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RATIONALITY OF INDUCED ORDERED WEIGHTED OPERATORS BASED ON THE RELIABILITY OF THE SOURCE OF INFORMATION IN GROUP DECISION-MAKING

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The aggregation of preference relations in group decision-making (GDM) problems can be carried out based on either the reliability of the preference values to be aggregated, as is the case with ordered weighted averaging operators, or on the reliability of the source of information that provided the preferences, as is the case with weighted mean operators.

In this paper, we address the problem of aggregation based on the reliability of the source of information, with a double aim: a) To provide a general framework for induced ordered weighted operators based upon the source of information, and b) to provide a study of their rationality. We study the conditions which need to be verified by an aggregation operator in order to maintain the rationality assumptions on the individual preferences in the aggregation phase of the selection process of alternatives. In particular, we show that any aggregation operator based on the reliability of the source of information does verify these conditions.

Keywords: aggregation operators, induced aggregation, group decision-making, preference relations, rationality, consistency

AMS Subject Classification: 91B06, 91B10

1. INTRODUCTION

Preference relations are the most common representation structures of information used in decision-making problems because they are a useful tool in modelling decision processes, above all when we want to aggregate experts' preferences into group preferences [14, 22, 23]. Many important decision models have been developed using mainly two kinds of preference relations: *fuzzy preference relations* [7, 14, 21, 24] and *multiplicative preference relations* [22].

Fuzzy preference relations: (See [14, 24].) A fuzzy preference relation P on a set of alternatives X is a fuzzy set on the product set $X \times X$, that is characterized by a membership function

$$\mu_P : X \times X \longrightarrow [0, 1].$$

When cardinality of X is small, the preference relation may be conveniently represented by the $n \times n$ matrix $P = (p_{ij})$ being $p_{ij} = \mu_P(x_i, x_j)$ $\forall i, j \in \{1, \ldots, n\}$. p_{ij} is interpreted as the preference degree of the alternative x_i over x_j: $p_{ij} = 1/2$ indicates indifference between x_i and x_j ($x_i \sim x_j$), $p_{ij} = 1$ indicates that x_i is absolutely preferred to x_j, and $p_{ij} > 1/2$ indicates that x_i is preferred to x_j ($x_i \succ x_j$). In this case, the preference matrix, P, is usually assumed additive reciprocal, i. e.,

$$p_{ij} + p_{ji} = 1 \;\; \forall i, j \in \{1, \ldots, n\}.$$

Multiplicative preference relations: (See [22].) A multiplicative preference relation A on a set of alternatives X is represented by a matrix $A \subset X \times X$, $A = (a_{ij})$, being a_{ij} interpreted as the ratio of the preference intensity of alternative x_i to that of x_j, i. e., it is interpreted as x_i is a_{ij} times as good as x_j. Saaty suggests measuring a_{ij} using a ratio-scale, and precisely the 1 to 9 scale: $a_{ij} = 1$ indicates indifference between x_i and x_j, $a_{ij} = 9$ indicates that x_i is absolutely preferred to x_j, and $a_{ij} \in \{2, \ldots, 8\}$ indicates intermediate preference evaluations. In this case, the preference relation, A, is usually assumed multiplicative reciprocal, i. e.,

$$a_{ij} \cdot a_{ji} = 1 \;\; \forall i, j \in \{1, \ldots, n\}.$$

The aggregation of a set of preference relations can be done taking into account the reliability of the preference values to be aggregated, as is the case with ordered weighted averaging (OWA) operators, or taking into account the reliability of the source of information, as is the case with weighted mean (WM) operators (see Appendix A.1).

A fundamental aspect of the OWA operators is the reordering of the arguments to be aggregated, based upon the magnitude of their respective values, which allows an importance to be given to the values to be aggregated. However, it is clear that a set of values can be reordered in a different way to the one used by the ordered weighted (OW) operators. This is the idea on which Yager and Filev based the definition of the induced OWA (IOWA) operator [28]. Motivated by this idea, and the fact that OWA operators are not appropriate aggregation operators for ratio-scale measurements (see Appendix A.2), we introduced the ordered weighted geometric (OWG) operator [16, 17] and the induced OWG (IOWG) operator [10]. The class of induce OW (IOW) operators includes both classes of OW and WM operators.

In this paper, we address the problem of aggregation based on the reliability of the source of information, with a double aim:

- To provide a general framework for IOW operators based upon the source of information. In particular, we present the importance IOW (I-IOW) operator, when dealing with heterogeneous GDM, which induces the ordering of the argument values based upon the importance of the source of information, and the consistency IOW (C-IOW) operator, when dealing with homogeneous

GDM, which induces the ordering of the argument values based upon the consistency of the source of information. We also show that these IOW operators when guided by a linguistic quantifier allow the introduction of the importance and consistency concepts in the aggregation phase of a selection process of the alternatives in GDM.

- Secondly, and as the main novelty of this paper, we study the conditions which need to be verified by an aggregation operator in order to maintain the rationality assumptions on the individual preferences in the aggregation phase of a selection process. In particular, we show that any aggregation operator based on the reliability of the source of information does verify these conditions, as do the I-IOW and the C-IOW operators.

In order to do this, the paper is set out as follows. In Section 2, we deal with the issue of rationality in the aggregation of preference relations in group decision-making. In Section 3, we justify the election of IOW operators based upon the reliability of the source of information in order to get rational aggregation results, and shortly introduce the basic IOW operators: the IOWA and the IOWG operators. In Section 4, we present two different IOW operators to aggregate preference relations in GDM problems based upon the reliability of the source of information, the I-IOW and the C-IOW operators. In Section 5, we study the conditions needed to guarantee both indifference, reciprocity and consistency properties of the individual preference relations trough the aggregation phase. Our concluding remarks are given in Section 6. Finally, in the appendix we provide some definitions needed throughout this paper.

2. THE PROBLEM OF RATIONALITY OF INFORMATION IN GDM

In this section, we analize the fundamental rationality assumptions when dealing with preference relations in GDM, as well as the necessary compatibility between them.

In a preference relation an expert associates a real number to each pair of alternatives that reflects the preference degree, or the ratio of preference intensity, of the first alternative over, or to that of, the second one. When doing this, a first and natural question immediately arises: Which conditions have to be verified in order to obtain consistent results?

There are three fundamental and hierarchical levels of rationality assumptions when dealing with preference relations [15]:

- The first level of rationality requires indifference between any alternative and itself.

- The second one assumes the property of reciprocity in the pairwise comparison between any two alternatives.

- Finally, the third one is associated with the transitivity in the pairwise comparison among any three alternatives.

The mathematical modelling of all these rationality assumptions obviously depends on the scales used for providing the preference values [12, 14, 20, 22, 24].

A preference relation verifying the third level of rationality is called a *consistent preference relation* and any property that guarantees the transitivity of the preferences is called a consistency property. The lack of consistency in decision making can lead to inconsistent conclusions; that is why it is important, in fact crucial, to study conditions under which consistency is satisfied [14, 20, 22].

Clearly, the problem of consistency itself includes two problems [5, 6, 18]:

(i) when an expert, considered individually, is said to be consistent and,

(ii) when a whole group of experts are considered consistent.

The first problem was addressed in [20], and thus in this paper we focus on the second one. We address the problem of rationality in the aggregation of rational (consistent) information in GDM problems.

Due to the hierarchical structure of the three rationality assumptions for a preference relation, the verification of a particular level of rationality should be a necessary condition in order to verify the next level of rationality. This means that the third level of rationality, transitivity of preferences, should imply or be compatible with the second level of rationality, reciprocity of preferences, and the second level with the first one, indifference of any alternative with itself.

This necessary compatibility between the rationality assumptions can be used as a criterion for considering a particular condition modelling any one of the rationality levels as adequate or inadequate. In the case of fuzzy (multiplicative) preference relations, the indifference between any alternative, x_i, and itself is modelled by associating the preference value $p_{ii} = 0.5$ ($a_{ii} = 1$). The reciprocity of fuzzy (multiplicative) preferences is modelled using the property $p_{ij} + p_{ji} = 1,\ \forall i, j$ ($a_{ij} \cdot a_{ji} = 1,\ \forall i, j$). A necessary condition for a preference relation to verify reciprocity should be that indifference between any alternative and itself holds. Because reciprocity property implies the indifference of preferences, we conclude that both properties are compatible.

In the case of multiplicative preference relations Saaty means by *consistency* what he calls *cardinal transitivity* in the strength of preferences,

$$a_{ij} \cdot a_{jk} = a_{ik} \quad \forall i, j, k = 1, \dots, n,$$

which is a stronger condition than the traditional requirement of the transitivity of preferences. Inconsistency for Saaty is a violation of proportionality which may not entail violation of transitivity [22]. Furthermore, consistency implies reciprocity, and therefore, they are both compatible.

In [22] Saaty shows that a reciprocal multiplicative preference relation is consistent if and only if its maximum or principal eigenvalue $\lambda_{\max}$ is equal to the number of alternatives n. Perfect consistency is however difficult to obtain in practice, especially when measuring preferences on a set with a large number of alternatives. For measuring consistency we can use Saaty's consistency index

$$CI^k = \frac{\lambda^k_{\max} - n}{n - 1}$$

where $\lambda^k_{\max}$ is the maximum or principal eigenvalue of A^k. The closer CI^k to 0 the more consistent the information provided by the expert e^k is, and thus, more importance should be given to this information.

In decision-making problems based on fuzzy preference relations, the study of consistency is associated with the study of the *transitivity property*, and more than one condition has been suggested for modelling the transitivity of preferences [20]. Thus, a study of the compatibility between them and the reciprocity property would be of great help in deciding which one of them is the most adequate to model the transitivity of preferences. Using the transformation function of proposition A (Appendix A), the additive transitivity property

$$p_{ij} + p_{jk} + p_{ki} = \frac{3}{2} \quad \forall i, j, k$$

is the corresponding concept to use to model the additive consistency property for fuzzy preference relations [20]. Furthermore, additive transitivity implies reciprocity of fuzzy preference relations, and therefore, they are both compatible. In [20], Herrera-Viedma et al. gave a characterization of the consistency property defined by the additive transitivity property of a fuzzy preference relation $P^k = (p^k_{ij})$. Using this characterization method, a procedure was given to construct a consistent fuzzy preference relation $\widetilde{P}^k$ from a non-consistent fuzzy preference relation P^k.

3. THE ROLE OF IOW OPERATORS BASED UPON THE SOURCE OF INFORMATION IN GDM

The classical GDM procedure follows two steps [14]: *aggregation* and *exploitation*. The aggregation of experts' preferences, consisting of combining the individual preferences into a collective one in such a way that it summarizes or reflects all the properties contained in all the individual preferences, is a necessary and very important task to carry out when we want to obtain a final solution of GDM problems [13, 14, 21].

The aggregation of a set of preference relations can be done taking into account the reliability of the preference values to be aggregated, as is the case with OW operators, or taking into account the reliability of the source of information, as is the case with WM operators. Furthermore, in any GDM process the final solution must be accepted by a *majority* of experts. The majority is traditionally defined as a threshold number of elements. However, this concept is not always included in the GDM process. *Fuzzy logic* provides one possible way of modelling it.

Fuzzy majority is a soft majority concept expressed by a *fuzzy quantifier* [30], which is manipulated via a fuzzy logic based calculus of linguistically quantified propositions. Therefore, using fuzzy majority guided aggregation operators we can incorporate the concept of majority into the computation of the solution. The OWA operators have been extensively implemented in the last few years in the resolution process of different problems and have also proved to be very important in solving GDM problems because they allow the implementation of the concept of fuzzy majority, which is fundamental when looking for a final solution of consensus [7, 21]. However, as far as we are aware no resolution process using WM operators has, so far, been proposed that implements the concept of fuzzy majority.

As shown in [3, 4], the proper aggregation operator of ratio-scale measurements is not the *arithmetic mean* but the *geometric mean*. However, this operator does not allow the concept of fuzzy majority in the decision processes to be implemented. We could use the OWA operator but as it presents a similar behaviour to the arithmetic mean this is not advisable. In [7] the OWG operator based on the OWA operator and the geometric mean was introduced. These operators allow the implementation of the concept of fuzzy majority in the decision processes of a GDM problem with ratio-scale measurements in a similar way to OWA operators [7, 16, 17].

As we have mentioned, a fundamental aspect of the OW operators is the reordering of the arguments to be aggregated, based upon the magnitude of their respective values, which allows an importance to be given to the values to be aggregated. However, it is clear that a set of values can be reordered in a different way to the one used by the OW operators. This is the idea on which Yager and Filev based the definition of the IOWA operator [28]. Motivated by this idea, and the aforementioned fact that OWA operators are not appropriate aggregation operators for ratio-scale measurements, in [10] we introduced the IOWG operator.

The OW operators allow the implementation of the concept of fuzzy majority but fail to maintain the rationality assumptions [9, 17], in contrast to WM operators, which maintain the rationality assumptions but do not allow the implementation of the fuzzy majority concept. The solution to this situation would be the use of IOW operators in the resolution process. In particular, the type of IOW operator that induces the ordering of the argument to be aggregated based upon the reliability of the source of information, will, on the one hand, guarantee the conservation of the rationality assumptions because they act in the same way as a WM operator, and, on the other hand, will allow the implementation of the concept of fuzzy majority because they are based on the OW operator.

Therefore, in this paper, we focus on the aggregation of preference relations based upon the reliability of the source of information (the experts). This would allow us to design a rational resolution process based on IOW aggregation that both implements the concept of fuzzy majority, and maintains the rationality assumption of the individual preference relations. Before that, we provide the basic definitions of the IOWA and IOWG operators.

3.1. The IOWA operator

Definition 1. (See [28].) An IOWA operator of dimension n is a function

$$\Phi_W : (\mathbb{R} \times \mathbb{R})^n \longrightarrow \mathbb{R},$$

to which a set of weights or weighting vector is associated, $W = (w_1, \ldots, w_n)$, such that $w_i \in [0,1]$ and $\Sigma_i w_i = 1$, and it is defined to aggregate the set of second arguments of a list of n 2-tuples $\{\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle\}$ according to the following expression,

$$\Phi_W (\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle) = \sum_{i=1}^{n} w_i \cdot p_{\sigma(i)}$$

being σ is a permutation of $\{1, \ldots, n\}$ such that $u_{\sigma(i)} \geq u_{\sigma(i+1)}$, $\forall i = 1, \ldots, n-1$, i. e., $\langle u_{\sigma(i)}, p_{\sigma(i)}\rangle$ is the 2-tuple with $u_{\sigma(i)}$ the ith highest value in the set $\{u_1, \ldots, u_n\}$.

In the above definition the reordering of the set of values to be aggregated, $\{p_1, \ldots, p_n\}$, is induced by the reordering of the set of values $\{u_1, \ldots, u_n\}$ associated to them, which is based upon their magnitude. Due to this use of the set of values $\{u_1, \ldots, u_n\}$, Yager and Filev called them the values of an order inducing variable and $\{p_1, \ldots, p_n\}$ the values of the argument variable [27, 28, 29]. As we have mentioned, the main difference between the OWA operator and the IOWA operator resides in the reordering step of the argument variable. In the case of OWA operator this reordering is based upon the magnitude of the values to be aggregated, while in the case of IOWA operator an order inducing variable has to be defined as the criterion to induce that reordering.

An immediate consequence of this definition is that if the order inducing variable is the argument variable then the IOWA operator is reduced to the OWA operator. A detailed list of properties of the IOWA operator and some of their uses can be consulted in [27, 28, 29].

Note 1. In this paper we focus on the aggregation of numerical preferences, which is why we assume that the nature of the first argument of the IOWA operators is also numeric, although it could be linguistic [27, 28, 29].

Note 2. In the case of using an IOWA operator in the aggregation phase of a GDM problem, the concept of fuzzy majority can be implemented by means of *fuzzy linguistic quantifiers* [30]. When a fuzzy linguistic quantifier Q is used to compute the weights of the IOWA operator Φ, then it is symbolized by Φ_Q.

Example 1. Suppose three experts provide the following fuzzy preference relations on a set of three alternatives

$$P^1 = \begin{pmatrix} 0.5 & 0.75 & 0.87 \\ 0.25 & 0.5 & 0.66 \\ 0.13 & 0.34 & 0.5 \end{pmatrix} \quad P^2 = \begin{pmatrix} 0.5 & 0.66 & 0.94 \\ 0.34 & 0.5 & 0.87 \\ 0.06 & 0.13 & 0.5 \end{pmatrix} \quad P^3 = \begin{pmatrix} 0.5 & 0.66 & 0.75 \\ 0.34 & 0.5 & 0.66 \\ 0.25 & 0.34 & 0.5 \end{pmatrix},$$

and have the following values associated to them $\mathbf{b} = (0.65, 0.13, 0.22)$. Using them to induce the ordering of the fuzzy preference values to be aggregated, and the fuzzy linguistic quantifier *"most of"*, we obtain the following collective fuzzy preference relation

$$P^c = \Phi_{\text{most}}\left(\langle 0.65, P^1\rangle, \langle 0.13, P^2\rangle, \langle 0.22, P^3\rangle\right) = \begin{pmatrix} 0.5 & 0.67 & 0.81 \\ 0.33 & 0.5 & 0.72 \\ 0.19 & 0.28 & 0.5 \end{pmatrix}.$$

For example, the value p^c_{13} is obtained as follows:

$$\begin{aligned} p^c_{13} &= \Phi_{\text{most}}\left(\langle 0.65, 0.87\rangle, \langle 0.13, 0.94\rangle, \langle 0.22, 0.75\rangle\right) \\ &= \frac{1}{15} \cdot 0.87 + \frac{10}{15} \cdot 0.75 + \frac{4}{15} \cdot 0.94 = \frac{12.13}{15} \simeq 0.81. \end{aligned}$$

3.2. The IOWG operator

Suppose that we want to aggregate a set of two-tuples $\{\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle\}$ where $\{u_1, \ldots, u_n\}$ is the set of order inducing values associated to the set of argument values $\{a_1, \ldots, a_n\}$, which are given on the basis of a positive ratio-scale. In this case, we can use the IOWA operator on the set $\{\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle\}$, where the argument values $\{p_1, \ldots, p_n\}$ are obtained using the transformation function f (see Appendix A.2), i.e., $p_i = f(a_i) = \frac{1}{2}(1 + \log_9 a_i)$. Thus, we obtain:

$$p = \Phi_W\left(\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle\right) = \sum_{i=1}^{n} w_i \cdot p_{\sigma(i)}$$

where $\langle u_{\sigma(i)}, p_{\sigma(i)}\rangle$ is the two-tuple with $u_{\sigma(i)}$ the ith highest value in the set $\{u_1, \ldots, u_n\}$.

The set of two-tuples $\{\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle\}$ and $\{\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle\}$ have the same set of order inducing values, and therefore the same order of the arguments $\{a_1, \ldots, a_n\}$ and $\{p_1, \ldots, p_n\}$ is induced:

$$\begin{aligned} p &= \sum_{i=1}^{n} w_i \cdot \frac{1}{2}\left(1 + \log_9 a_{\sigma(i)}\right) = \frac{1}{2}\left(\sum_{i=1}^{n} w_i + \sum_{i=1}^{n} w_i \cdot \log_9 a_{\sigma(i)}\right) \\ &= \frac{1}{2}\left(1 + \sum_{i=1}^{n} \log_9 \left(a_{\sigma(i)}\right)^{w_i}\right) = \frac{1}{2}\left(1 + \log_9 \prod_{i=1}^{n} \left(a_{\sigma(i)}\right)^{w_i}\right). \end{aligned}$$

This last expression justifies the definition of the IOWG operator as follows:

Definition 2. (See [10].) An IOWG operator of dimension n is a function

$$\Phi^G_W : (\mathbb{R} \times \mathbb{R}_+)^n \longrightarrow \mathbb{R}_+,$$

to which a set of weights or weighting vector is associated, $W = (w_1, \ldots, w_n)$, such that $w_i \in [0, 1]$ and $\Sigma_i w_i = 1$, and it is defined to aggregate the set of second arguments of a list of n two-tuples $\{\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle\}$, given on the basis of a positive ratio-scale, according to the following expression,

$$\Phi_W^G(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle) = \prod_{i=1}^{n} \left(a_{\sigma(i)}\right)^{w_i}$$

being σ a permutation of $\{1, \ldots, n\}$ verifying $u_{\sigma(i)} \geq u_{\sigma(i+1)}$, $\forall i=1, \ldots, n-1$, that is $\langle u_{\sigma(i)}, a_{\sigma(i)}\rangle$ is the two-tuple with $u_{\sigma(i)}$ the ith highest value in the set $\{u_1, \ldots, u_n\}$.

Example 2. Suppose a set of three experts provide the following multiplicative preference relations on a set of three alternatives

$$A^1 = \begin{pmatrix} 1 & 3 & 5 \\ 1/3 & 1 & 2 \\ 1/5 & 1/2 & 1 \end{pmatrix} \quad A^2 = \begin{pmatrix} 1 & 2 & 7 \\ 1/2 & 1 & 5 \\ 1/7 & 1/5 & 1 \end{pmatrix} \quad A^3 = \begin{pmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 2 \\ 1/3 & 1/2 & 1 \end{pmatrix}.$$

If these experts are associated with the following values $\mathbf{b} = (0.65, 0.13, 0.22)$, then the collective multiplicative preference relation obtained using them to induce the ordering, and the same fuzzy linguistic quantifier *"most of"*, is

$$A^c = \Phi_{\text{most}}^G\left(\langle 0.65, A^1\rangle, \langle 0.13, A^2\rangle, \langle 0.22, A^3\rangle\right) = \begin{pmatrix} 1 & 1.08 & 3.89 \\ 1/1.08 & 1 & 2.55 \\ 1/3.89 & 1/2.55 & 1 \end{pmatrix}.$$

4. IOW OPERATORS BASED ON THE RELIABILITY OF THE SOURCE OF INFORMATION

In this section we present a general framework for the definition of IOW operators for GDM problems based on the reliability of the source of information.

In [10, 11] we present two general studies on the IOWG and IOWA operators, respectively. Here, we present a particular study focused on the induced aggregation based on the reliability of the source of information. In particular, we study the importance IOW (I-IOW) operator, which induces the ordering of the argument values based upon the importance of the source of information; and the consistency IOW (C-IOW) operator, which induces the ordering of the argument values based upon the consistency of the source of information. These IOW operators allow the introduction of some semantics or meaning in the aggregation.

4.1. The Importance IOW operator

In many cases, each expert $e_k \in E$ has an *importance degree* associated to them. This *importance degree* can be interpreted as a fuzzy subset, $\mu_I : E \longrightarrow [0,1]$, in such a way that $\mu_I(e_k) \in [0,1]$ denotes the importance degree of the opinion provided by the expert e_k. When this is the case, we call this a heterogeneous GDM problem [19]. The general procedure for the inclusion of these importance values in the aggregation process involves the transformation of the preference values, p_{ij}^k, under the importance degree $\mu_I(e_k)$ to generate a new value, $\bar{p}_{ij}^k$. This activity is carried out by means of a transformation function g :

$$\bar{p}_{ij}^k = g\left(p_{ij}^k, \mu_I(e_k)\right).$$

Examples of functions g used in these cases include the minimum operator [19], the exponential function $g(x,y) = x^y$ [25], or generally any t-norm operator [31].

In our case, we can implement this importance degree variable by an alternative method, which consists of using it as the order inducing variable of the IOW operator to be applied in the aggregation stage of our resolution process. Thus, the ordering of the preference values is induced by the ordering of the experts from the most to the least important one. We call this importance degree based IOW operator the I-IOW operator and denote it as Φ_W^I.

Definition 3. If a set of of experts, $E = \{e_1, \ldots, e_m\}$, provide preferences about a set of alternatives, $X = \{x_1, \ldots, x_n\}$, by means of the set of preference relations, $\{R^1, \ldots, R^m\}$, and each expert e_k has an importance degree, $\mu_I(e_k) \in [0,1]$, associated to them, then an I-IOW operator of dimension n, Φ_W^I, is an IOW operator whose set of order inducing values is the set of importance degrees.

Example 3. Suppose that the importance pairwise comparisons of the set of three experts of example 1 are given in the following fuzzy preference relation

$$I = \begin{pmatrix} 0.5 & 0.87 & 0.75 \\ 0.23 & 0.5 & 0.38 \\ 0.25 & 0.62 & 0.5 \end{pmatrix}.$$

As shown in [7], the vector of importance of a consistent fuzzy preference relation induces the same ordering among the set of experts as the vector of *quantifier guided dominance degrees*, no matter which linguistic quantifier is used. For this reason, we propose to calculate the importance associated to the expert e_i as the total sum of the values of the row i, i.e., $\mu_I(e_k) = \sum_j p_{ij}^k$. The normalized vector of importance for this matrix is $\mathbf{I} = (0.46, 0.24, 0.30)$.

Using the fuzzy linguistic quantifier *"most of"*, the collective fuzzy preference relation obtained using the corresponding Importance IOWA (I-IOWA) operator Φ_{most}^I is

$$P^c = \Phi_{\text{most}}^I\left(\langle 0.46, P^1\rangle, \langle 0.24, P^2\rangle, \langle 0.30, P^3\rangle\right) = \begin{pmatrix} 0.5 & 0.67 & 0.81 \\ 0.33 & 0.5 & 0.72 \\ 0.19 & 0.28 & 0.5 \end{pmatrix}$$

whose elements can be considered as the preference of one alternative over another for *most of the more important experts.*

Example 4. Suppose that we have a set of three experts $E = \{e_1, e_2, e_3\}$ and a set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$. Suppose that the importance pairwise comparisons of these three experts are given in the following reciprocal multiplicative preference relation

$$I = \begin{pmatrix} 1 & 6 & 4 \\ 1/6 & 1 & 3 \\ 1/4 & 1/3 & 2 \end{pmatrix}.$$

According to Saaty, the next step would be the computation of a vector of priorities, in our case of importance, from the given matrix, for which the principal eigenvector is computed and normalized. The vector of importance for this matrix is $\mathbf{I} = (0.70, 0.19, 0.11)$.

Suppose that these experts provide the following reciprocal multiplicative preference relation on the set of alternatives

$$A^1 = \begin{pmatrix} 1 & 6 & 6 & 3 \\ 1/6 & 1 & 4 & 3 \\ 1/6 & 1/4 & 1 & 1/2 \\ 1/3 & 1/3 & 2 & 1 \end{pmatrix} A^2 = \begin{pmatrix} 1 & 6 & 6 & 8 \\ 1/6 & 1 & 2 & 3 \\ 1/6 & 1/2 & 1 & 1/2 \\ 1/8 & 1/3 & 2 & 1 \end{pmatrix} A^3 = \begin{pmatrix} 1 & 1/5 & 1/3 & 1 \\ 5 & 1 & 4 & 1/5 \\ 3 & 1/4 & 1 & 1/4 \\ 1 & 1/5 & 4 & 1 \end{pmatrix}.$$

Using the the fuzzy linguistic quantifier *"most of"*, the collective multiplicative preference relation using the I-IOWG operator Φ^{I-G}_{most} is:

$$A^c = \Phi^{I-G}_{\text{most}} \left(\langle 0.701, A^1 \rangle, \langle 0.193, A^2 \rangle, \langle 0.106, A^3 \rangle \right) = \begin{pmatrix} 1 & 2.42 & 3.65 & 4.3 \\ 0.41 & 1 & 2.52 & 1.46 \\ 0.27 & 0.4 & 1 & 0.42 \\ 0.23 & 0.68 & 2.38 & 1 \end{pmatrix}$$

whose elements can be considered as the preference of one alternative over another for *most of the more important experts.*

4.2. The Consistency IOW operator

When the experts have equal importance, i. e., in a homogeneous GDM problem, the I-IOW operator is reduced to the Average Mean (AM) operator. Thus, in this case the application of the I-IOW operator does not introduce any new meaning and its application is not advisable. However, in a homogeneous situation, each expert can always have a consistency index (CI) value associated to them, with the following interpretation: the closer CI to 0 the more consistent the expert is. Usually, for each expert this consistency index value is obtained by analyzing the consistency of the preference relation provided. These values can be used as the order inducing variable in the aggregation of preferences by means of IOW operators. In this case, we call this a C-IOW operator.

Definition 4. If a set of experts, $E = \{e_1, \dots, e_m\}$, provides preferences about a set of alternatives, $X = \{x_1, \dots, x_n\}$, by means of the preference relations, $\{R^1, \dots, R^m\}$, then a C-IOW operator of dimension n, Φ_W^C, is an IOW operator whose set of order inducing values is the set of consistency index values, $\{CI^1, \dots, CI^m\}$, associated to the set of experts.

Example 5. (See [21].) Suppose a set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$ and a set of four experts $E = \{e_1, e_2, e_3, e_4\}$, whose fuzzy preference relations on X are:

$$P^1 = \begin{pmatrix} 0.5 & 0.3 & 0.7 & 0.1 \\ 0.7 & 0.5 & 0.6 & 0.6 \\ 0.3 & 0.4 & 0.5 & 0.2 \\ 0.9 & 0.4 & 0.8 & 0.5 \end{pmatrix} \qquad P^2 = \begin{pmatrix} 0.5 & 0.4 & 0.6 & 0.2 \\ 0.6 & 0.5 & 0.7 & 0.4 \\ 0.4 & 0.3 & 0.5 & 0.1 \\ 0.8 & 0.6 & 0.9 & 0.5 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0.5 & 0.5 & 0.7 & 0 \\ 0.5 & 0.5 & 0.8 & 0.4 \\ 0.3 & 0.2 & 0.5 & 0.2 \\ 1 & 0.6 & 0.8 & 0.5 \end{pmatrix} \qquad P^4 = \begin{pmatrix} 0.5 & 0.4 & 0.7 & 0.8 \\ 0.6 & 0.5 & 0.4 & 0.3 \\ 0.3 & 0.6 & 0.5 & 0.1 \\ 0.7 & 0.7 & 0.9 & 0.5 \end{pmatrix}.$$

The consistency indexes are $\mathbf{CI} = (-0.6, -0.14, -0.73, -0.77)$. The collective fuzzy preference relation obtained by using a C-IOWA operator guided by the same linguistic quantifier *"most of"*, with weighting vector $(0, 0.4, 0.5, 0.1)$, is

$$\begin{aligned} P^c &= \Phi^C_{\text{most}}\left(\langle -0.6, P^1\rangle, \langle -0.14, P^2\rangle, \langle -0.73, P^3\rangle, \langle -0.77, P^4\rangle\right) \\ &= \begin{pmatrix} 0.5 & 0.41 & 0.7 & 0.12 \\ 0.59 & 0.5 & 0.68 & 0.47 \\ 0.3 & 0.32 & 0.5 & 0.19 \\ 0.88 & 0.53 & 0.81 & 0.5 \end{pmatrix}, \end{aligned}$$

whose elements can be considered as the preference of one alternative over another for *most of the more consistent experts.*

Example 6. If we take the same data as in Example 2, the consistency index values associated to these experts are $\mathbf{CI} = (0.002, 0.007, 0.005)$, and the collective multiplicative preference relation obtained by using a C-IOWG operator guided by the same linguistic quantifier *"most of"* is

$$A^c = \Phi^{C-G}_{\text{most}}\left(\langle -0.002, A^1\rangle, \langle -0.007, A^2\rangle, \langle -0.005, A^3\rangle\right) = \begin{pmatrix} 1 & 1.08 & 3.89 \\ 0.93 & 1 & 2.55 \\ 0.26 & 0.39 & 1 \end{pmatrix}$$

whose elements can be interpreted as the preference intensity, measured in $[1/9, 9]$ [22], of one alternative over another for *most of the more consistent experts.*

5. RATIONALITY OF IOW OPERATORS BASED ON THE RELIABILITY OF THE SOURCE OF INFORMATION

In GDM models we normally assume that preference relations are reciprocal. However, it is well known that reciprocity is not generally maintained after aggregation is carried out in the resolution process [9, 17]. An aggregation operator that maintains the rationality assumption is called a rational aggregation operator. In the following, we study the conditions needed to guarantee that an aggregation operator is rational.

Definition 5. An aggregation operator F of preference relations is a rational aggregation operator when it maintains the indifference, the reciprocity and the consistency properties.

A desirable property to be verified by a rational aggregation operator is that a small change in the arguments to be aggregated should produce a small change in the value of the operator. In other words, we consider rational aggregation operators to be continuous.

Assuming that F is a rational aggregation operator and $R^c = (r^c_{ij})$ is the collective preference relation obtained from the set of m individual consistent preference relations $\{R^1, \ldots, R^m\}$, the above definition of a rational aggregation operator implies that F has to verify the following properties:

1. Fuzzy preference relations:

   (a) *Indifference property:* $r^c_{ii} = 0.5 \ \forall \ i = 1, 2, \ldots, n$. In terms of function F :

   $$F(0.5, 0.5, \ldots, 0.5) = 0.5.$$

   (b) *Reciprocity property:* $r^c_{ij} + r^c_{ji} = 1 \ \forall \ i, j = 1, 2, \ldots, n$. In terms of function F :

   $$F(r^1_{ij}, r^2_{ij}, \ldots, r^m_{ij}) + F(r^1_{ji}, r^2_{ji}, \ldots, r^m_{ji}) = 1.$$

   (c) *Additive consistency property:* $r^c_{ij} + r^c_{jk} + r^c_{ki} = 1.5 \ \forall \ i, j = 1, 2, \ldots, n$. In terms of function F :

   $$F(r^1_{ij}, r^2_{ij}, \ldots, r^m_{ij}) + F(r^1_{jk}, r^2_{jk}, \ldots, r^m_{jk}) + F(r^1_{ki}, r^2_{ki}, \ldots, r^m_{ki}) = 1.5$$

2. *Multiplicative preference relations:*

   (a) *Indifference property:* $r^c_{ii} = 1 \ \forall \ i = 1, 2, \ldots, n$, or equivalently

   $$F(1, 1, \ldots, 1) = 1$$

   (b) *Reciprocity property:* $r^c_{ij} \cdot r^c_{ji} = 1 \ \forall \ i, j = 1, 2, \ldots, n$, or equivalently

   $$F(r^1_{ij}, r^2_{ij}, \ldots, r^m_{ij}) \cdot F(r^1_{ji}, r^2_{ji}, \ldots, r^m_{ji}) = 1.$$

(c) *Multiplicative consistency property:* $r_{ij}^c \cdot r_{jk}^c \cdot r_{ki}^c = 1 \; \forall \; i,j = 1,2,\ldots,n$, or equivalently

$$F(r_{ij}^1, r_{ij}^2, \ldots, r_{ij}^m) \cdot F(r_{jk}^1, r_{jk}^2, \ldots, r_{jk}^m) \cdot F(r_{ki}^1, r_{ki}^2, \ldots, r_{ki}^m) = 1.$$

The following result holds:

Proposition 1. A continuous aggregation operator of fuzzy preference relations that maintains the additive consistency property is a rational aggregation operator.

Proof. Assuming that F is a continuous aggregation operator verifying

$$F(r_{ij}^1, r_{ij}^2, \ldots, r_{ij}^m) + F(r_{jk}^1, r_{jk}^2, \ldots, r_{jk}^m) + F(r_{ki}^1, r_{ki}^2, \ldots, r_{ki}^m) = 1.5,$$

firstly, we have

$$F(0.5, 0.5, \ldots, 0.5) + F(0.5, 0.5, \ldots, 0.5) + F(0.5, 0.5, \ldots, 0.5) = 1.5$$

and thus $F(0.5, 0.5, \ldots, 0.5) = 0.5$, that is, F maintains the indifference property. Secondly, taking $(r_{jk}^1, r_{jk}^2, \ldots, r_{jk}^m) = (0.5, 0.5, \ldots, 0.5)$ and $k = j$ we have

$$F(r_{ij}^1, r_{ij}^2, \ldots, r_{ij}^m) + F(0.5, 0.5, \ldots, 0.5) + F(r_{ji}^1, r_{ji}^2, \ldots, r_{ji}^m) = 1.5,$$

which implies

$$F(r_{ij}^1, r_{ij}^2, \ldots, r_{ij}^m) + F(r_{ji}^1, r_{ji}^2, \ldots, r_{ji}^m) = 1,$$

that is, F maintains the reciprocity property. □

The dual result for multiplicative preference relations also holds:

Proposition 2. An aggregation operator of multiplicative preference relations that maintains the multiplicative consistency property is a rational aggregation operator.

The following result characterises the aggregation operators that maintain the reciprocity property, and therefore, also the indifference property.

Proposition 3. An aggregation operator verifying

$$F(1 - x_1, \ldots, 1 - x_n) = 1 - F(x_1, \ldots, x_n) \; \forall \, (x_1, \ldots, x_n) \in \mathbb{R}^n$$

$$\left(F(1/x_1, \ldots, 1/x_n) = 1/F(x_1, \ldots, x_n) \; \forall \, (x_1, \ldots, x_n) \in \mathbb{R}_+^n\right)$$

maintains the additive (multiplicative) reciprocity.

Proof. If F verifies

$$F(1 - x_1, \ldots, 1 - x_n) = 1 - F(x_1, \ldots, x_n) \; \forall \, (x_1, \ldots, x_n) \in \mathbb{R}^n$$

then

$$
\begin{aligned}
F(r_{ij}^1, r_{ij}^2, \dots, r_{ij}^m) + F(r_{ji}^1, r_{ji}^2, \dots, r_{ji}^m) &= F(r_{ij}^1, r_{ij}^2, \dots, r_{ij}^m) \\
&\quad + F(1 - r_{ij}^1, 1 - r_{ij}^2, \dots, 1 - r_{ij}^m) \\
&= F(r_{ij}^1, r_{ij}^2, \dots, r_{ij}^m) + 1 - F(r_{ij}^1, r_{ij}^2, \dots, r_{ij}^m) \\
&= 1.
\end{aligned}
$$

that is, F maintains the reciprocity property.

The proof for the multiplicative case follows a similar reasoning to this one. □

In the case of OW operators guided by a linguistic quantifier, in [9, 17] a similar result was obtained:

Proposition 4. If Q is a linguistic quantifier with a membership function verifying

$$Q(1 - x) = 1 - Q(x), \ \forall x$$

then the collective preference relation, obtained by aggregating a set of reciprocal preference relations, using an OW operator guided by Q, is reciprocal.

Moreover, in the case of Q being a non-decreasing relative fuzzy quantifier with membership function:

$$Q(x) = \begin{cases} 0 & 0 \leq x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & b < x \leq 1 \end{cases}$$

$a, b \in [0, 1]$, the following characterization result was also obtained in [9, 17]:

Proposition 5. If Q is a relative non-decreasing linguistic quantifier with parameters a and b then the collective preference relation, obtained by aggregating a set of reciprocal preference relations, using an OW operator guided by Q, is reciprocal if and only if $a + b = 1$.

The corresponding results to guarantee the consistency property are not as straightforward as the previous ones. However, the following general condition guarantees both reciprocity and consistency properties (see Appendix A).

Proposition 6. A $+$-separable ($\times$-separable) mean aggregation operator is a rational aggregation operator for fuzzy (multiplicative) preference relations.

Proof. If $X = (r_{ij}^1, r_{ij}^2, \dots, r_{ij}^m)$, $Y = (r_{jk}^1, r_{jk}^2, \dots, r_{jk}^m)$ and $Z = (r_{ki}^1, r_{ki}^2, \dots, r_{ki}^m)$, then the consistency of individual fuzzy preference relations imply

$$X + Y + Z = (r_{ij}^1 + r_{jk}^1 + r_{ki}^1, r_{ij}^2 + r_{jk}^2 + r_{ki}^2, \dots, r_{ij}^m + r_{jk}^m + r_{ki}^m) = (1.5, 1.5, \dots, 1.5).$$

Because F is +-separable then:

$$F(1.5, 1.5, \ldots, 1.5) = F((X+Y)+Z) = F(X+Y)+F(Z) = F(X)+F(Y)+F(Z).$$

F is a mean operator then $F(X+Y+Z) = 1.5$. All these considerations together imply that

$$F(r_{ij}^1, r_{ij}^2, \ldots, r_{ij}^m) + F(r_{jk}^1, r_{jk}^2, \ldots, r_{jk}^m) + F(r_{ki}^1, r_{ki}^2, \ldots, r_{ki}^m) = 1.5,$$

which proves the rationality of the operator F.

The proof in the case of a ×-separable mean aggregation operator follows a similar reasoning. □

The following result characterises the +-separable (×-separable) continuous mean aggregation operators.

Proposition 7. A +-separable (×-separable) continuous mean aggregation operator is a weighted averaging (geometric) operator.

Proof. We will prove only the part corresponding to +-separable, as the ×-separable part is straightforward using the logarithmic function. Therefore, we have to prove that if a continuous mean aggregation operator verifies

$$F(X+Y) = F(X) + F(Y) \quad \forall X, Y \in \mathbb{R}^n$$

then

$$F(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i \cdot x_i,$$

with $W = (w_1, \ldots, w_n)$ a weighting vector verifying $\sum_{i=1}^{n} w_i = 1$.

Proof by induction will be used.

1. *Basis:* For $n = 1$, we have that $F(x+y) = F(x) + F(y)\ \forall x, y \in \mathbb{R}$. Because F is continuous then there exists a constant $a \in \mathbb{R}$ such that $F(x) = ax\ \forall x \in \mathbb{R}$ [1, 2].

2. *Induction hypothesis:* Let's assume that any +-separable continuous mean aggregation operator of dimension k is a weighted averaging operator, i.e., $F(X+Y) = F(X) + F(Y)\ \ \forall X, Y \in \mathbb{R}^k$ then

$$F(x_1, \ldots, x_k) = \sum_{i=1}^{k} w_i \cdot x_i,$$

with $W = (w_1, \ldots, w_k)$ a weighting vector verifying $\sum_{i=1}^{n} w_i = 1$.

3. *Induction step:* Let's assume that F is a +-separable continuous mean aggregation operator of dimension $k+1$ such that $F(X+Y) = F(X) + F(Y) \ \ \forall X, Y \in \mathbb{R}^{k+1}$. Then, we have:

$$\begin{aligned} F(x_1, \dots, x_k, x_{k+1}) &= F\left((x_1, \dots, x_k, 0) + (0, \dots, 0, x_{k+1})\right) \\ &= F(x_1, \dots, x_k, 0) + F(0, \dots, 0, x_{k+1}). \end{aligned}$$

We define $G(x_1, \dots, x_k) = F(x_1, \dots, x_k, 0)$ and $H(x_{k+1}) = F(0, \dots, 0, x_{k+1})$.

Function G is a +-separable continuous aggregation operator of dimension k, and therefore, by applying the induction hypothesis, we obtain: $G(x_1, \dots, x_k) = \sum_{i=1}^{k} w_i \cdot x_i$.

Function H verifies $H(x+y) = H(x) + H(y) \ \forall x, y \in \mathbb{R}$, and therefore, there exists a constant $w_{k+1} \in \mathbb{R}$ such that $H(x) = w_{k+1} \cdot x \ \forall x \in \mathbb{R}$.

Both results together imply:

$$F(x_1, \dots, x_k, x_{k+1}) = \sum_{i=1}^{k} w_i \cdot x_i + w_{k+1} \cdot x_{k+1} = \sum_{i=1}^{k+1} w_i \cdot x_i$$

Finally, because F is a mean aggregation operator $1 = F(1, \dots, 1, 1) = \sum_{i=1}^{k+1} w_i$. □

This result guarantees that the IOW operators that induce the ordering of the arguments based on the reliability of the information source are rational aggregation operators. In particular, both the I-IOW and C-IOW operators are rational aggregation operators. We note that the same cannot be assured in the case of OW operators as has been shown in [9, 17].

6. CONCLUDING REMARKS

In this paper we have studied the use of IOW operators in the aggregation of preference relations in GDM problems: the I-IOW operator, which induces the ordering of the argument values based upon the importance of the source of information; and the C-IOW operator, which induces the ordering of the argument values based upon the consistency of the source of information.

Conditions have been given to assure the rationality of an aggregation operator of preference relations. In particular, we have shown that IOW operators inducing the ordering of the arguments based on the reliability of the source of information are rational ones.

APPENDIX A. ORDERED WEIGHTED OPERATORS

In this appendix, we present the OWA operator used to aggregate measurements given on a difference scale. When the measurements are given on a ratio-scale, the OWG Operator is the appropriate one. Firstly, we set out some definitions of aggregation operators.

Definition A.1. A function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called an aggregation operator of dimension n.

Definition A.2. An aggregation operator of dimension n is $*$-separable if

$$F(X * Y) = F(X) * F(Y) \;\; \forall X, Y \in \mathbb{R}^n$$

where $* : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous, commutative and associative operation such that $x * z = y * z \; \forall z \in \mathbb{R}$ then $x = y$.

Definition A.3. An aggregation operator $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a mean operator if

$$\min\{x_1, \dots, x_n\} \leq F(x_1, \dots, x_n) \leq \max\{x_1, \dots, x_n\}.$$

In the following we define the two special cases of weighted mean operators:

Definition A.4. An aggregation operator $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a weighted averaging operator if

$$F(x_1, \dots, x_n) = \sum_{i=1}^{n} w_i \cdot x_i,$$

with $W = (w_1, \dots, w_n)$ a weighting vector verifying $\sum_{i=1}^{n} w_i = 1$.

Definition A.5. An aggregation operator $F : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+$ is a weighted geometric operator if

$$F(x_1, \dots, x_n) = \prod_{i=1}^{n} {x_i}^{w_i},$$

with $W = (w_1, \dots, w_n)$ a weighting vector verifying $\sum_{i=1}^{n} w_i = 1$.

Weighted averaging operators are separable with respect to the addition operation (+-separable), while weighted geometric operators are separable with respect to the product operation ($\times$-separable).

APPENDIX A.1. THE OWA OPERATOR

In [7] Chiclana et al. considered GDM problems where the information about the alternatives is represented using *fuzzy preference relations* and designed a *fuzzy majority* guided choice scheme that follows two steps to achieve a final decision from the synthesis of performance degrees of the majority of criteria: i) aggregation and ii) exploitation. This choice scheme is based on the OWA operator [26].

Definition A.6. (See [26].) An OWA operator of dimension n is a function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$, that has associated to it a set of weights or weighting vector $W = (w_1, \ldots, w_n)$ such that, $w_i \in [0,1]$ and $\sum_{i=1}^n w_i = 1$, and is defined to aggregate a list of values $\{p_1, \ldots, p_n\}$ according to the following expression,

$$\phi(p_1, \ldots, p_n) = \sum_{i=1}^{n} w_i \cdot p_{\sigma(i)}$$

being σ a permutation of $\{1, \ldots, n\}$ such that $p_{\sigma(i)} \geq p_{\sigma(i+1)}$, $\forall i = 1, \ldots, n-1$, that is $p_{\sigma(i)}$ is the ith highest value in the set $\{p_1, \ldots, p_n\}$.

In [26], Yager proposed two ways to obtain the weighting vector associated to an OWA operator. The first approach is to use some kind of learning mechanism using some sample data; and the second approach is to try to give some semantic meaning to the weights. In this last case, the OWA operator can be used to implement the concept of fuzzy majority in the aggregation phase by means of the *fuzzy quantifiers* [30] which are used to calculate its weights, which in the case of a non-decreasing relative quantifier Q, is expressed as follows [26]:

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right), \quad i = 1, \ldots, n.$$

When a fuzzy quantifier Q is used to compute the weights of the OWA operator ϕ, then it is symbolized by ϕ_Q.

APPENDIX A.2. THE OWG OPERATOR

The GDM problem when the experts express their preferences using multiplicative preference relations has been studied by Saaty using the decision analytic hierarchical process (AHP), which obtains the set of solution alternatives by means of the eigenvector method [22]. However, this decision process is not guided by the concept of fuzzy majority. As shown in [3, 4], the proper aggregation operator of ratio-scale measurements is not the *arithmetic mean* but the *geometric mean.* However, the geometric mean does not allow the concept of fuzzy majority to be incorporated in the decision process. Therefore, if we want to design a decision scheme for multiplicative preference relations that allows decision makers to implement the concept of fuzzy majority to obtain the final solution, then it is necessary to introduce a new class of operator to aggregate ratio-scale measurements allowing the implementation of the fuzzy majority concept.

In [7] we obtained the transformation function between multiplicative and fuzzy preference relations, which is given in the following result:

Proposition A. (See [7].) Suppose that we have a set of alternatives, $X = \{x_1, \ldots, x_n\}$, and associated with it a multiplicative reciprocal preference relation $A = (a_{ij})$, with $a_{ij} \in [1/9, 9]$ and $a_{ij} \cdot a_{ji} = 1, \forall i, j$. Then the corresponding fuzzy reciprocal preference relation, $P = (p_{ij})$, associated to A, with $p_{ij} \in [0,1]$ and $p_{ij} + p_{ji} = 1, \forall\, i,j$, is given as follows:

$$p_{ij} = f(a_{ij}) = \frac{1}{2}\left(1 + \log_9 a_{ij}\right).$$

The above transformation function is bijective and, therefore, allows us to transpose concepts that have been defined for fuzzy preference relations to multiplicative preference relations. In this way, for example, if we want to aggregate a set of values $\{a_1, \ldots, a_n\}$ given on the basis of a positive ratio-scale we can use the OWA operator on the set of values $\{p_1, \ldots, p_n\}$ obtained using the above transformation function f, i.e., $p_i = f(a_i) = \frac{1}{2}(1 + \log_9 a_i)$. Thus, we obtain:

$$p = \phi(p_1, \ldots, p_n) = \sum_{i=1}^{n} w_i \cdot p_{\sigma(i)}$$

being σ a permutation of $\{1, \ldots, n\}$ such that $p_{\sigma(i)}$ is the ith highest value in the set $\{p_1, \ldots, p_n\}$. Because f is an increasing function, then $a_{\sigma(i)}$ is the ith highest value in the set $\{a_1, \ldots, a_n\}$, and therefore

$$\begin{aligned} p = \sum_{i=1}^{n} w_i \cdot \frac{1}{2}(1 + \log_9 a_{\sigma(i)}) &= \frac{1}{2}\left(1 + \sum_{i=1}^{n} w_i \cdot \log_9 a_{\sigma(i)}\right) \\ &= \frac{1}{2}\left(1 + \sum_{i=1}^{n} \log_9 (a_{\sigma(i)})^{w_i}\right) \\ &= \frac{1}{2}\left(1 + \log_9 \prod_{i=1}^{n} (a_{\sigma(i)})^{w_i}\right) \end{aligned}$$

This last expression justifies the definition of the OWG operator as an aggregation operator of information given on a ratio-scale:

Definition A.7. (See [8].) An OWG operator of dimension n is a function $\phi^G : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+$, to which a set of weights or weighting vector is associated, $W = (w_1, \ldots, w_n)$ such that $w_i \in [0,1]$ and $\Sigma_i w_i = 1$, and it is defined to aggregate a list of values $\{a_1, \ldots, a_n\}$ according to the following expression,

$$\phi^G(a_1, \ldots, a_n) = \prod_{i=1}^{n} \left(a_{\sigma(i)}\right)^{w_i}$$

where σ is a permutation of $\{1, \ldots, n\}$ such that $a_{\sigma(i)} \geq a_{\sigma(i+1)}$, $\forall\, i = 1, \ldots, n-1$, that is $a_{\sigma(i)}$ is the ith highest value in the set $\{a_1, \ldots, a_n\}$.

As the OWG operator is based on the OWA operator, it is clear that the weighting vector W can be obtained by the same method used in the case of the OWA operator,

i. e., the vector may be obtained using a fuzzy quantifier, Q, representing the concept of fuzzy majority. When a fuzzy quantifier Q is used to compute the weights of the OWG operator ϕ^G, then, it is symbolized by ϕ_Q^G. In [8, 17], a *fuzzy majority* guided choice scheme based on the quantifier guided OWG operator was presented.

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