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THE CANCELLATION LAW FOR PSEUDO-CONVOLUTION[1]

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Cancellation law for pseudo-convolutions based on triangular norms is discussed. In more details, the cases of extremal t-norms T_M and T_D, of continuous Archimedean t-norms, and of general continuous t-norms are investigated. Several examples are included.

Keywords: cancellation law, t-norm, pseudo-convolution

AMS Subject Classification: 03E72, 28E10

1. INTRODUCTION

In algebraic structures, a commutative binary operation $*$ is said to be cancellative if for all elements g, h, v it holds

$$g * v = h * v \Rightarrow g = h.$$

The cancellation law ensures for example the uniqueness of solution of equation $x * v = u$ (if a solution exists).

The aim of this paper is investigation of the cancellativity of pseudo-convolutions introduced in [16]. Recall that the standard probabilistic convolution of distribution functions is cancellative.

The paper is organized as follows. In the next section, pseudo-convolutions are introduced. In Section 3, cancellation law for pseudo-convolutions based on boundary t-norms is discussed. Section 4 and Section 5 are devoted to the study of cancellation law in the case of continuous Archimedean t-norms and more general continuous t-norms based pseudo-convolutions.

2. PSEUDO-CONVOLUTIONS

2.1. Pseudo-convolution of real functions

Let $[a, b]$ be a closed subinterval of the extended real line (sometimes also semiclosed subintervals are taken into account).

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Definition 1. A binary operation $\oplus$ on $[a,b]$ is called a *pseudo-addition* on $[a,b]$ if it is commutative, nondecreasing, associative, continuous (possibly up to the points (a,b), (b,a)) and with a neutral element, denoted by $\mathbf{0}$, i.e., for each $x \in [a,b]$ $\mathbf{0} \oplus x = x$ holds.

So, $\oplus$ is either a t-norm, or a t-conorm or a uni-norm on $[a,b]$, see [4]. Because of the duality, it is sufficient to deal with t-conorms and uninorms only. Denote $[a,b]_+ = \{x;\ x \in [a,b], x \geq \mathbf{0}\}$.

Definition 2. A binary operation $\otimes$ on $[a,b]$ is called a *pseudo-multiplication* with respect to $\oplus$ if it is commutative, associative, distributive with respect to $\oplus$, positively nondecreasing (i.e., $x \leq y \Rightarrow x \otimes z \leq y \otimes z$ if $z \in [a,b]_+$) with a unit element, denoted by $\mathbf{1}$, (i.e., for each $x \in [a,b]$ $\mathbf{1} \otimes x = x$ holds). We suppose, further, $\mathbf{0} \otimes x = \mathbf{0}$, i.e., $\mathbf{0}$ is annihilator.

The structure $([a,b], \oplus, \otimes)$ is called *a semiring*, see e.g., [2].

Let $([a,b], \oplus, \otimes)$ be a semiring with continuous operations (possibly up to the continuity of $\otimes$ in points $(\mathbf{0}, a)$, $(\mathbf{0}, b)$, $(a, \mathbf{0})$ and $(b, \mathbf{0})$). The standart building up of an integral with respect to $\oplus$-decomposable measures based on the pseudo-addition and pseudo-multiplication leads to the definition of a pseudo-integral [12]. *The pseudo-convolution* of the functions defined on $[0,\infty[$ with values in $[a,b]$ was introduced in [16], see also [12, 14], by means of the corresponding pseudo-integral,

$$g * h(z) = \int_{[0,z]} g(z-x) \otimes h(x)\mathrm{d}x. \tag{1}$$

In our paper we will deal with the special semiring only, so we will not describe some details here. (It is possible to find them in [14, 16].)

2.2. Pseudo-convolution with respect to the semiring $([0,1], \vee, T)$

One of typical examples of a semiring is $([0,1], \vee, T)$, where $\vee = \sup$ and T is a t-norm, see [4]. This is the semiring with $\mathbf{0} = 0$ and $\mathbf{1} = 1$. In this case the formula for convolution (1) can be rewritten to

$$g * h(z) = \sup_{x \in [0,z]} T(g(z-x), h(x)), \tag{2}$$

where T is a t-norm.

Observe that the pseudo-convolution $*$ is commutative due to the commutativity of T, however, it need not be associative, in general. Nevertheless, for t-norms continuous on $[0,1[^2$, $*$ is also associative.

Note that the kernel of a function $g : [0,\infty[\to [0,1]$ is defined as

$$\ker(g) = \{x \in [0,\infty[; g(x) = 1\}.$$

Denote by $\mathcal{D}$ the class of all continuous distribution functions on $[0,\infty[$ and by $\mathcal{S}$ the subclass of $\mathcal{D}$ such that the restriction of g on $]a_g, b_g[:= \operatorname{supp} g \setminus \ker(g)$ (if $\ker(g) = \emptyset$ then $b_g = \infty$) is strictly increasing, i. e.,

$$\mathcal{S} = \left\{ g : [0,\infty[\to [0,1];\ g(0) = 0,\ g\,|_{]a_g,b_g[} \to]0,1[\text{ is increasing bijection} \right\}.$$

Lemma 1. Let $\ker(v) \neq \emptyset$ for a function $v \in \mathcal{D}$. Then for all g, $h \in \mathcal{S}$ the following implication holds:

$$g * v = h * v \Rightarrow \ker(g) = \ker(h), \quad \text{i. e., } b_g = b_h. \tag{3}$$

Proof. Let $\ker(v) \neq \emptyset$. We can get the formula (3) from the property $\ker(g*v) = \ker(g) + \ker(v)$. First we suppose that $b_g < \infty$.

- Let $z \geq b_g + b_v$. Then

$$\begin{aligned} g * v(z) &= \sup_{x\in[0,z]} T(g(z-x), v(x)) \\ &= \max\left\{ \sup_{x\in[0,z-b_g[} T(g(z-x), v(x)),\ \sup_{x\in[z-b_g,z]} T(g(z-x), v(x)) \right\}, \end{aligned}$$

  – if $0 \leq x < \underbrace{z - b_g}_{\geq b_v}$, i. e., $z - x > b_g$

$$g * v(z) = \sup_{x\in[0,z-b_g[} T(1, v(x)) = \sup_{x\in[0,z-b_g[} v(x) = 1,$$

  – if $b_v \leq z - b_g \leq x \leq z$, i. e., $z - x \leq b_g$

$$g * v(z) = \sup_{x\in]z-b_g,z]} T(g(z-x), 1) = \sup_{x\in]z-b_g,z]} g(z-x) = g(b_g) = 1.$$

- Let $0 \leq z < b_g + b_v$. Then

$$g * v(z) = \max\left\{ \sup_{x\in[0,z-b_g]} T(g(z-x), v(x)),\ \sup_{x\in]z-b_g,b_v[} T(g(z-x), v(x)), \right.$$
$$\left. \sup_{x\in[b_v,z]} T(g(z-x), v(x)) \right\}$$

  – if $0 \leq x \leq z - b_g$, i. e., $z - x \geq b_g$

$$g * v(z) = \sup_{x\in[0,z-b_g]} T(1, v(x)) = \sup_{x\in[0,z-b_g]} v(x) = v(\underbrace{z - b_g}_{<b_v}) < 1,$$

– if $z - b_g < x < b_v$, i.e., $z - x < b_g$

$$g * v(z) = \sup_{x \in]z-b_g, b_v[} T(g(z-x), v(x)) < 1,$$

– if $b_v \leq x \leq z$

$$g * v(z) = \sup_{x \in [b_v, z]} T(g(z-x), 1) = \sup_{x \in [b_v, z]} g(z-x) = g(\underbrace{z - b_v}_{<b_g}) < 1.$$

It is easy to see that if $b_v < \infty$ then $b_g = \infty$ if and only if $b_{g*v} = \infty$, i.e., if $b_g = \infty$ then $b_h = \infty$ too. □

3. THE CANCELLATION LAW FOR PSEUDO-CONVOLUTION BASED ON THE SEMIRING $([0,1], \vee, T_M)$ AND $([0,1], \vee, T_D)$, RESPECTIVELY

Recall that T is a t-norm if it is associative, commutative, non-decreasing binary operation on $[0,1]$ with neutral element 1. For more details we recommend [4]. For any t-norm T it holds $T_D \leq T \leq T_M$, where the strongest t-norm $T_M = \min$ and the weakest t-norm T_D (the drastic product) is given by

$$T_D(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Theorem 1. Consider the strongest t-norm T_M. Let g, h, $v \in \mathcal{D}$. Then the cancellation law holds,i.e.,

$$g * v = h * v \;\Rightarrow g = h.$$

Proof. We denote $g^{(c)}$ the c-cut of function g, i.e., $g^{(c)} = \{x; g(x) \geq c\}$ for $c \in]0,1]$. Then for convolution based on the T_M it holds

$$(g * h)^{(c)} = g^{(c)} + h^{(c)} \text{ for any } c \in]0,1].$$

An arbitrary c-cut of function g from $\mathcal{D}$ is interval $[a_g^{(c)}, \infty[$. Suppose $g * v = h * v$. Then

$$\begin{aligned} & [a_g^{(c)}, \infty[+ [a_v^{(c)}, \infty[= [a_h^{(c)}, \infty[+ [a_v^{(c)}, \infty[\text{ for all } c \in]0,1]. \text{ Thus } [a_g^{(c)} + a_v^{(c)}, \infty[\\ = \; & [a_h^{(c)} + a_v^{(c)}, \infty[\Rightarrow a_g^{(c)} = a_h^{(c)} \;\Rightarrow g^{(c)} = h^{(c)} \text{ for all } c \in]0,1] \;\Rightarrow g = h, \end{aligned}$$

i.e., the cancellation law holds. □

Remark 1. The cancellation law with respect to T_M fails if $\sup v < 1$ or $\inf v > 0$ or if we deal with non-monotone functions. See Example 1.

Example 1. Consider the t-norm T_M. Let $v(x) = \begin{cases} x, & x \in [0,1] \\ 1, & x \in]1,\infty[, \end{cases}$

$$g(x) = \begin{cases} x, & x \in [0,\frac{1}{2}] \\ \frac{1}{2}, & x \in]\frac{1}{2},2] \\ x-\frac{3}{2}, & x \in]2,\frac{5}{2}] \\ 1, & x \in]\frac{5}{2},\infty[\end{cases} \quad \text{and} \quad h(x) = \begin{cases} x, & x \in [0,\frac{1}{2}] \\ \frac{1}{2}, & x \in]\frac{1}{2},1] \\ -x+\frac{3}{2}, & x \in]1,\frac{5}{4}] \\ x-1, & x \in]\frac{5}{4},\frac{3}{2}] \\ \frac{1}{2}, & x \in]\frac{3}{2},2] \\ x-\frac{3}{2}, & x \in]2,\frac{5}{2}] \\ 1, & x \in]\frac{5}{2},\infty[. \end{cases}$$

The function h is not a monotone function. Then pseudo-convolutions of functions g, v and h, v based on semiring $([0,1],\vee,T_M)$ are the same, i.e.

$$g * v(x) = h * v(x) = \begin{cases} \frac{1}{2}x, & x \in [0,1] \\ \frac{1}{2}, & x \in]1,\frac{5}{2}] \\ \frac{x}{2}-\frac{3}{4}, & x \in]\frac{5}{2},\frac{7}{2}] \\ 1, & x \in]\frac{7}{2},\infty[. \end{cases}$$

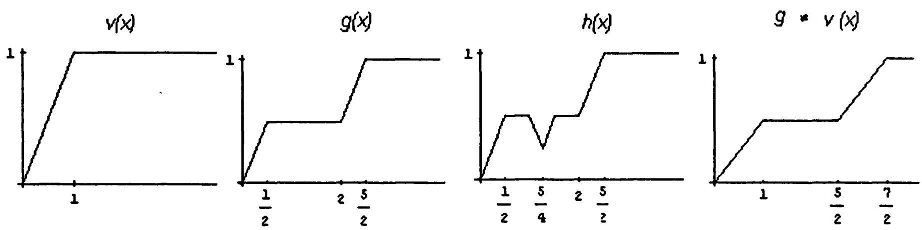

Fig. 1.

On the other hand, consider the weakest t-norm T_D. Then the cancellation law holds only in special cases.

Theorem 2. Consider the pseudo-convolution based on the T_D. Let g, h, $v \in \mathcal{D}$. Moreover, let

$$v(b_v - x) \le \min\left(g(b_g - x), h(b_h - x)\right) \quad \text{for all } x \in [0,b],$$

where $b := \min\{b_v, b_g, b_h\}$. Then $g * v = h * v \Leftrightarrow g = h$.

Proof. Applying the formula for sum of fuzzy quantities based on the drastic product from [9], we get

$$g * v(x) = \max\{g(/x - b_v/), v(/x - b_g/)\}$$

for all $x \in [0, \infty[$, where $/x/ = \min\{\max\{0, x\}, 1\}$. Now we can easily get condition for cancellativity. $\square$

4. THE CANCELLATION LAW FOR PSEUDO-CONVOLUTION BASED ON $([0,1], \vee, T)$, WHERE T IS AN ARCHIMEDEAN CONTINUOUS t-NORM

In this section at first we describe some Zagrodny's results [20]. Further we will apply them for investigation of validity of cancellation law for pseudo-convolution of functions based on a strict t-norm. Finally, the case of nilpotent t-norms will be discussed.

4.1. The cancellation law for inf-convolution – Zagrodny's results

Definition 3. Let $g,\ h : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$. The inf-convolution of g and h at $x \in \mathbb{R}$ is defined by

$$g \,\square\, h(x) := \inf_{y+z=x} (g(y) + h(z)).$$

Definition 4. Let $h : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$. The function h is said to be uniformly convex if for $\forall \varepsilon \geq 0\ \exists \delta > 0$

$$|a-b| \geq \varepsilon \ \Rightarrow h\left(\frac{a+b}{2}\right) \leq \frac{h(a)+h(b)-\delta|a-b|}{2},\ \forall\, a, b \in \mathrm{dom}\ h.$$

Note that the domain of functions g, h can be restricted to some intervals. Zagrodny in [20] deal with more general functions on Banach space.

Theorem 3. Let X be a reflexive Banach space. If $q,\ g,\ h : X \to \mathbb{R} \cup \{\infty\}$ are proper lower semicontinuous convex functions such that h is strictly convex and $\lim_{\|x\| \to \infty} \frac{h(x)}{\|x\|} = \infty$ then $q \,\square\, h = g \,\square\, h$ implies $q = g$.

Theorem 4. Let X be a Banach space and $q,\ g,\ h : X \to \mathbb{R} \cup \{\infty\}$ be proper lower semicontinuous convex functions. Moreover, suppose h is uniformly convex. Then $q \,\square\, h = g \,\square\, h$ implies $q = g$.

4.2. The cancellation law for pseudo-convolution based on a strict t-norm

Recall that the pseudo-convolution of functions based on semiring $([0,1], \vee, T)$ with some Archimedean continuous t-norm T can be expressed by

$$g * h(x) = f^{[-1]}\left(\inf_{y+z=x} (f(g(y)) + f(h(z))) = f^{[-1]}(f \circ g \,\square\, f \circ h\ (x)\right),\ x \in [0, \infty[,$$

where f is additive generator of t-norm T, i. e., $f : [0,1] \to [0,\infty]$ is continuous strictly decreasing mapping verifying $f(1) = 0$, and pseudo-inverse $f^{[-1]} : [0,\infty] \to [0,1]$ of f is defined by

$$f^{[-1]}(x) = f^{-1}(\min(f(0), x)).$$

Archimedean continuous t-norms can be divided into two classes: strict and nilpotent. An additive generator of a strict t-norm is unbounded, and then $f^{[-1]} = f^{-1}$.

Theorem 5. Consider a strict t-norm T with an additive generator f. Let g, h, $v \in \mathcal{S}$ such that $f \circ g$ and $f \circ h$ are convex on $[a_g, b_g[$ and $[a_h, b_h[$, respectively and $f \circ v$ is either

(i) uniformly convex on $[a_v, b_v[$ or

(ii) strictly convex on $[a_v, b_v[$

and if $b_v = \infty$ then $\lim\limits_{x \to \infty} \frac{f \circ v(x)}{x} = \infty$.

Then $g * v = h * v$ implies $g = v$.

P r o o f. Assume $f \circ g$, $f \circ h$ and $f \circ v$ verify conditions from theorem. Let $g * v = h * v$. This imply $f \circ g \,\square\, f \circ v = f \circ h \,\square\, f \circ v$ and by Zagrodny's results $f \circ g = f \circ h \Rightarrow g = h$, i. e., the cancellativity is valid. □

4.3. The cancellation law for pseudo-convolution based on a nilpotent t-norm

The case of nilpotent t-norm is more complicated. Conditions from Theorem 5 are deficient. See Example 2.

Example 2. Consider the Łukasiewicz t-norm T_L with additive generator

$$f(x) = 1 - x \qquad \text{and functions} \qquad g(x) = \begin{cases} x, & x \in [0,1] \\ 1, & x \in]1,\infty[, \end{cases}$$

$$h(x) = \begin{cases} 0, & x \in [0, 0.1] \\ 2x - 0.2, & x \in]0.1, 0.2] \\ x, & x \in]0.2, 1] \\ 1, & x \in]1,\infty[\end{cases} \quad \text{and } v(x) = \begin{cases} 1 - (x-1)^2, & x \in [0,1] \\ 1, & x \in]1,\infty[. \end{cases}$$

The interval $[a_v, b_v[= [0,1[$ and $f \circ v$ is given by formula $f \circ v(x) = 1 - (x-1)^2$ (i. e., strictly convex function).

The interval $[a_g, b_g[= [0,1[$ too and $f \circ g(x) = 1 - x$ on $[0,1[$ (i. e., convex function). Finally, $[a_h, b_h[= [0.1, 1[$ and

$$f \circ h = \begin{cases} 1.2 - 2x, & x \in [0.1, 0.2[\\ 1 - x, & x \in [0.2, 1[, \end{cases}$$

(i. e., convex function).

However, the pseudo-convolution based on $([0,1], \vee, T_L)$ of functions v and g is the same as pseudo-convolution (based on the same semiring) of functions v and h.

$$g * v(x) = h * v(x) = \begin{cases} 0, & x \in [0, \frac{3}{4}] \\ x - \frac{3}{4}, & x \in]\frac{3}{4}, \frac{3}{2}] \\ 1 - (x-2)^2, & x \in]\frac{3}{2}, 2] \\ 1, & x \in]2, \infty[. \end{cases}$$

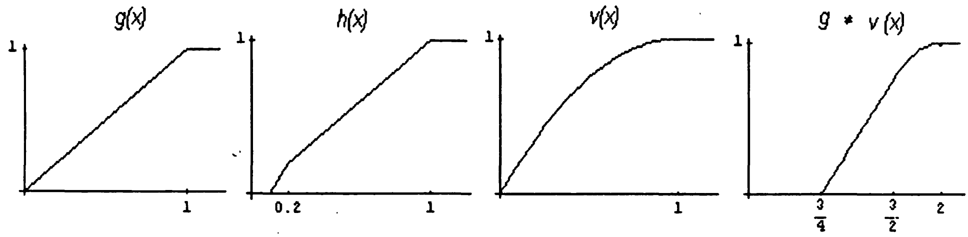

Fig. 2.

Thus Theorem 5 is not valid in the case when T is a nilpotent t-norm, in general. For nilpotent t-norms, we have only the following special cancellation theorems.

Theorem 6. Consider a nilpotent t-norm T with normed additive generator f. Let g, h, $v \in \mathcal{S}$, such that $f \circ g$, $f \circ h$ and $f \circ v$ are concave on the interval $[a_g, b_g[$, $[a_h, b_h[$ and $[a_v, b_v[$ respectively. Moreover,

$$v(b_v - x) \leq \min\left(g(b_g - x), h(b_h - x)\right) \quad \text{for all } x \in [0, b],$$

where $b := \min\{b_v, b_g, b_h\}$. Then $g * v = h * v \Leftrightarrow g = h$.

The proof follows from the fact that under requirements of the theorem, the pseudo-convolution of function based on semiring $([0,1], \vee, T)$ with some nilpotent t-norm T behaves as the pseudo-convolution of function based on semiring $([0,1], \vee, T_D)$, see [7, 9]. Note that the same claim is true also for strict t-norms. However then $b_g = b_h = b_v = \infty$.

Consider $(a, b) \in \mathbb{R}^2$, $a \neq b$, then $\phi_{(a,b)}$ is the linear transformation defined by

$$\phi_{(a,b)}(x) = \frac{x-a}{b-a}.$$

Note that the inverse mapping $\phi^{-1}_{(a,b)}$ of $\phi_{(a,b)}$ is given by $\phi^{-1}_{(a,b)}(x) = a + (b-a)x$.□

Theorem 7. Consider a nilpotent t-norm T with normed additive generator f. Let g, h, $v \in \mathcal{S}$, such that b_g, b_h, $b_v < \infty$ and $f \circ v \circ \phi_v^{-1}(x) = f \circ g \circ \phi_g^{-1}(x) = f \circ h \circ \phi_h^{-1}(x) = 1 - (1-x)^p$ on the interval $(0,1)$ for some $p \in (1,\infty)$, where $\phi_v = \phi_{(a_v,b_v)}$ and similarly for functions g, h. Then $g * v = h * v \Rightarrow g = h$.

Proof. Following [8], under requirements of the theorem,

$$f \circ (g * v) \circ \phi_{g*v}^{-1}(x) = 1 - (1-x)^p,$$

where $b_{g*v} = b_g + b_v$ and

$$(b_{g*v} - a_{g*v})^{\frac{1}{p}-1} = (b_g - a_g)^{\frac{1}{p}-1} + (b_v - a_v)^{\frac{1}{p}-1}.$$

Similarly,

$$f \circ (h * v) \circ \phi_{h*v}^{-1}(x) = 1 - (1-x)^p,$$

where $b_{h*v} = b_h + b_v$ and

$$(b_{h*v} - a_{h*v})^{\frac{1}{p}-1} = (b_h - a_h)^{\frac{1}{p}-1} + (b_v - a_v)^{\frac{1}{p}-1}.$$

Now, it is evident that $g * v = h * v$ if and only if $a_g = a_h$, $b_g = b_h$, i.e., $g = h$.□

5. THE CANCELLATION LAW FOR PSEUDO-CONVOLUTION BASED ON $([0,1], \vee, T)$, WHERE T IS A CONTINUOUS t-NORM

5.1. Ordinal sums of t-norms

Definition 5. Consider a family $(T_k)_{k \in K}$ of t-norms and a family $(]\alpha_k, \beta_k[)_{k \in K}$ of pairwise disjoint open non-degenerate subintervals of $[0,1]$. The $[0,1]^2 \to [0,1]$ mapping T defined by

$$T(x,y) = \begin{cases} \phi_k^{-1}\left(T_k\left(\phi_k(x), \phi_k(y)\right)\right), & \text{if } (x,y) \in [\alpha_k, \beta_k]^2 \\ T_M(x,y), & \text{elsewhere,} \end{cases}$$

where $\phi_k = \phi_{(\alpha_k,\beta_k)}$, is a t-norm. T is called the ordinal sum of the summands $\langle \alpha_k, \beta_k, T_k \rangle$, and is denoted by $T \equiv (\langle \alpha_k, \beta_k, T_k \rangle \mid k \in K)$.

Note that in the foregoing proposition the case of an empty index set is also allowed, and obviously leads to the minimum operator T_M. The notion 'ordinal sum' has led to the following important characterization of continuous t-norms.

Theorem 8. A $[0,1]^2 \to [0,1]$ mapping T is a continuous t-norm if and only if it is an ordinal sum of continuous Archimedean t-norms.

5.2. Cancellation law for pseudo-convolution

Theorem 8 and the results from [1] allow to transform the cancellation law for pseudo-convolution based on a continuous t-norm T to the cases discussed in the previous sections.

Definition 6. Consider a real function g and $(a,b) \in [0,1]^2$, $a < b$.

(i) The function $g^{[a,b]}$ is defined as

$$g^{[a,b]} = /\phi_{(a,b)} \circ g/,$$

i.e. $g^{[a,b]}(x) = \Big/ \frac{g(x)-a}{b-a} \Big/$, where $/x/ = \min\{\max\{0,x\},1\}$

(ii) The function $g_{[a,b]}$ is defined by

$$g_{[a,b]}(x) = \begin{cases} \phi^{-1}_{(a,b)}\left(g(x)\right), & \text{if } g(x) > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Theorem 9. Consider an ordinal sum $T \equiv (\langle a_i, b_i, T_i \rangle \mid i \in I)$ written in such a way that $\bigcup_{i\in I}[a_i,b_i] = [0,1]$, and functions $g,\ h \in \mathcal{S}$, then the pseudo-convolution based on the semiring $([0,1], \vee, T)$ is given by

$$g * h(x) = \sup_{i\in I} \left(g^{[a_i,b_i]} *_{T_i} h^{[a_i,b_i]} \right)_{[a_i,b_i]}(x),$$

where $*_{T_i}$ is pseudo-convolution based on semiring $([0,1], \vee, T_i)$.

Theorem 10. Let T be a continuous t-norm represented as an ordinal sum of Archimedean continuous t-norms, $T \equiv (\langle a_i, b_i, T_i \rangle \mid i \in I)$ and let $g,\ h,\ v \in \mathcal{S}$. Then cancellation law for pseudo-convolution based on the semiring $([0,1], \vee, T)$ is valid iff for $\forall\, i \in I$ holds

$$g^{[a_i,b_i]} *_{T_i} v^{[a_i,b_i]} = h^{[a_i,b_i]} *_{T_i} v^{[a_i,b_i]} \Rightarrow g^{[a_i,b_i]} = h^{[a_i,b_i]}.$$

Example 3. Consider the continuous t-norm $T = \left\{ \left\langle 0, \frac{1}{2} \right\rangle, T_P \right\}$ and $g,\ h,\ v \in \mathcal{S}$. Let $-\ln v(x)$ be strictly convex on the interval $[a_v, v^{-1}(\frac{1}{2})]$ and $-\ln g(x)$ and $-\ln h(x)$ be convex on $[a_g, g^{-1}(\frac{1}{2})]$ and $[a_h, h^{-1}(\frac{1}{2})]$ respectively. Then $g * v = h * v \Rightarrow g = h$.

6. CONCLUSION

We have discussed the cancellation law for pseudo-convolutions based on triangular norms. While for the case of T_M the cancellation law is valid without special requirements, in all other cases it holds only under special restrictions. Note that T-based pseudo-convolutions acting on (continuous) distribution functions are special triangle functions, see e. g. [4, Chapter 9], and thus our results provide a partial answer to an open problem of V. Höhle posed in [3, Problem 13]. As a continuation of our work, we aim to discuss the cancellation law for another types of triangle functions.

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