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NEUTRAL FUNCTIONAL INTEGRODIFFERENTIAL CONTROL SYSTEMS IN BANACH SPACES

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Sufficient conditions for controllability of neutral functional integrodifferential systems in Banach spaces with initial condition in the phase space are established. The results are obtained by using the Schauder fixed point theorem. An example is provided to illustrate the theory.

Keywords: controllability, phase space, neutral functional integrodifferential system, Schauder fixed point theorem

AMS Subject Classification: 93B05

1. INTRODUCTION

Several authors [5–8, 12] have studied the theory of neutral functional differential equations. These type of equations occur in the study of heat conduction in materials with memory. So it is interesting to study the controllability problem for such systems. There are several papers appeared on the controllability of linear and nonlinear systems in both finite and infinite dimensional spaces. Balachandran et al [1] studied the controllability problem for nonlinear and semilinear integrodifferential systems while in [2] they established a set of sufficient conditions for the controllability of nonlinear functional differential systems in Banach spaces. Controllability of nonlinear functional integrodifferential systems in Banach spaces has been studied by Park and Han [10]. Balachandran and Sakthivel [3] discussed the controllability of neutral functional integrodifferential systems in Banach spaces by using the semigroup theory and the Schaefer fixed point theorem. Recently Han et al [4] investigated the problem of controllability of integrodifferential systems in Banach spaces in which the initial condition is in some approximated phase space. The purpose of this paper is to derive a set of sufficient conditions for the controllability of neutral functional integrodifferential systems in Banach spaces with the initial condition taken in some approximated phase space by using the Schauder fixed point theorem.

2. PRELIMINARIES

Let X be a Banach space with norm $\|\cdot\|$ and let $\mathcal{B}$ be an abstract phase space which will be defined later. Consider the neutral functional integrodifferential system of the form

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}[x(t)-g(t,x_t)] &= Ax(t)+f\left(t,x_t,\int_0^t k(t,s,x_s)\,\mathrm{d}s\right)+Bu(t), \quad t\geq 0,\\ x_0 &= \phi\in\mathcal{B}, \end{aligned} \tag{1}$$

where the state $x(\cdot)$ takes values in the Banach space X, x_t represents the function $x_t : (-h,0] \to X$ defined by $x_t(\theta) = x(t+\theta)$, $-h < \theta \leq 0$ which belongs to the phase space $\mathcal{B}$, the control $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space and $J = [0,b]$ and B is a bounded linear operator from U into X. Here $A : D(A) \to X$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$ on X and the nonlinear operators $g : J\times\mathcal{B} \to X$, $k : J\times J\times\mathcal{B} \to X$ and $f : J\times\mathcal{B}\times X \to X$ are given.

The axiomatic definition of the phase space $\mathcal{B}$ is introduced by Hale and Kato [5]. The phase space $\mathcal{B}$ is a linear space of functions mapping $(-h,0] \to X$ endowed with the seminorm $|\cdot|$ and assume that $\mathcal{B}$ satisfies the following axioms [9]:

(A1) If $x : (-h,b) \to X$, $b > 0$ is continuous on $[0,b)$ and $x_0 \in \mathcal{B}$, then for every t in $[0,b)$ the following conditions hold:

- (i) x_t is in $\mathcal{B}$.
- (ii) $\|x(t)\| \leq H|x_t|$.
- (iii) $|x_t| \leq K(t)\sup\{\|x(s)\| : 0 \leq s \leq t\} + M(t)|x_0|$.

Here $H \geq 0$ is a constant, $K, M : [0,\infty) \to [0,\infty)$, K is locally bounded and H, K and M are independent of $x(\cdot)$.

(A2) For the function $x(\cdot)$ in (A1), x_t is a $\mathcal{B}$ valued continuous function on $[0,b)$.

(A3) The space $\mathcal{B}$ is complete.

Let $B_r[x]$ be the closed ball with center at x and radius r. We shall assume the following hypotheses:

(i) A is the infinitesimal generator of a C_0 semigroup $T(t)$ and there exist constants $M_1, M_2 > 0$ such that $\|T(t)\| \leq M_1$ and $\|AT(t)\| \leq M_2$.

(ii) There exists a positive constant $0 < b_0 < b$ and for each $0 < t \leq b_0$, there is a compact set $V_t \subset X$ such that $T(t)\,f\left(s,\eta,\int_0^s k(s,\tau,\eta)\,\mathrm{d}\tau\right)$, $AT(t)g(s,\eta)$, $T(t)\,Bu(s) \in V_t$ for every $\eta \in B_r[\phi]$ and all $0 \leq s \leq b_0$.

(iii) $g : J\times\mathcal{B} \to X, k : J\times J\times\mathcal{B} \to X$ and $f : J\times\mathcal{B}\times X \to X$ are continuous and there exist constants $0 \leq r_1 \leq r$, $K_1 \geq 0$ and $K_2 \geq 0$ such that

$$\left\|f(s,\eta,\int_0^s k(s,\tau,\eta)\,\mathrm{d}\tau)\right\| \leq K_1$$

and

$$\|g(s,\eta)\| \le K_2,$$

for every $0 \le s \le b_0$ and $\eta \in B_{r_1}[\phi]$.

(iv) The linear operator W from $L^2(J,U)$ into X defined by

$$Wu = \int_0^b T(b-s)\,Bu(s)\,\mathrm{d}s$$

induces an invertible operator $\tilde{W}$ defined on $L^2(J,U)/kerW$ and there exist positive constants $L_1, L_2 > 0$ such that $\|\tilde{W}^{-1}\| \le L_1$ and $\|B\| \le L_2$. The construction of the bounded invertible operator $\tilde{W}^{-1}$ in general Banach space is outlined in the Remark.

Then the mild solution of the system (1) is given by [8]

$$\begin{aligned} x(t) &= T(t)[\phi(0) - g(0,\phi)] + g(t,x_t) + \int_0^t AT(t-s)\,g(s,x_s)\,\mathrm{d}s \\ &\quad + \int_0^t T(t-s)\left[Bu(s) + f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)\,\mathrm{d}\tau\right)\right]\mathrm{d}s. \end{aligned} \tag{2}$$

We say the system (1) is said to be *controllable* on the interval J, if for every initial function $\phi \in \mathcal{B}$, $x_1 \in X$, there exists a control $u \in L^2(J,U)$ such that the solution $x(\cdot)$ of (1) satisfies $x(b) = x_1$.

3. MAIN RESULT

Theorem. If the hypotheses (i) – (iv) are satisfied, then the system (1) is controllable on J.

Proof. Let $y(\cdot,\phi) : (-h,b] \to X$ be the function defined by

$$y(t) = \begin{cases} \phi(t) & -h < t < 0, \\ T(t)\,\phi(0) & t \ge 0. \end{cases}$$

From the axioms of $\mathcal{B}$, we see that the map $t \to y_t$ is continuous and for every $0 < r_2 < r_1$ there exists $b_1 > 0$ such that

$$|y_t - \phi| \le \frac{r_2}{2}, \text{ for all } 0 \le t \le b_1.$$

Let b be any constant such that

$$0 < b \le \min\left\{b_1, \frac{1}{N^* K'}\left[r_1 - \frac{r_2}{2} - K_2(1+M_1)\right]\right\},$$

where $N^* = M_2K_2 + M_1L_1L_2[\|x_1\| + M_1[\|\phi(0)\| + K_2] + K_2 + bM_2K_2 + bM_1K_1] + M_1K_1$ and $K' = \max\{K(t);\, 0 \le t \le b\}$.

Using the hypothesis (iv), for an arbitrary function $x(\cdot)$ define the control

$$\begin{aligned} u(t) &= \tilde{W}^{-1}\Big[x_1 - T(b)[\phi(0) - g(0,\phi)] - g(b,x_b) - \int_0^b AT(b-s)\, g(s,x_s)\, \mathrm{d}s \\ &\quad - \int_0^b T(b-s)\, f\Big(s, x_s, \int_0^s k(s,\tau,x_\tau)\, \mathrm{d}\tau\Big)\, \mathrm{d}s\Big]\,(t). \end{aligned} \tag{3}$$

Let Y be the space of all functions $x : (-h,b] \to X$ such that $x_0 \in \mathcal{B}$ and the restriction $x : [0,b] \to X$ is continuous and let $|||\cdot|||$ be the seminorm in Y defined by

$$|||x||| = |x_0| + \sup\{\|x(s)\|;\ 0 \le s \le b\}.$$

Define

$$Y_b = \{x \in Y : |x_0 - \phi| = 0 \text{ and } |x_t - \phi| \le r_1,\ 0 \le t \le b\}.$$

Clearly Y_b is a nonempty, bounded, convex and closed subset of Y. Using the control $u(t)$, we shall show that the operator Ψ defined on Y_b by

$$\begin{aligned} \Psi x(t) &= T(t)[\phi(0) - g(0,\phi)] + g(t,x_t) + \int_0^t AT(t-s)\, g(s,x_s)\, \mathrm{d}s \\ &\quad + \int_0^t T(t-s)\, B\tilde{W}^{-1}[x_1 - T(b)[\phi(0) - g(0,\phi)] - g(b,x_b) \\ &\quad - \int_0^b AT(b-\tau)g(\tau,x_\tau)\, \mathrm{d}\tau - \int_0^b T(b-\tau)\, f\left(\tau, x_\tau, \int_0^\tau k(\tau,\theta,x_\theta)\, \mathrm{d}\theta\right)\, \mathrm{d}\tau](s)\, \mathrm{d}s \\ &\quad + \int_0^t T(t-s)\, f(s, x_s, \int_0^s k(s,\tau,x_\tau)\, \mathrm{d}\tau)\, \mathrm{d}s \end{aligned} \tag{4}$$

has a fixed point. This fixed point is then a solution of (2). Clearly $\Psi x(b) = x_1$, which means that the control u steers the system from the initial function ϕ to x_1 at time b, provided we can obtain a fixed point of the nonlinear operator Ψ.

First we show that Ψ maps Y_b into Y_b. Take

$$\begin{aligned} v_1(t) &= -T(t)g(0,\phi) + g(t,x_t). \\ v_2(t) &= \int_0^t AT(t-s)\, g(s,x_s)\, \mathrm{d}s. \\ v_3(t) &= \int_0^t T(t-s)\, B\tilde{W}^{-1}[x_1 - T(b)\,[\phi(0) - g(0,\phi)] - g(b,x_b) \\ &\quad - \int_0^b AT(b-\tau)\, g(\tau,x_\tau)\, \mathrm{d}\tau - \int_0^b T(b-\tau)\, f\left(\tau, x_\tau, \int_0^\tau k(\tau,\theta,x_\theta)\, \mathrm{d}\theta\right)\, \mathrm{d}\tau]\,(s)\, \mathrm{d}s \\ &\quad + \int_0^t T(t-s) f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)\, \mathrm{d}\tau\right)\, \mathrm{d}s. \end{aligned}$$

Then

$$\Psi x(t) = y(t) + v_1(t) + v_2(t) + v_3(t).$$

Let us consider $z = \Psi x$, then we can write the map as

$$z = y + v_1 + v_2 + v_3.$$

We have

$$\begin{aligned} |z_t - \phi| &\le |y_t - \phi| + |v_{1t}| + |v_{2t}| + |v_{3t}| \\ &\le \frac{r_2}{2} + |v_{1t}| + |v_{2t}| + |v_{3t}|. \end{aligned}$$

Now,

$$\begin{aligned} & |v_{1t}| + |v_{2t}| + |v_{3t}| \\ \le\ & K(t)\sup\{\|v_1(s)\| : 0 \le s \le t\} + K(t)\sup\{\|v_2(s)\| : 0 \le s \le t\} \\ & + K(t)\sup\{\|v_3(s)\| : 0 \le s \le t\} \end{aligned}$$

and

$$\begin{aligned} & \|v_1(s)\| + \|v_2(s)\| + \|v_3(s)\| \\ \le\ & M_1K_2 + K_2 + bM_2K_2 + bM_1L_1L_2[\|x_1\| + M_1[\|\phi(0)\| + K_2] + K_2 \\ & \quad + bM_2K_2 + bM_1K_1] + bM_1K_1 \\ \le\ & K_2(1 + M_1) + bN^* \\ \le\ & \frac{1}{K'}\left[r_1 - \frac{r_2}{2}\right]. \end{aligned}$$

Thus we have $|z_t - \phi| \le r_1$ and hence Ψ maps Y_b into itself.

Next we shall prove that Ψ maps Y_b into a precompact subset of Y_b. First we show that, for every fixed $t \in J$ the set $Y_b(t) = \{(\Psi x(t));\ x \in Y_b\}$ is precompact in X. Obviously for $t = 0$, $Y_b(0) = \{\phi(0)\}$. Let $t > 0$ be fixed and for $0 < \epsilon < t$, define

$$\begin{aligned} \Psi_\epsilon x(t) &= T(t)[\phi(0) - g(0,\phi)] + g(t, x_t) + \int_0^{t-\epsilon} AT(t-s)\, g(s, x_s)\, \mathrm{d}s \\ &+ \int_0^{t-\epsilon} T(t-s)\, B\tilde{W}^{-1}\,[x_1 - T(b)[\phi(0) - g(0,\phi)] - g(b, x_b) \\ &- \int_0^b AT(b-\tau)\, g(\tau, x_\tau)\, \mathrm{d}\tau - \int_0^b T(b-\tau)\, f\left(\tau, x_\tau, \int_0^\tau k(\tau,\theta,x_\theta)\,\mathrm{d}\theta\right)\mathrm{d}\tau\Bigg](s)\, \mathrm{d}s \\ &+ \int_0^{t-\epsilon} T(t-s)\, f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)\,\mathrm{d}\tau\right)\mathrm{d}s. \end{aligned}$$

Since $AT(t)g(s,x_s)$, $T(t)f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)\,\mathrm{d}\tau\right)$ and $T(t)\,Bu(s)$ belong to the compact set V_t, the set

$$Y_\epsilon(t) = \{\Psi_\epsilon x(t); x \in Y_b\}$$

is precompact in X for every ϵ, $0 < \epsilon < t$. Further for $x \in Y_b$, we have

$$\|\Psi x(t) - \Psi_\epsilon x(t)\|$$

$$
\begin{aligned}
&\leq \left\| \int_{t-\epsilon}^{t} AT(t-s)\, g(s,x_s)\, \mathrm{d}s \right\| \\
&\quad + \left\| \int_{t-\epsilon}^{t} T(t-s)\, B\tilde{W}^{-1} \left[x_1 - T(b)\,[\phi(0) - g(0,\phi)] - g(b,x_b) \right. \right. \\
&\quad \left. \left. - \int_0^b AT(b-s)\, g(\tau,x_\tau)\, \mathrm{d}\tau - \int_0^b T(b-\tau)\, f\left(\tau, x_\tau, \int_0^\tau k(\tau,\theta,x_\theta)\, \mathrm{d}\theta\right) \mathrm{d}\tau \right] (s)\, \mathrm{d}s \right\| \\
&\quad + \left\| \int_{t-\epsilon}^{t} T(t-s)\, f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)\, \mathrm{d}\tau\right) \mathrm{d}s \right\| \\
&\leq \epsilon M_2 K_2 + \epsilon M_1 L_1 L_2 [\|x_1\| + M_1[\|\phi(0)\| + K_2] + K_2 + bM_2K_2 + bM_1K_1] + \epsilon M_1 K_1.
\end{aligned}
$$

Since there are precompact sets arbitrarily close to the set $Y_b(t)$, it is totally bounded, that is, precompact in X. We now show that the image

$$
\Psi(Y_b) = \{\Psi x;\ x \in Y_b\}
$$

is an equicontinuous family of functions. Let $0 < t_1 < t_2$,

$$
\begin{aligned}
&\|\Psi x(t_1) - \Psi x(t_2)\| \\
\leq\ & \|[T(t_1) - T(t_2)]\,[\phi(0) - g(0,\phi)]\| + \|g(t_1,x_{t_1}) - g(t_2,x_{t_2})\| \\
&+ \left\| \int_0^{t_1} [AT(t_1-s) - AT(t_2-s)]\, g(s,x_s)\, \mathrm{d}s - \int_{t_1}^{t_2} AT(t_2-s)\, g(s,x_s)\, \mathrm{d}s \right\| \\
&+ \left\| \int_0^{t_1} [T(t_1-s) - T(t_2-s)]\, B\tilde{W}^{-1} \Big[x_1 - T(b)\,[\phi(0) - g(0,\phi)] - g(b,x_b) \right. \\
&- \int_0^b AT(b-\tau)\, g(\tau,x_\tau)\, \mathrm{d}\tau - \int_0^b T(b-\tau)\, f\left(\tau, x_\tau, \int_0^\tau k(\tau,\theta,x_\theta)\, \mathrm{d}\theta\right) \mathrm{d}\tau \Big] (s)\, \mathrm{d}s \\
&- \int_{t_1}^{t_2} T(t_2-s)\, B\tilde{W}^{-1} [x_1 - T(b)[\phi(0) - g(0,\phi)] - g(b,x_b) \\
&\left. - \int_0^b AT(b-\tau)\, g(\tau,x_\tau)\, \mathrm{d}\tau - \int_0^b T(b-\tau)\, f\left(\tau, x_\tau, \int_0^\tau k(\tau,\theta,x_\theta)\, \mathrm{d}\theta\right) \mathrm{d}\tau \right] (s)\, \mathrm{d}s \Big\| \\
&+ \left\| \int_0^{t_1} [T(t_1-s) - T(t_2-s)]\, f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)\, \mathrm{d}\tau\right) \mathrm{d}s \right. \\
&\left. - \int_{t_1}^{t_2} T(t_2-s)\, f\left(s, x_s, \int_0^s g(s,\tau,x_\tau)\, \mathrm{d}\tau\right) \mathrm{d}s \right\| \\
\leq\ & \|T(t_1) - T(t_2)\|[\|\phi(0)\| + \|g(0,\phi)\|] + \|g(t_1,x_{t_1}) - g(t_2,x_{t_2})\| \\
&+ \int_0^{t_1} \|[AT(t_1-\epsilon-s) - AT(t_2-\epsilon-s)]T(\epsilon)\, g(s,x_s)\|\, \mathrm{d}s \\
&+ |t_1 - t_2| M_2 K_2 + \int_0^{t_1} \Big\| [T(t_1-\epsilon-s) - T(t_2-\epsilon-s)]\, T(\epsilon)\, B\tilde{W}^{-1}[x_1 \\
&\qquad - T(b)[\phi(0) - g(0,\phi)] - g(b,x_b) - \int_0^b AT(b-\tau) g(\tau,x_\tau)\, \mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
& - \int_0^b T(b-\tau)\, f\left(\tau, x_\tau, \int_0^\tau k(\tau,\theta,x_\theta)\,\mathrm{d}\theta\right)\mathrm{d}\tau](s)\Bigg\|\,\mathrm{d}s \\
& + |t_1 - t_2| M_1 L_1 L_2 [\|x_1\| + M_1[\|\phi(0)\| + \|g(0,\phi(0))\|] \\
& \quad + K_2 + bM_2K_2 + bM_1K_1] \\
& + \int_0^{t_1} \left\| [T(t_1-\epsilon-s) - T(t_2-\epsilon-s)]\, T(\epsilon) f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)\,\mathrm{d}\tau\right)\right\| \mathrm{d}s \\
& \quad + |t_1 - t_2| M_1 K_1.
\end{aligned}
$$

Since $T(\epsilon)g(s,x_s)$, $T(\epsilon)f\left(s,x_s,\int_0^s k(s,\tau,x_\tau)\,\mathrm{d}\tau\right)$ are in the compact set V_t for all $0 \le s \le b$ and all $x \in Y_b$ and the functions $g(s,x_s)$ and $T(\cdot)x$, $x \in V_t$ are continuous, the right hand side of the above inequality tends to zero as $t_1 \to t_2$. Hence $\Psi(Y_b)$ is an equicontinuous family of functions. Also $\Psi(Y_b)$ is bounded in Y and so by the Arzela–Ascoli theorem, $\Psi(Y_b)$ is precompact. Hence it follows from the Schauder fixed point theorem, Ψ has a fixed point in Y_b. Any fixed point of Ψ is a mild solution of (1) on J satisfying $(\Psi x)(t) = x(t) \in X$. Thus the system (1) is controllable on J.

4. EXAMPLE

Consider the partial neutral integrodifferential equation of the form

$$
\begin{aligned}
\frac{\partial}{\partial t}[z(t,y) - p(t, z(t-h,y))] &= \frac{\partial}{\partial y}\left(k(y)\frac{\partial z}{\partial y}(t,y)\right) + u(t,y) \\
&\quad + q\left(t, z(t-h,y), \int_0^t a(t,s,z(s-h,y))\,\mathrm{d}s\right), \\
&\qquad y \in [0,1],\ t \in J = [0,b], \\
z(t,y) &= \phi(t,y) \in \mathcal{C} \qquad (5)
\end{aligned}
$$

where the function ϕ is continuous and $u \in L^2(J,U)$.

Let $X = U = L^2(J,R)$ and $\mathcal{C} = L^p([-h,0];X)$ be the space endowed with the seminorm as in [9]

$$
|\phi| = \left(\|\phi(0)\|^p + \int_{-h}^0 \|\phi(\theta)\|^p\,\mathrm{d}\theta\right)^{\frac{1}{p}}.
$$

Then it is clear that the space $\mathcal{C}$ satisfies the axioms (A1) – (A3) of the phase space $\mathcal{B}$ [8]. Define the operator $A : X \to X$ by

$$
(Az)(t)(y) = \frac{\partial}{\partial y}\left(k(y)\frac{\partial z}{\partial y}(t,y)\right)
$$

with $k \in H^1(0,1) = \{z \in L^2(0,1);\ z \text{ is absolutely continuous on } (0,1) \text{ with } \frac{\partial z}{\partial y} \in L^2(0,1)\}$ and domain

$$
D(A) = \left\{ z \in X; \frac{\partial}{\partial y}\left(k(y)\frac{\partial z}{\partial y}(\cdot,y)\right) \in X,\ z(\cdot,0) = z(\cdot,1) = 0 \right\}.
$$

It is well known that the operator A generates a compact semigroup $T(t)$, $t \geq 0$ [8] and there exist constants M_1, M_2 such that $\|T(t)\| \leq M_1$ and $\|AT(t)\| \leq M_2$ for each $t > 0$.

Assume that there exists an invertible operator $\tilde{W}^{-1}$ on $L^2(J,U)/\ker W$ defined by

$$(Wu)(y) = \int_0^b T(b-s)\,Bu(s,y)\,\mathrm{d}s$$

satisfies the condition (iv). Also the nonlinear operators $p : J \times \mathcal{C} \to X$, $q : J \times \mathcal{C} \times X \to X$ and $a : J \times J \times \mathcal{C} \to X$ are continuous. Let $p(t,\phi) = g(t,\phi(-h))$ and $q\left(t,\phi,\int_0^t a(t,s,\phi)\,\mathrm{d}s\right) = f\left(t,\phi(-h),\int_0^t k(t,s,\phi(-h))\,\mathrm{d}s\right)$ be such that the condition (iii) is satisfied. We can easily see that the equation (5) is an abstract formulation of (1). Also all the conditions stated in the theorem are satisfied. Hence the system (5) is controllable on J.

Remark. (See also [11].)
Construction of $\tilde{W}^{-1}$.
Let $Y = L^2[J,U]/\ker W$. Since $\ker W$ is closed, Y is a Banach space under the norm

$$\|[u]\|_Y = \inf_{u \in [u]} \|u\|_{L^2[J,U]} = \inf_{W\hat{u}=0} \|u + \hat{u}\|_{L^2[J,U]}$$

where $[u]$ are the equivalence classes of u.

Define $\tilde{W} : Y \to X$ by

$$\tilde{W}[u] = Wu, \ u \in [u].$$

Now $\tilde{W}$ is one-to-one and

$$\|\tilde{W}[u]\|_X \leq \|W\|\,\|[u]\|_Y.$$

We claim that $V = \text{Range } W$ is a Banach space with the norm

$$\|v\|_V = \|\tilde{W}^{-1}v\|_Y.$$

This norm is equivalent to the graph norm on $D(\tilde{W}^{-1}) = \text{Range } W$, $\tilde{W}$ is bounded and since $D(\tilde{W}) = Y$ is closed, $\tilde{W}^{-1}$ is closed and so the above norm makes Range $W = V$ a Banach space.

Moreover,

$$\begin{aligned} \|Wu\|_V &= \|\tilde{W}^{-1}Wu\|_Y = \|\tilde{W}^{-1}\tilde{W}[u]\| \\ &= \|[u]\| = \inf_{u \in [u]} \|u\| \leq \|u\|, \end{aligned}$$

so

$$W \in \mathcal{L}(L^2[J,U], V).$$

Since $L^2[J,U]$ is reflexive and ker W is weakly closed, so that the infimum is actually attained. For any $v \in V$, we can therefore choose a control $u \in L^2[J,U]$ such that $u = \tilde{W}^{-1}v$.

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