(\mathcal{T})$. Moreover, the mapping

$$f^{-1} : \mathcal{B}([0,1]) \to \mathcal{B}(\mathcal{T})$$

is a σ-homomorphism of Boolean σ-algebras.

Lemma 3.2. Let $\mathcal{T}$ be a tribe of fuzzy subsets of a set $\Omega \neq \emptyset$. For every $f, g \in \mathcal{T}$, $f = g$ if and only if $f^{-1}(X) = g^{-1}(X)$ for all $X \in \mathcal{B}([0,1])$.

Proof. If $f = g$, then $f^{-1}(X) = g^{-1}(X)$ for all X is clear. If $f \neq g$, there is $\omega \in \Omega$ such that $f(\omega) \neq g(\omega)$. Assume $f(\omega) < g(\omega)$, then $g(\omega) > f(\omega) + \frac{1}{n}$ for some integer n. Putting $f(\omega) = \alpha$, we have $\omega \in f^{-1}[0, \alpha]$, while $\omega \notin g^{-1}[0, \alpha]$. □

By the Butnariu–Klement theorem [3], [22, Th. 8.1.12], [4] for every σ-additive state m on $\mathcal{T}$ we have

$$m(f) = \int_\Omega f(\omega)\, \mathrm{d}\mu(\omega), \tag{1}$$

where $\mu(A) = m(\chi_A)$, $A \in \mathcal{B}(\mathcal{T})$ is a probability measure.

Let M be an MV-algebra. On the set $\mathcal{M}(M)$ of all maximal ideals of M a topology τ_M is introduced as the collection of all subsets of the form

$$O(I) := \{A \in \mathcal{M}(M) : A \not\supseteq I\}, I \text{ is an ideal of } M.$$

It was shown that τ_M makes $\mathcal{M}(M)$ a compact Hausdorff topological space. For any $a \in M$, we put

$$M(a) := \{A \in \mathcal{M}(M) : a \notin A\}.$$

Then $\{M(a) : a \in M\}$ is a base of τ_M, and for $a, b \in M$, (i) $M(0) = \emptyset$, (ii) $M(a) \subseteq M(b)$ whenever $a \leq b$, (iii) $M(a \wedge b) = M(a) \cap M(b)$, $M(a \vee b) = M(a) \cup M(b)$, (iv) $M(a)^c \subseteq M(a^*)$ ($M(a)^c$ is the set-theoretical complement of $M(a)$).

Recall that an element $a \in M$ is *idempotent* iff $a \oplus a = a$, equivalently, iff $a \wedge a^* = 0$. Denote by $\mathcal{B}(M)$ the set of all idempotent elements of M. We have the following facts ([13, Prop. 7.1.12]): if a is idempotent then $M(a)^c = M(a^*)$, moreover, if M is semisimple, then $M(a)^c = M(a^*)$ iff a is idempotent.

By [6], for every MV-algebra, the set of idempotent elements $\mathcal{B}(M)$ is a Boolean algebra. If M is a σ-MV-algebra, then $\mathcal{B}(M)$ is a Boolean σ-algebra [13, Th. 7.1.12].

Let M be a σ-MV-algebra. With the topology τ_M, the space $\Omega := \mathcal{M}(M)$ is basically disconnected, that is, the closure of every F_σ-subset of Ω is open.

Denote by $\mathrm{Ext}(\mathcal{S}(M))$ the set of all extremal states on M. There is a one-to-one correspondence between $\mathrm{Ext}(\mathcal{S}(M))$ and $\mathcal{M}(M)$ given by the homeomorphism $m \mapsto \mathrm{Ker}_m$ [13, Th. 7.1.2].

For $a \in M$, define $a \mapsto \bar{a}$, where $\bar{a} \in [0,1]^{\mathcal{M}(M)}$ by

$$\bar{a} := a/A, A \in \mathcal{M}(M),$$

and $a \mapsto \hat{a}$, where $\hat{a} \in [0,1]^{\mathrm{Ext}(\mathcal{S}(M))}$ by

$$\hat{a}(m) := m(a),\ m \in \mathrm{Ext}(\mathcal{S}(M)).$$

In view of the correspondence $m \mapsto \mathrm{Ker}_m$, we have $\hat{a}(m) = \bar{a}(\mathrm{Ker}_m)$.

Notice that by [13, Prop. 7.1.20], $a \in M$ is idempotent if and only if $\hat{a}$ is a characteristic function.

Let f be a real function on $\Omega \neq \emptyset$. Define

$$N(f) := \{\omega \in \Omega : |f(\omega)| > 0\}.$$

Let M be a σ-MV-algebra. Let $\mathcal{T}$ be the tribe of fuzzy sets defined on $\Omega := \mathrm{Ext}(\mathcal{S}(M))$ generated by the set $\{\hat{a} : a \in M\}$. Denote by $\mathcal{T}'$ the class of all functions $f \in \mathcal{T}$ with the property that for some $b \in M$, $N(f - \hat{b})$ is a meager set. It can be shown that if for some b_1 and b_2 and $f \in \mathcal{T}'$ we have $N(f - \hat{b}_i)$ is a meager set for $i = 1, 2$, then $b_1 = b_2$. Moreover, $\mathcal{T}' = \mathcal{T}$. Due to the definition of $\mathcal{T}'$, for any $f \in \mathcal{T}$ there is a unique element $h(f) := b \in M$ such that $N(f - \hat{b})$ is meager. Consequently, the following generalization of the Loomis–Sikorski theorem can be proved [12, 18].

Theorem 3.3. For every σ-MV-algebra M there exist a tribe $\mathcal{T}$ of fuzzy sets and an MV-σ-homomorphism h from $\mathcal{T}$ onto M.

4. SPECTRAL THEOREM FOR σ–MV–ALGEBRAS

Let M be a σ-MV-algebra. A triple $(\Omega, \mathcal{T}, h)$ where $\mathcal{T}$ is a tribe of fuzzy sets on Ω and h is a σ-homomorphism from $\mathcal{T}$ onto M will be called a *representation* of M. If $\Omega = \mathrm{Ext}(\mathcal{S}(M))$ and $\mathcal{T}$ and $h : M \to \mathcal{T}$ are given by the construction in the proof of the Loomis–Sikorski Theorem 3.3, then the triple $(\Omega, \mathcal{T}, h)$ will be called the *canonical representation* of M.

Let $(\Omega, \mathcal{T}, h)$ be any representation of M. Let us identify the elements in $\mathcal{B}(\mathcal{T})$ with their characteristic functions, which are elements of $\mathcal{T}$. Then the restriction of h to $\mathcal{B}(\mathcal{T})$ is a σ-homomorphism (of Boolean σ-algebras) from $\mathcal{B}(\mathcal{T})$ to $\mathcal{B}(M)$.

For every $f \in \mathcal{T}$ and every $X \in \mathcal{B}([0,1])$ we have

$$h((f^{-1}(X)) \in \mathcal{B}(M),$$

and the map $X \mapsto h((f^{-1}(X))$ from $X \in \mathcal{B}([0,1])$ to $\mathcal{B}(M)$ is a σ-homomorphism of Boolean σ-algebras [10].

The following theorem can be considered as an analogue of a spectral theorem for self-adjoint Hilbert space operators.

Recall that a *symmetric difference* on an MV-algebra M is a map $\Delta : M \times M \to M$ defined as follows:

$$a\Delta b = (a \vee b) \ominus (a \wedge b),$$

and equivalent expressions are $a\Delta b = (a \ominus a \wedge b) \vee (b \ominus a \wedge b) = ((a \vee b) \ominus b) \vee ((a \vee b) \ominus a)$. We have $a\Delta b = 0$ iff $a = b$.

Theorem 4.1. Let M be a σ-MV-algebra. To every $a \in M$ a σ-homomorphism $\Lambda_a : \mathcal{B}([0,1]) \to \mathcal{B}(M)$ can be constructed such that the map $a \mapsto \Lambda_a$ is one-to-one and for every σ-additive state m on M we have

$$m(a) = \int_0^1 \lambda m(\Lambda_a(\mathrm{d}\lambda)). \tag{2}$$

Proof. Let $(\Omega, \mathcal{T}, h)$ be the canonical representation of M.

Choose $f \in \mathcal{T}$ such that $h(f) = a$ and define, for $X \in \mathcal{B}([0,1])$,

$$\Lambda_a(X) := h(f^{-1}(X)).$$

Then $\Lambda_a : \mathcal{B}([0,1]) \to \mathcal{B}(M)$ is a σ-homomorphism.

Let $g \in \mathcal{T}$ be another element such that $h(g) = a$. Then $0 = h(f)\Delta h(g) = h(f \Delta g)$. Moreover, $f^{-1}(X)\Delta g^{-1}(X)$, where Δ is the set-theoretical symmetric difference, is a subset of $N(f-g) \subseteq N(f-\hat{a}) \cup N(g-\hat{a})$, that is a meager set. Therefore $h(f^{-1}(X)) = h(g^{-1}(X))$, which proves that Λ_a is well defined.

Assume that for $a, b \in M$ with $a = h(f)$, $b = h(g)$ we have $\Lambda_a = \Lambda_b$. Then for every $X \in \mathcal{B}([0,1])$ we have $h(f^{-1}(X)) = h(g^{-1}(X))$. This implies that $f^{-1}(X)\Delta g^{-1}(X)$ is a meager set. For every rational number $k \in [0,1]$ put $D_k = [0,k)$, $D'_k = [k,1]$. Then

$$\begin{aligned} N(f-g) &= \{\omega \in \Omega : f(\omega) \neq g(\omega)\} \\ &= \bigcup_k [f^{-1}(D_k) \cap g^{-1}(D'_k) \cup f^{-1}(D'_k) \cap g^{-1}(D_k)] \\ &= \bigcup_k f^{-1}(D_k) \Delta g^{-1}(D_k), \end{aligned}$$

which implies that $N(f-g)$ is a meager set, and consequently $a=b$.

Let m be a σ-additive state on M. Define $\hat{m} : \mathcal{T} \to [0,1]$ by $\hat{m}(f) = m(h(f))$. Clearly, $\hat{m}(1_\Omega) = m(h(1_\Omega)) = 1$ and if $f_i \in \mathcal{T}$, $i = 1,2,\dots$, are such that $\sum_{i=1}^\infty f_i \le 1$, then $\sum_{i=1}^\infty f_i \in \mathcal{T}$ and

$$\begin{aligned}\hat{m}\Big(\sum_{i=1}^\infty f_i\Big) &= m\Big(h\Big(\sum_{i=1}^\infty f_i\Big)\Big)\\ &= m(\oplus_{i=1}^\infty h(f_i)) = \sum_{i=1}^\infty m(h(f_i)) = \sum_{i=1}^\infty \hat{m}(f_i),\end{aligned}$$

hence $\hat{m}$ is a σ-additive state on $\mathcal{T}$ vanishing on Ker_h. For every $a \in M$ we have $m(a) = \hat{m}(f)$, where $f \in \mathcal{T}$ is such that $a = h(f)$. The mapping $\hat{m}\circ f^{-1} : \mathcal{B}([0,1]) \to [0,1]$ is a probability measure. By [3] and integral transformation theorem,

$$\hat{m}(f) = \int_\Omega f(\omega)\hat{m}(\mathrm{d}\omega) = \int_0^1 \lambda\hat{m}(f^{-1}(\mathrm{d}\lambda).$$

That is,

$$m(a) = \int_0^1 \lambda m(h(f^{-1}(\mathrm{d}\lambda)) = \int_0^1 \lambda m(\Lambda_a(\mathrm{d}\lambda)).$$

□

The mapping $a \mapsto \Lambda_a$ will be called the *spectral measure* of a, or an *observable* on $\mathcal{B}(M)$ corresponding to a. In the sequel, we will make use of the following definition.

Definition 4.2. Let M be a σ-MV algebra. An injective mapping $a \mapsto \Lambda_a$, where $a \in M$ and $\Lambda_a : \mathcal{B}([0,1]) \to \mathcal{B}(M)$ is a σ-homomorphism, will be called a *spectral representation* of M. The spectral representation constructed in Theorem 4.1 will be called the *canonical spectral representation* of M.

Theorem 4.3. Let M be a σ-MV-algebra. Every probability measure on the Boolean σ-algebra $\mathcal{B}(M)$ of idempotent elements in M uniquely extends to a σ-additive state on M.

Proof. Let $\mathcal{T}$ be the tribe and $h : \mathcal{T} \to M$ the σ-homomorphism from the Loomis–Sikorski theorem. If m is a σ-additive state (probability measure) on $\mathcal{B}(M)$, then $m \circ h : \mathcal{B}(\mathcal{T}) \to [0,1]$ is a σ-additive state on $\mathcal{B}(\mathcal{T})$.

Without loss of generality, we may write $a = h(\hat{a})$ for every $a \in M$. Then we have for $E \in \mathcal{B}([0,1])$, $\Lambda_a(E) = h(\hat{a}^{-1}(E)) \in \mathcal{B}(M)$.

Define a map $\tilde{m} : M \to [0,1]$ by putting

$$\begin{aligned}\tilde{m}(a) &= \int_0^1 \lambda m(\Lambda_a(\mathrm{d}\lambda))\\ &= \int_0^1 \lambda m(h(\hat{a}^{-1}(\mathrm{d}\lambda)) = \int_\Omega \hat{a}(\omega)m(h(\mathrm{d}\omega)),\end{aligned}$$

using the integral transformation theorem.

Let $a, b \in M$ be such that $a \le b^*$. Then we have

$$\begin{aligned}\tilde{m}(a \oplus b) &= \int_\Omega (\widehat{a \oplus b})(\omega) m(h(\mathrm{d}\omega)) = \int_0^1 (\hat{a}(\omega) + \hat{b}(\omega)) m(h(\mathrm{d}\omega)) \\ &= \int_0^1 (\hat{a}(\omega) m(h(\mathrm{d}\omega)) + \int_0^1 (\hat{b}(\omega) m(h(\mathrm{d}\omega)) = \tilde{m}(a) + \tilde{m}(b),\end{aligned}$$

as for every $\omega \in \mathrm{Ext}(S(M)$, $(\widehat{a \oplus b})(\omega) = \omega(a \oplus b) = \omega(a) + \omega(b) = \hat{a}(\omega) + \hat{b}(\omega)$. This entails that the map $\tilde{m}$ is finitely additive.

Assume that $b, b_n \in M$, $n = 1, 2, \ldots$, and $b_n \nearrow b$. According to [13], proof of the Loomis–Sikorski theorem on p. 464, the mapping $a \longrightarrow \hat{a}$ preserves countable suprema and infima. Moreover, putting $b_0 = \lim_n \hat{b}_n$, we have $N(\hat{b} - b_0)$ is a meager set, hence $b = h(\hat{b}) = h(b_0)$. So we have

$$\tilde{m}(b_n) = \int_\Omega \hat{b}_n(\omega) m(h(\mathrm{d}\omega)),$$

and $\lim_n \hat{b}_n = \hat{b}$ a.e. $m \circ h$ on $(\Omega, \mathcal{B}(\mathcal{T}))$. By [15, §27, Th. B], then

$$\lim_n \int_\Omega \hat{b}_n(\omega) m(h(\mathrm{d}\omega)) = \int_\Omega \hat{b}(\omega) m(h(\mathrm{d}\omega)) = \tilde{m}(b).$$

This proves that $\tilde{m}$ is a σ-additive state on M. If $a \in \mathcal{B}(M)$, then $\hat{a} = \chi_A$ for some $A \in \mathcal{B}(\mathcal{T})$, and thus $\tilde{m}(a) = \int_\Omega \chi_A(\omega) m(h(\mathrm{d}\omega)) = m(h(A)) = m(h(\chi_A)) = m(a)$. So $\tilde{m}$ extends m.

Let $\tilde{m}_1$ be any other σ-additive extension of m. Then for all $a \in M$,

$$\tilde{m}_1(a) = \int_0^1 \lambda \tilde{m}_1(\Lambda(\mathrm{d}\lambda)) = \int_0^1 \lambda m(\Lambda(\mathrm{d}\lambda) = \tilde{m}(a),$$

so $\tilde{m}_1 = \tilde{m}$. □

5. REGULAR REPRESENTATIONS

In this sequel, we introduce the notion of a regular representation and show that every regular representation gives rise to a spectral representation.

Theorem 5.1. Let M be a σ-MV algebra, and let $(\Omega, \mathcal{T}, h)$ be a representation of M. Assume that the σ-homomorphism $h : \mathcal{T} \to M$ has the following property:

$$h(f) = 0 \quad \text{if and only if} \quad h(\chi_{N(f)}) = 0. \tag{3}$$

Define for $a \in M$, $\beta_a : \mathcal{B}([0,1]) \to \mathcal{B}(M)$ by putting $\beta_a(X) = h(f^{-1}(X))$, where $f \in \mathcal{T}$ is such that $h(f) = a$. Then β_a is a well-defined observable on $\mathcal{B}(M)$ and the map $a \mapsto \beta_a$ is one-to-one.

Proof. Take $a \in M$ and choose its representative $f \in \mathcal{T}$. For every $X \in \mathcal{B}([0,1])$ put $\beta_a(X) := h(f^{-1}(X))$. Clearly, $X \mapsto \beta_a(X)$ is a σ-homomorphism (of Boolean

σ-algebras). Let g be any other representative of a in $\mathcal{T}$. Then $h(f\Delta g) = 0$ and by our hypotheses, $h(N(f-g)) = 0$, and since $f^{-1}(X)\Delta g^{-1}(X) \subseteq N(f-g)$, we have $h(f^{-1}(X)) = h(g^{-1}(X))$, $X \in \mathcal{B}([0,1])$, hence β_a does not depend on the choice of the representative.

Now assume that for $a, b \in M$ we have $\beta_a = \beta_b$. Then for every $X \in \mathcal{B}([0,1])$ and any representatives f and g of a and b, respectively, we have $h(f^{-1}(X))\Delta h(g^{-1}(X)) = 0$, i. e. $f^{-1}(X)\Delta g^{-1}(X) \in \mathrm{Ker}_h$. As Ker_h is a σ-ideal, and $N(f-g)$ can be expressed as in the proof of Theorem 4.1, we get $N(f-g) \in \mathrm{Ker}_h$, which entails $h(f) = h(g)$, i. e., $a = b$. This proves that the map $a \mapsto \beta_a$ is one-to-one. □

A representation $(\Omega, \mathcal{T}, h)$ of M satisfying the requirement (3) will be called a *regular* representation. It is clear by the construction that the canonical representation is regular.

Theorem 5.2. Let $(\Omega, \mathcal{T}, h)$ be a regular representation of M. Then h maps $\mathcal{B}(\mathcal{T})$ onto $\mathcal{B}(M)$.

Proof. Since h maps $\mathcal{T}$ onto M, for every $a \in \mathcal{B}(M)$ there is an element $f \in \mathcal{T}$ with $h(f) = a$. Since h is a homomorphism of MV-algebras, we have

$$h(f) = a = a \oplus a = h(\min(f+f,1)).$$

We have $\{x \in \Omega : f(x) = \min(f+f,1)(x)\} = \{x : f(x) = 1\} \cup \{x : f(x) = 0\}$. By regularity of the representation, $\{x : f(x) \neq \min(f+f,1)(x)\} \in \mathrm{Ker}_h$. Put $A = \{x : f(x) = 1\}$. Then $\{x : f(x) \neq \chi_A(x)\} = \{x : f(x) \neq 0\} \cap \{x : f(x) \neq 1\} \in \mathrm{Ker}_h$. This entails $h(f) = h(\chi_A) = a$, and $A \in \mathcal{B}(\mathcal{T})$. □

It can be easily seen that if $(\Omega, \mathcal{T}, h)$ is a regular representation, then M is isomorphic with classes of functions from $\mathcal{T}$ modulo h, where we define $f = g$ modulo h if $h(\{\omega \in \Omega : f(\omega) \neq g(\omega)\}) = 0$.

Remark 1. From Theorem 5.1 we see that the spectral representation need not be unique, in general. In the subsequent paper [20] we prove that the spectral representation of M does not depend on a regular representation, and that it is uniquely defined.

6. OBSERVABLES AND FUNCTIONAL CALCULUS

Let M be a σ-MV algebra and let $(Y, \mathcal{F})$ be a measurable space, where $\mathcal{F}$ is a σ-algebra. An *observable* associated with M is a mapping $\xi : \mathcal{F} \to M$ such that

(i) $\xi(\Omega) = 1$;

(ii) $\xi(A \cup B) = \xi(A) \dot{+} \xi(B)$ whenever $A, B \in \mathcal{F}$, $A \cap B = \emptyset$;

(iii) $A, A_i \in \mathcal{F}$, $A_i \nearrow A$ implies $\xi(A_i) \nearrow \xi(A)$.

The pair $(Y, \mathcal{F})$ is called the *value space* of ξ. An observable with the value space $(R, \mathcal{B}(R))$ is called a *real* observable. If ξ is an observable associated with M and s is a σ-additive state on M, the mapping $s \circ \xi : \mathcal{F} \to [0,1]$ is a probability measure on $(Y, \mathcal{F})$.

In the next theorem we show how observables on M are related with regular representations.

Theorem 6.1. Let $(\Omega, \mathcal{T}, h)$ be a regular representation of M and let $(Y, \mathcal{F})$ be a measurable space. The following statements are equivalent:

(i) a mapping $\xi : \mathcal{F} \to M$ is an observable;

(ii) there is a mapping $\nu : \Omega \times \mathcal{F} \to [0,1]$ such that (a) for any fixed $X \in \mathcal{F}$, $\omega \to \nu(\omega, X) \in \mathcal{T}$ and (b) for every sequence $(X_i)_{i=1}^{\infty}$ of pairwise disjoint subsets $X_i \in \mathcal{F}$, $h(\{\omega \in \Omega : \sum_{i=1}^{\infty} \nu(\omega, X_i) \neq \nu(\omega, \bigcup_{i=1}^{\infty} X_i)\}) = 0$.

Proof. (i)$\Rightarrow$(ii): Let $\xi : \mathcal{F} \to M$ be an observable. For every $X \in \mathcal{F}$ we have $\xi(X) \in M$, and hence there is a function $f_X \in \mathcal{T}$ with $h(f_X) = \xi(X)$. Define $\nu(\omega, X) := f_X(\omega)$. Then (a) is satisfied. To prove (b), let $(X_i)_i$ be a sequence of pairwise disjoint elements of $\mathcal{F}$. Then $\xi(\bigcup X_i) = \oplus_i \xi(X_i)$, which entails that $h(f_{\bigcup X_i}) = \oplus_i h(f_{X_i}) = h\Big(\min(\sum_i f_{X_i}, 1)\Big)$, and we can choose representatives f_{X_i} such that $\sum_i f_{X_i} \leq 1$. The proof of the converse implication is similar. $\square$

Let $R(\xi)$ denote the range of ξ, i.e. $R(\xi) = \{\xi(X) : X \in \mathcal{F}\}$.

An observable ξ is called *sharp* if its range consists of idempotent (sharp) elements. A sharp observable $\xi : \mathcal{F} \to \mathcal{B}(M)$ can be considered as an observable associated with the Boolean σ-algebra $\mathcal{B}(M)$. It is well known that such an observable is a σ-homomorphism of Boolean σ-algebras and the range of ξ is a Boolean sub-σ-algebra of $\mathcal{B}(M)$ ([23, 19]).

A *spectrum* of a sharp observable $\xi : \mathcal{B}(R^n) \to \mathcal{B}(M)$ is defined as the smallest closed subset C of $\mathcal{B}(R^n)$ such that $\xi(C) = 1$. Let $\sigma(\xi)$ denote the spectrum of ξ. Then we have [23, 19],

$$\sigma(\xi) = \cap\{C \text{ closed} : \xi(C) = 1\}.$$

Remark 2. If ξ is a sharp observable with spectrum $\sigma(\xi)$, then for every $X \in \mathcal{B}(R^n)$ we have $\xi(X) = \xi(X \cap \sigma(\xi))$. Therefore ξ can be considered as an observable from $\mathcal{B}(\sigma(\xi))$ to $\mathcal{B}(M)$. Conversely, if ξ is a sharp observable defined on $\mathcal{B}([0,1]^n)$, the prescription $\tilde{\xi}(X) = \xi(X \cap [0,1]^n)$ for all $X \in \mathcal{B}(R^n)$ defines an observable from $\mathcal{B}(R^n)$ to $\mathcal{B}(M)$.

Let $(Y_1, \mathcal{F}_1)$ and $(Y_2, \mathcal{F}_2)$ be measurable spaces, and let $\xi_i : \mathcal{F}_i \to \mathcal{B}(M)$, $i = 1, 2$ be sharp observables. We say that ξ_2 is a *function* of ξ_1 if there is a measurable function $f : Y_1 \to Y_2$ such that for every $X \in \mathcal{F}_2$, $\xi_2(X) = \xi_1 \circ f^{-1}(X)$. If $(Y_2, \mathcal{F}_2) = (R, \mathcal{B}(R))$, then according to [23, Th. 1.4], ξ_2 is a function of ξ_1 iff $R(\xi_2) \subseteq R(\xi_1)$. Moreover, the function f is essentially unique in the sense that if $g : Y_1 \to R$

is another measurable function such that $\xi_2(X) = \xi_1 \circ g^{-1}(X)$, $X \in \mathcal{B}(R)$, then $\xi_1(\{x \in Y_1 : f(x) \neq g(x)\}) = 0$.

Theorem [23, Th. 1.6] (i) and (ii) enables us to define functions of real observables associated with a Boolean σ-algebra. For the convenience of readers, we repeat this important theorem below. We recall that a Boolean σ-algebra is *separable* if it has a countable generator.

Theorem 6.2. (i) Let $\mathcal{L}$ be a Boolean σ-algebra and u a real observable associated with $\mathcal{L}$. Then the range $R(u)$ of u is a separable Boolean sub-σ-algebra of $\mathcal{L}$. Conversely, if $\mathcal{L}_1$ is a separable Boolean sub-σ-algebra of $\mathcal{L}$, then there exists a real observable u associated with $\mathcal{L}$ such that $\mathcal{L}_1$ is the range of u.

(ii) Let $\mathcal{L}$ be a Boolean σ-algebra, and let u_i $i = 1, 2, \dots, n$ be real observables associated with $\mathcal{L}$, and $\mathcal{L}_i$ $i = 1, 2, \dots, n$ their respective ranges. Suppose that $\mathcal{L}_0$ is the smallest sub-σ-algebra containing all the $\mathcal{L}_i$. Then there exists a unique σ-homomorphism u of $\mathcal{B}(R^n)$ (the σ-algebra of Borel subsets of R^n) onto $\mathcal{L}_0$ such that for any Borel set E of R^1, $u_i(E) = u(p_i^{-1}(E))$, where p_i is the projection $(t_1, t_2, \dots, t_n) \to t_i$ of R^n to R^1. If $\phi : R^n \to R^1$ is any Borel function, the map $E \mapsto u(\phi^{-1}(E))$ $(E \in \mathcal{B}(R^1))$ is an observable associated with $\mathcal{L}$ whose range is contained in $\mathcal{L}_0$. Conversely, if v is any observable associated with $\mathcal{L}$ such that the range of v is contained in $\mathcal{L}_0$, there exists a real valued Borel function ϕ on R^n such that $v(E) = u(\phi^{-1}(E))$ for all E.

The observable $u : \mathcal{B}(R^n) \to \mathcal{L}$ from (ii) of the above theorem is called the *joint observable* of the observables u_i, $i = 1, 2, \dots, n$. The observable $u \circ \phi^{-1}$ is called the *ϕ-function* of u_i, $i = 1, 2, \dots, n$, and is denoted by $\phi(u_1, \dots, u_n)$.

Let B be a Boolean σ-algebra, and ξ_n, ξ $(n \in N)$ be real observables associated with B. According to [19, Def. 6.1.2], we say that ξ_n converge to ξ *everywhere* if for every $\epsilon > 0$ we have $\liminf((\xi_n - \xi)(-\epsilon, \epsilon)) = 1$.

Assume that for some observable z and measurable functions f_n, f we have $\xi_n = z \circ f_n^{-1}, \xi = z \circ f^{-1}$. Then according to [19, Th. 6.1.3], ξ_n converge to ξ everywhere iff there is a subset $Z \in \mathcal{B}(R)$ such that $z(Z) = 1$ and $f_n \to f$ everywhere on Z.

Using the above theorem, we can prove the following statement.

Theorem 6.3. Let $(\Omega, \mathcal{T}, h)$ be a regular representation of a σ-MV algebra M, and let $a \mapsto \beta_a$ be the corresponding spectral representation of M. The following statements hold.

(i) For every $a, b \in M$, the observable $\beta_{a \oplus b}$ is a ϕ-function of the observables β_a and β_b, where $\phi(t_1, t_2) = \min(t_1 + t_2, 1)$, $t_1, t_2 \in [0, 1] \subset R$;

(ii) for every $a, b \in M$, the observable $\beta_{a \vee b}$ $(\beta_{a \wedge b})$ is a ϕ-function of the observables β_a and β_b, where $\phi(t_1, t_2) = \max(t_1, t_2)$ $(\phi(t_1, t_2) = \min(t_1, t_2))$;

(iii) the observable β_{a^*} is the function ϕ of the observable β_a, where $\phi(t) = 1 - t$;

(iv) for any sequence $a_i \in M$, $i = 1, 2, \dots$, $a_i \nearrow a$ implies $\beta_{a_i} \to \beta_a$ everywhere.

Proof. (i) Put $\mathcal{L} = \mathcal{B}(M)$, $\mathcal{S} = \mathcal{B}(\mathcal{T})$. Then h maps $\mathcal{S}$ onto $\mathcal{L}$. Let $f_a, f_b \in \mathcal{T}$ be such that $h(f_a) = a$, $h(f_b) = b$. Then $\beta_a(X) = h(f_a^{-1}(X))$, $\beta_b(X) = h(f_b^{-1}(X))$ for all $X \in \mathcal{B}([0,1])$. In view of Remark 2, we may consider observables β_a and β_b as σ-homomorphisms from $\mathcal{B}(R^1)$ to $\mathcal{B}(M)$. Starting with $(\Omega, \mathcal{B}(\mathcal{T}), h)$, we will follow the construction in the proof of Theorem 1.6 (ii) in [23].[1] Let $\mathcal{L}_0$ denote the smallest sub-σ-algebra of $\mathcal{L}$ which contains the ranges of β_a and β_b. Define $\tilde{f} : \Omega \to R^2$ by $\tilde{f}(x) = (f_a(x), f_b(x))$. Then $\tilde{f}$ is $\mathcal{S}$-measurable. Let $u = h \circ \tilde{f}^{-1}$. Then $u : \mathcal{B}(R^2) \to \mathcal{L}$ is a σ-homomorphism such that $\beta_a(X) = u(p_1^{-1}(X))$, $\beta_b(X) = u(p_2^{-1}(X))$ for all $X \in \mathcal{B}(R)$. Since $\mathcal{B}(R^2)$ is the smallest σ-algebra of subsets of R^2 containing all the sets $p_i^{-1}(X)$, $i = 1, 2$, the range of u is $\mathcal{L}_0$. The uniqueness of u is obvious. For any Borel function $\phi : R^2 \to R$, the mapping $u \circ \phi^{-1}$ is an observable on $\mathcal{L}$ whose range is contained in $\mathcal{L}_0$. In particular, we may put $\phi(t_1, t_2) = \min(t_1 + t_2, 1)$, and we obtain the function $\phi(\beta_a, \beta_b)$.

On the other hand, we have $\beta_{a\oplus b} = h \circ f_{a\oplus b}^{-1}$, where $f_{a\oplus b} \in \mathcal{T}$ is any function such that $h(f_{a\oplus b}) = a \oplus b$. Since $h : \mathcal{T} \to M$ is a σ-homomorphism, we have $h(\min(f_a + f_b, 1)) = a \oplus b = h(f_{a\oplus b})$. Hence $\beta_{a\oplus b} = h \circ f_{a\oplus b}^{-1} = h \circ \min(f_a + f_b, 1)^{-1}$. Therefore $R(\beta_{a\oplus b}) \subset \mathcal{L}_0$. Now we have for every $X \in \mathcal{B}(R)$,

$$
\begin{aligned}
\phi(\beta_a, \beta_b)(X) &= u \circ \phi^{-1}(X) = h \circ \tilde{f}^{-1}(\phi^{-1}(X)) \\
&= h((\phi \circ \tilde{f})^{-1}(X)) = h(\min(f_a + f_b, 1)^{-1}(X)) \\
&= \beta_{a\oplus b}(X).
\end{aligned}
$$

(ii), (iii) Similarly we can prove the corresponding functional relations between β_a, β_b and $\beta_{a\vee b}$, $\beta_{a\wedge b}$ and also that $\beta_{a^*} = \phi(\beta_a)$, where $\phi(t) = 1 - t, t \in [0,1]$.

(iv) Let a_i, $i = 1, 2, \ldots$ be a nondecreasing sequence of elements of M such that $\bigvee_i a_i = a$ and let $\beta_{a_i} = h \circ f_{a_i}^{-1}$ $i = 1, 2, \ldots$ and $\beta_a = h \circ f_a^{-1}$ be the corresponding observables.

Put $f_n = \sup\{f_{a_i} : i \le n\}$, then $(f_n)_n$ is a nondecreasing sequence of functions in $\mathcal{T}$ with $h(f_n) = \bigvee_{i\le n} h(f_{a_i}) = a_n$ for every n. Put $V_n = \{x \in \Omega : f_n(x) \ne f_{a_n}(x)\}$, then $h(V_n) = 0$

Let $f = \lim_n f_n = \sup_n f_n$ be their pointwise limit. Then $f \in \mathcal{T}$, and by the properties of h, $h(f) = h(\sup_n f_n) = \bigvee_n h(f_n) = \bigvee_n a_n = a$. Put $V = \{x \in \Omega : f(x) \ne f_a(x)\}$, then $h(V) = 0$.

So we have $h(\{x \in \Omega : f(x) \ne f_a(x)\}) = 0$, and hence $f_{a_n} \to f_a$ pointwise on the set $\Omega \setminus V \cup (\bigcup_n V_n)$, which entails that $\xi_{a_n} \to \xi_a$ everywhere. □

ACKNOWLEDGEMENT

This work was partially supported by Science and Technology Assistance Agency under the contract No. APVT-51-032002 and grant VEGA 2/3163/23.

(Received April 13, 2004.)

[1] Notice that the function of observables does not depend on the choice of a triple $(X, \mathcal{S}, h)$.

REFERENCES

[1] L. P. Belluce: Semisimple algebras of infinite valued logic and Bold fuzzy set theory. Canad. J. Math. *38* (1986), 1356–1379.

[2] P. Busch, P. J. Lahti, and P. Mittelstaedt: The Quantum Theory of Measurement. Springer–Verlag, Berlin 1991.

[3] D. Butnariu and E. Klement: Triangular-norm-based measures and their Markov kernel representation. J. Math. Anal. Appl. *162* (1991), 111-143.

[4] G. Barbieri and H. Weber: Measures on clans and on MV-algebras. In: Handbook of Measure Theory, Vol. II (E. Pap, ed.), Elsevier, Amsterdam 2002, Chapt. 22, pp. 911–945.

[5] G. Cattaneo, R. Giuntini, and S. Pulmannová: Pre-BZ and degenerate BZ posets: Applications to fuzzy sets and unsharp quantum theories. Found. Phys. *30* (2000), 1765–1799.

[6] C. C. Chang: Algebraic analysis of many-valued logics. Trans. Amer. Math. Soc. *88* (1958), 467–490.

[7] C. C. Chang: A new proof of the completeness of the Lukasiewicz axioms. Trans. Amer. Math. Soc. *93* (1959), 74-80.

[8] F. Chovanec and F. Kôpka: D-lattices. Internat. J. Theor. Phys. *34* (1995), 1297–1302.

[9] R. Cignoli, I. M. L. D'Ottaviano, and D. Mundici: Algebraic Foundation of Many-Valued Reasoning. Kluwer Academic Publishers, Dordrecht 2000.

[10] A. Di Nola, A. Dvurečenskij, M. Hyčko, and C. Manara: Entropy on effect algebras with the Riesz decomposition property I, II. Kybernetika *41* (2005), 143–160, 161–176.

[11] M. Dalla Chiara, R. Giuntini, and R. Greechie: Reasoning in Quantum Theory. Kluwer Academic Publishers, Dordrecht 2004.

[12] A. Dvurečenskij: Loomis–Sikorski theorem for σ-complete MV-algebras and ℓ-groups. J. Austral. Math. Soc. Ser. A *68* (2000), 261–277.

[13] A. Dvurečenskij and S. Pulmannová: New Trends in Quantum Structures. Kluwer Academic Publishers, Dordrecht and Ister Science, Bratislava 2000.

[14] D. J. Foulis and M. K. Bennett: Effect algebras and unsharp quantum logic. Found. Phys. *24* (1994), 1325–1346.

[15] P. R. Halmos: Measure Theory. Van Nostrand, Princeton, New Jersey 1962.

[16] F. Kôpka and F. Chovanec: D-posets. Math. Slovaca *44* (1994), 21–34.

[17] D. Mundici: Interpretation of AF C*-algebras in Lukasiewicz sentential calculus. J. Funct. Anal. *65* (1986), 15–63.

[18] D. Mundici: Tensor products and the Loomis–Sikorski theorem for MV-algebras. Adv. Appl. Math. *22* (1999), 227–248.

[19] P. Pták and S. Pulmannová: Orthomodular Structures as Quantum Logics. Kluwer Academic Publishers, Dordrecht and VEDA, Bratislava 1991.

[20] S. Pulmannová: Spectral resolutions in Dedekind σ-complete ℓ-groups. J. Math. Anal. Appl. (to appear).

[21] B. Riečan and D. Mundici: Probability on MV-algebras. In: Handbook of Measure Theory, Vol. II (E. Pap, ed.), Elsevier, Amsterdam 2002, Chapt. 21, pp. 869–909.

[22] B. Riečan and T. Neubrunn: Integral, Measure and Ordering. Kluwer Academic Publishers, Dordrecht and Ister Science, Bratislava 1997.

[23] V. S. Varadarajan: Geometry of Quantum Theory. Springer–Verlag, New York 1985.

Sylvia Pulmannová, Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, SK-814 73 Bratislava. Slovak Republic.
e-mail: pulmann@mat.savba.sk