Kybernetika

Eva Fišerová; Lubomír Kubáček

Sensitivity analysis in singular mixed linear models with constraints

Kybernetika, Vol. 39 (2003), No. 3, [317]--332

Persistent URL: http://dml.cz/dmlcz/135534

KYBERNETIKA — VOLUME 39 (2003), NUMBER 3, PAGES 317–332

SENSITIVITY ANALYSIS IN SINGULAR MIXED LINEAR MODELS WITH CONSTRAINTS[1]

EVA FIŠEROVÁ AND LUBOMÍR KUBÁČEK

The singular mixed linear model with constraints is investigated with respect to an influence of inaccurate variance components on a decrease of the confidence level. The algorithm for a determination of the boundary of the insensitivity region is given. It is a set of all shifts of variance components values which make the tolerated decrease of the confidence level only. The problem about geometrical characterization of the confidence domain is also presented.

Keywords: mixed linear model with constraints, confidence region, sensitiveness
AMS Subject Classification: 62J05, 62F10, 62F25

1. INTRODUCTION

Let the singular mixed linear model with constraints with inaccurate variance components be under consideration. An attention is focused on a problem of the confidence region.

Two special problems arise. The first one is how can the confidence domain be geometrically characterized. In regular linear models confidence regions, in the case of the normality, are ellipsoids either given by a positive definite matrix (in models without constraints), or by a positive semidefinite matrix (in models with constraints). However, in the case of the singularity of models, confidence regions can have another shape. It will be shown that this shape is a cylinder.

The second problem is connected with inaccurate variance components. Shifts between true and approximate values of variance components can caused a decrease of the confidence level. The sensitivity analysis approach can be used in a determination of a set of admissible shifts of variance components. This set is called an insensitivity region, which is defined as a set of all shifts of variance components values which make the tolerated decrease of the confidence level only.

In [2, 5, 6] this sensitivity problem has been studied in the case of the regularity of the model (the model with or without constraints). In the singular mixed model,

---

[1]Presented at the Workshop "Perspectives in Modern Statistical Inference II" held in Brno on August 14–17, 2002.
The research was supported by the Council of Czech Government under Project J14/98:153100011.

analogous problems connected with the variance of the estimator have been studied in [3, 4].

The aim of the paper is to study geometrical characterization of the confidence domain and to find an algorithm for a determination of the boundary of the insensitivity region for the confidence domain.

2. NOTATIONS AND AUXILIARY STATEMENTS

Let $\mathbf{A}$ be an $m \times n$ matrix. Let $\mathcal{M}(\mathbf{A}) = \{\mathbf{A}\boldsymbol{u} : \ \boldsymbol{u} \in \mathbb{R}^n\} \subset \mathbb{R}^m$ and $Ker(\mathbf{A}) = \{\boldsymbol{u} : \ \boldsymbol{u} \in \mathbb{R}^n, \ \mathbf{A}\boldsymbol{u} = \mathbf{0}\} \subset \mathbb{R}^n$ denote the column space and the null space of the matrix $\mathbf{A}$, respectively. Let $\mathbf{W}$ be an $m \times m$ symmetric positive semidefinite matrix such that $\mathcal{M}(\mathbf{A}) \subset \mathcal{M}(\mathbf{W})$. Then $\mathbf{P}_A^W = \mathbf{A}(\mathbf{A}'\mathbf{W}\mathbf{A})^-\mathbf{A}'\mathbf{W}$ denotes a projector on $\mathcal{M}(\mathbf{A})$ in the $\mathbf{W}$-seminorm. The symbol $\mathbf{M}_A^W$ means $\mathbf{I} - \mathbf{P}_A^W$. If $\mathbf{W} = \mathbf{I}$ (identity matrix), symbols $\mathbf{P}_A$ and $\mathbf{M}_A$ are used. The $\mathbf{W}$-seminorm of $\boldsymbol{x}$, $\boldsymbol{x} \in \mathbb{R}^m$, is given by $\|\boldsymbol{x}\|_W = \sqrt{\boldsymbol{x}'\mathbf{W}\boldsymbol{x}}$. Symbols $\mathbf{A}^-$ and $\mathbf{A}^+$ mean the g-inverse and the Moore–Penrose inverse of the matrix $\mathbf{A}$, respectively.

Let $\mathbf{N}$ be an $n \times n$ symmetric positive semidefinite matrix. The symbol $\mathbf{A}^-_{m(N)}$ denotes the minimum $\mathbf{N}$-seminorm g-inverse of the matrix $\mathbf{A}$, i. e., the matrix $\mathbf{A}^-_{m(N)}$ satisfies equations

$$\mathbf{A}\mathbf{A}^-_{m(N)}\mathbf{A} = \mathbf{A}, \quad \mathbf{N}\mathbf{A}^-_{m(N)}\mathbf{A} = \mathbf{A}'[\mathbf{A}^-_{m(N)}]'\mathbf{N}.$$

One of representations of the matrix $\mathbf{A}^-_{m(N)}$ is

$$\mathbf{A}^-_{m(N)} = \begin{cases} \mathbf{N}^-\mathbf{A}'(\mathbf{A}\mathbf{N}^-\mathbf{A}')^- & \text{if } \mathcal{M}(\mathbf{A}') \subset \mathcal{M}(\mathbf{N}), \\ (\mathbf{N} + \mathbf{A}'\mathbf{A})^-\mathbf{A}'[\mathbf{A}(\mathbf{N} + \mathbf{A}'\mathbf{A})^-\mathbf{A}']^- & \text{otherwise.} \end{cases}$$

In more detail cf. [7].

3. UNIVERSAL MODEL WITH CONSTRAINTS

The universal model with constraints is

$$\boldsymbol{Y} \sim_n \left(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}_{\vartheta}\right), \quad \boldsymbol{\beta} \in \{\boldsymbol{u} : \ \mathbf{B}\boldsymbol{u} + \boldsymbol{b} = \mathbf{0}\} = \mathcal{B}, \quad \boldsymbol{\vartheta} \in \underline{\vartheta}, \tag{1}$$

where $\boldsymbol{Y}$ is an n-dimensional random vector, $\mathbf{X}\boldsymbol{\beta}$ is the mean value of $\boldsymbol{Y}$ and $\boldsymbol{\Sigma}_{\vartheta}$ its covariance matrix. $\mathbf{X}$ and $\mathbf{B}$ are given matrices with the dimension $n \times k$ and $q \times k$, respectively, $\boldsymbol{b}$ is a known q-dimensional vector such that $\boldsymbol{b} \in \mathcal{M}(\mathbf{B})$, $\underline{\vartheta}$ is an open set in $\mathbb{R}^p$. The covariance matrix is considered in the form $\boldsymbol{\Sigma}_{\vartheta} = \sum_{i=1}^p \vartheta_i \mathbf{V}_i$, where $\mathbf{V}_1, \dots, \mathbf{V}_p$ are given $n \times n$ symmetric matrices and $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_p)' \in \underline{\vartheta}$ is unknown, and it is supposed that $\boldsymbol{\Sigma}_{\vartheta}$ is positive semidefinite for all $\boldsymbol{\vartheta} \in \underline{\vartheta}$.

A special case of the universal model is the mixed model. It is the model (1) if $\mathbf{V}_1, \dots, \mathbf{V}_p$ are symmetric positive semidefinite and $\vartheta_1, \dots, \vartheta_p$ are positive.

The equivalent expression of the universal model (1) is

$$\begin{pmatrix} \boldsymbol{Y} \\ -\boldsymbol{b} \end{pmatrix} \sim_{n+q} \left[\begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix}\boldsymbol{\beta}, \begin{pmatrix} \boldsymbol{\Sigma}_{\vartheta}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{0} \end{pmatrix}\right], \ \boldsymbol{\beta} \in \mathbb{R}^k, \ \boldsymbol{\vartheta} \in \underline{\vartheta}.$$

Lemma 3.1. In the universal model (1) a function $\boldsymbol{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{B}$, $\boldsymbol{h} \in \mathbb{R}^k$, is unbiasedly estimable if and only if $\boldsymbol{h} \in \mathcal{M}(\mathbf{X}', \mathbf{B}')$.

Proof. Cf. [1], p. 136. □

Lemma 3.2. Within the universal model (1) the ϑ-LBLUE (ϑ-locally best linear unbiased estimator) of a function $\boldsymbol{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{B}$, $\boldsymbol{h} \in \mathcal{M}(\mathbf{X}', \mathbf{B}')$, is

$$\widehat{\boldsymbol{h}'\boldsymbol{\beta}}(\vartheta) = \boldsymbol{h}'\mathbf{L}' \begin{pmatrix} \boldsymbol{Y} \\ -\boldsymbol{b} \end{pmatrix},$$

where

$$\begin{aligned} \mathbf{L}' &= [\mathbf{L}_1', \mathbf{L}_2'], \\ \mathbf{L}_1' &= (\mathbf{M}_{B'}\mathbf{W}_\vartheta\mathbf{M}_{B'})^+\mathbf{X}'(\boldsymbol{\Sigma}_\vartheta + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+, \\ \mathbf{L}_2' &= [\mathbf{I} - (\mathbf{M}_{B'}\mathbf{W}_\vartheta\mathbf{M}_{B'})^+\mathbf{X}'(\boldsymbol{\Sigma}_\vartheta + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+\mathbf{X}]\,\mathbf{B}'(\mathbf{B}\mathbf{B}')^-. \end{aligned}$$

Here

$$\begin{aligned} &\mathbf{W}_\vartheta = \mathbf{M}_{B'}\mathbf{X}'(\boldsymbol{\Sigma}_\vartheta + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+\mathbf{X}\mathbf{M}_{B'} + \mathbf{B}'\mathbf{B}, \\ &[\mathbf{M}_{B'}\mathbf{W}_\vartheta\mathbf{M}_{B'}]^+ = \mathbf{W}_\vartheta^+ - \mathbf{W}_\vartheta^+\mathbf{B}'(\mathbf{B}\mathbf{W}_\vartheta^+\mathbf{B}')^-\mathbf{B}\mathbf{W}_\vartheta^+. \end{aligned}$$

Proof. The vector function $\begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix}\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{B}$, represents the class of all unbiasedly estimable functions. The ϑ-LBLUE of $\begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix}\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{B}$, is

$$\widehat{\begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix}\boldsymbol{\beta}}(\vartheta) = \begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix} \left[(\mathbf{X}', \mathbf{B}')^-_{m\begin{pmatrix} \boldsymbol{\Sigma}_\vartheta, & 0 \\ 0, & 0 \end{pmatrix}} \right]' \begin{pmatrix} \boldsymbol{Y} \\ -\boldsymbol{b} \end{pmatrix}.$$

Since

$$\begin{aligned} \left[(\mathbf{X}', \mathbf{B}')^-_{m\begin{pmatrix} \boldsymbol{\Sigma}_\vartheta, & 0 \\ 0, & 0 \end{pmatrix}} \right]' &= \left[(\mathbf{X}', \mathbf{B}') \begin{pmatrix} \boldsymbol{\Sigma}_\vartheta + \mathbf{X}\mathbf{X}', & \mathbf{X}\mathbf{B}' \\ \mathbf{B}\mathbf{X}', & \mathbf{B}\mathbf{B}' \end{pmatrix}^- \begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix} \right]^- \\ &\quad \times (\mathbf{X}', \mathbf{B}') \begin{pmatrix} \boldsymbol{\Sigma}_\vartheta + \mathbf{X}\mathbf{X}', & \mathbf{X}\mathbf{B}' \\ \mathbf{B}\mathbf{X}', & \mathbf{B}\mathbf{B}' \end{pmatrix}^- \end{aligned} \tag{2}$$

and (cf. [1], p. 446)

$$\begin{aligned} &\begin{pmatrix} \boldsymbol{\Sigma}_\vartheta + \mathbf{X}\mathbf{X}', & \mathbf{X}\mathbf{B}' \\ \mathbf{B}\mathbf{X}', & \mathbf{B}\mathbf{B}' \end{pmatrix}^- = \begin{pmatrix} \mathbf{Q}_{11}, & \mathbf{Q}_{12} \\ \mathbf{Q}_{12}', & \mathbf{Q}_{22} \end{pmatrix}, \\ &\mathbf{Q}_{11} = (\boldsymbol{\Sigma}_\vartheta + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^-, \\ &\mathbf{Q}_{12}' = -(\mathbf{B}\mathbf{B}')^-\mathbf{B}\mathbf{X}'(\boldsymbol{\Sigma}_\vartheta + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^-, \\ &\mathbf{Q}_{22} = (\mathbf{B}\mathbf{B}')^- + (\mathbf{B}\mathbf{B}')^-\mathbf{B}\mathbf{X}'(\boldsymbol{\Sigma}_\vartheta + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^-\mathbf{X}\mathbf{B}'(\mathbf{B}\mathbf{B}')^-, \end{aligned}$$

we have

$$(\mathbf{X}',\mathbf{B}')\begin{pmatrix}\mathbf{\Sigma}_\vartheta+\mathbf{X}\mathbf{X}', & \mathbf{X}\mathbf{B}'\\ \mathbf{B}\mathbf{X}', & \mathbf{B}\mathbf{B}'\end{pmatrix}^- = [\mathbf{K}_1,\mathbf{K}_2], \tag{3}$$

$$\mathbf{K}_1 = \mathbf{M}_{B'}\mathbf{X}'(\mathbf{\Sigma}_\vartheta+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+,$$

$$\mathbf{K}_2 = \mathbf{B}'(\mathbf{B}\mathbf{B}')^- - \mathbf{M}_{B'}\mathbf{X}'(\mathbf{\Sigma}_\vartheta+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+\mathbf{X}\mathbf{B}'(\mathbf{B}\mathbf{B}')^-$$

and

$$\begin{aligned}&(\mathbf{X}',\mathbf{B}')\begin{pmatrix}\mathbf{\Sigma}_\vartheta+\mathbf{X}\mathbf{X}', & \mathbf{X}\mathbf{B}'\\ \mathbf{B}\mathbf{X}', & \mathbf{B}\mathbf{B}'\end{pmatrix}^-\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}\\ = \;& \mathbf{P}_{B'}+\mathbf{M}_{B'}\mathbf{X}'(\mathbf{\Sigma}_\vartheta+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+\mathbf{X}\mathbf{M}_{B'}.\end{aligned}$$

Thus

$$\begin{aligned}\widehat{\mathbf{X}\beta}(\vartheta) &= \mathbf{X}\left[\mathbf{P}_{B'}+\mathbf{M}_{B'}\mathbf{X}'(\mathbf{\Sigma}_\vartheta+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+\mathbf{X}\mathbf{M}_{B'}\right]^+\\ &\times \left[\mathbf{M}_{B'}\mathbf{X}'(\mathbf{\Sigma}_\vartheta+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+\boldsymbol{Y}-\mathbf{B}'(\mathbf{B}\mathbf{B}')^-\boldsymbol{b}\right.\\ &+ \left.\mathbf{M}_{B'}\mathbf{X}'(\mathbf{\Sigma}_\vartheta+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+\mathbf{X}\mathbf{B}'(\mathbf{B}\mathbf{B}')^-\boldsymbol{b}\right],\\ \widehat{\mathbf{B}\beta}(\vartheta) &= -\boldsymbol{b}.\end{aligned}$$

Further

$$\begin{aligned}&\left[\mathbf{P}_{B'}+\mathbf{M}_{B'}\mathbf{X}'(\mathbf{\Sigma}_\vartheta+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+\mathbf{X}\mathbf{M}_{B'}\right]^+\\ =\;& \mathbf{P}_{B'}+\left[\mathbf{M}_{B'}\mathbf{X}'(\mathbf{\Sigma}_\vartheta+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+\mathbf{X}\mathbf{M}_{B'}\right]^+\\ =\;& \mathbf{P}_{B'}+(\mathbf{M}_{B'}\mathbf{W}_\vartheta\mathbf{M}_{B'})^+.\end{aligned}$$

Since $(\mathbf{M}_{B'}\mathbf{W}_\vartheta\mathbf{M}_{B'})^+\mathbf{M}_{B'} = (\mathbf{M}_{B'}\mathbf{W}_\vartheta\mathbf{M}_{B'})^+$, it holds

$$\begin{aligned}\widehat{\mathbf{X}\beta}(\vartheta) &= \mathbf{X}(\mathbf{M}_{B'}\mathbf{W}_\vartheta\mathbf{M}_{B'})^+\mathbf{M}_{B'}\mathbf{X}'(\mathbf{\Sigma}_\vartheta+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+\boldsymbol{Y}\\ &\quad -\mathbf{X}\mathbf{B}'(\mathbf{B}\mathbf{B}')^-\boldsymbol{b}\\ &\quad +\mathbf{X}(\mathbf{M}_{B'}\mathbf{W}_\vartheta\mathbf{M}_{B'})^+\mathbf{M}_{B'}\mathbf{X}'(\mathbf{\Sigma}_\vartheta+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+\mathbf{X}\mathbf{B}'(\mathbf{B}\mathbf{B}')^-\boldsymbol{b}\\ &= \mathbf{X}[\mathbf{L}_1',\mathbf{L}_2']\begin{pmatrix}\boldsymbol{Y}\\ -\boldsymbol{b}\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}\mathbf{L}_1' &= (\mathbf{M}_{B'}\mathbf{W}_\vartheta\mathbf{M}_{B'})^+\mathbf{X}'(\mathbf{\Sigma}_\vartheta+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+,\\ \mathbf{L}_2' &= \left[\mathbf{I}-(\mathbf{M}_{B'}\mathbf{W}_\vartheta\mathbf{M}_{B'})^+\mathbf{X}'(\mathbf{\Sigma}_\vartheta+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+\mathbf{X}\right]\mathbf{B}'(\mathbf{B}\mathbf{B}')^-.\end{aligned}$$

Analogously

$$\widehat{\mathbf{B}\beta}(\vartheta) = \mathbf{B}[\mathbf{L}_1',\mathbf{L}_2']\begin{pmatrix}\boldsymbol{Y}\\ -\boldsymbol{b}\end{pmatrix} = -\boldsymbol{b}.$$ □

Lemma 3.3. Let $\boldsymbol{h}\in\mathcal{M}(\mathbf{X}',\mathbf{B}')$. Then in the universal model (1) it holds

$$\mathrm{Var}_\vartheta\left[\widehat{\boldsymbol{h}'\beta}(\vartheta)\right] = \boldsymbol{h}'\left[(\mathbf{M}_{B'}\mathbf{W}_\vartheta\mathbf{M}_{B'})^+ - \mathbf{M}_{B'}\right]\boldsymbol{h}.$$

Proof. The proof can be found in [1], p. 158. □

4. CONFIDENCE REGION

In what follows let the observation vector $\boldsymbol{Y}$ be normally distributed. According to Lemma 3.1

$$\boldsymbol{h} \in \mathcal{M}(\mathbf{X}', \mathbf{B}') \Leftrightarrow \boldsymbol{h}'\boldsymbol{\beta} \text{ is unbiasedly estimable.}$$

Any vector $\boldsymbol{h} \in \mathbb{R}^k$ can be considered in the form $\boldsymbol{h} = \boldsymbol{h}_1 + \boldsymbol{h}_2$, where $\boldsymbol{h}_1 \in \mathcal{M}(\mathbf{X}', \mathbf{B}')$ and $\boldsymbol{h}_2 \in [\mathcal{M}(\mathbf{X}', \mathbf{B}')]^{\perp} = \{Ker(\mathbf{X}) \cap Ker(\mathbf{B})\}$. If $\boldsymbol{h}'\boldsymbol{\beta}$ is not unbiasedly estimable, i.e., $\boldsymbol{h} = \boldsymbol{h}_2$, then we put $\text{Var}_{\vartheta}\left[\widehat{\boldsymbol{h}_2'\boldsymbol{\beta}(\vartheta)}\right] = \infty$. If $\boldsymbol{h}'\boldsymbol{\beta}$ is unbiasedly estimable, three following cases come into consideration

1. $\boldsymbol{h} \in \mathcal{M}(\mathbf{B}') \Rightarrow \text{Var}_{\vartheta}\left[\widehat{\boldsymbol{h}'\boldsymbol{\beta}(\vartheta)}\right] = 0,$

2. $\boldsymbol{h} \in \mathcal{M}(\mathbf{X}') \wedge \boldsymbol{h} \in \mathcal{M}(\mathbf{B}') \Rightarrow \text{Var}_{\vartheta}\left[\widehat{\boldsymbol{h}'\boldsymbol{\beta}(\vartheta)}\right] = 0,$

3. $\boldsymbol{h} \in \mathcal{M}(\mathbf{X}') \wedge \boldsymbol{h} \notin \mathcal{M}(\mathbf{B}') \Rightarrow$

$$\text{Var}_{\vartheta}\left[\widehat{\boldsymbol{h}'\boldsymbol{\beta}(\vartheta)}\right] = \boldsymbol{h}'\left[(\mathbf{M}_{B'}\mathbf{W}_{\vartheta}\mathbf{M}_{B'})^{+} - \mathbf{M}_{B'}\right]\boldsymbol{h}.$$

Summarizing this, the space $\mathcal{M}(\mathbf{X}', \mathbf{B}')$ can be divided into three disjoint subspaces $\mathcal{M}(\mathbf{X}_1')$, $\mathcal{M}(\mathbf{X}_2')$ and $\mathcal{M}(\mathbf{B}_1')$ such that

$$\begin{aligned} \mathcal{M}(\mathbf{X}', \mathbf{B}') &= \mathcal{M}(\mathbf{X}_1', \mathbf{X}_2', \mathbf{B}_1') = \mathcal{M}(\mathbf{X}_1', \mathbf{B}'), \\ \mathcal{M}(\mathbf{X}_2') &= \mathcal{M}(\mathbf{X}') \cap \mathcal{M}(\mathbf{B}'). \end{aligned}$$

Hence, from the variance of the estimator viewpoint the whole parametric space $\mathbb{R}^k$ is divided into three subspaces

$$\mathbb{R}^k = \mathcal{M}(\mathbf{X}_1') \vee \mathcal{M}(\mathbf{B}') \vee \{Ker(\mathbf{X}) \cap Ker(\mathbf{B})\},$$

where the symbol $\vee$ is defined as follows

$$\begin{aligned} &\mathcal{M}(\mathbf{X}_1') \vee \mathcal{M}(\mathbf{B}') \vee \{Ker(\mathbf{X}) \cap Ker(\mathbf{B})\} \\ &= \{\boldsymbol{p} + \boldsymbol{q} + \boldsymbol{r} : \boldsymbol{p} \in \mathcal{M}(\mathbf{X}_1'),\ \boldsymbol{q} \in \mathcal{M}(\mathbf{B}'),\ \boldsymbol{r} \in \{Ker(\mathbf{X}) \cap Ker(\mathbf{B})\}\}. \end{aligned}$$

Consequently, the $(1-\alpha)$-confidence interval for the function $\boldsymbol{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{B}$,

$$\mathcal{E}_{1-\alpha}(\boldsymbol{h}'\boldsymbol{\beta}) = \left\{u:\ u \in \mathbb{R}^1,\ \left|u - \widehat{\boldsymbol{h}'\boldsymbol{\beta}(\vartheta)}\right| \le u\left(1 - \frac{\alpha}{2}\right)\sqrt{\text{Var}_{\vartheta}\left[\widehat{\boldsymbol{h}'\boldsymbol{\beta}(\vartheta)}\right]}\right\},$$

where the symbol $u\left(1-\frac{\alpha}{2}\right)$ denotes $\left(1-\frac{\alpha}{2}\right)$-quantile of $N(0,1)$ distribution, is represented by a point if $\boldsymbol{h} \in \mathcal{M}(\mathbf{B}')$ and by the whole real line $\mathbb{R}^1$ if $\boldsymbol{h} \in \{Ker(\mathbf{X}) \cap Ker(\mathbf{B})\}$.

It remains to analyse the space $\mathcal{M}(\mathbf{X}_1')$. In the following, the universal model (1) will be considered in the partitioned form (after the suitable reindexing)

$$\begin{pmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \\ -\boldsymbol{b} \end{pmatrix} \sim_{n_1+n_2+q_1} \left[\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{B}_1 \end{pmatrix}\boldsymbol{\beta}, \begin{pmatrix} \boldsymbol{\Sigma}_{\vartheta,11}, & \boldsymbol{\Sigma}_{\vartheta,12}, & \mathbf{0} \\ \boldsymbol{\Sigma}_{\vartheta,21}, & \boldsymbol{\Sigma}_{\vartheta,22}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{0}, & \mathbf{0} \end{pmatrix}\right], \tag{4}$$

$$n_1 + n_2 + q_1 = n + q, \quad n_1 \le n, \quad q_1 \le q.$$

Lemma 4.1. The dimension of the space $\mathcal{M}(\mathbf{X}_1')$ is equal to $r(\mathbf{X}\mathbf{M}_{B'})$. Here the symbol $r(\mathbf{X}\mathbf{M}_{B'})$ means the rank of the matrix $\mathbf{X}\mathbf{M}_{B'}$.

Proof. The proof follows from the following relations

$$r\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix} = r(\mathbf{X}\mathbf{M}_{B'}) + r(\mathbf{B}),$$
$$\mathcal{M}(\mathbf{X}',\mathbf{B}') = \mathcal{M}(\mathbf{X}_1',\mathbf{X}_2',\mathbf{B}_1'),\quad \mathcal{M}(\mathbf{X}_1'),\ \mathcal{M}(\mathbf{X}_2'),\ \mathcal{M}(\mathbf{B}_1')\ \text{disjoint},$$
$$Ker(\mathbf{B}) = \mathcal{M}(\mathbf{M}_{B'})$$

(cf. [7], p. 137). □

Lemma 4.2. Within the universal model (1) it holds

$$r\left(\mathrm{Var}_\vartheta\left[\widehat{\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}\beta(\vartheta)}\right]\right) = r\left(\mathbf{M}_{B'}\mathbf{X}'[\boldsymbol{\Sigma}_\vartheta + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}']^-\boldsymbol{\Sigma}_\vartheta\right).$$

Proof. With respect to Lemma 3.2 it holds

$$\mathrm{Var}_\vartheta\left[\widehat{\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}\beta(\vartheta)}\right] = \begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}\mathbf{L}'\begin{pmatrix}\boldsymbol{\Sigma}_\vartheta, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}\mathbf{L}(\mathbf{X}',\mathbf{B}'),$$

where

$$\mathbf{L}' = \left[(\mathbf{X}',\mathbf{B}')^-_{m\begin{pmatrix}\Sigma_\vartheta, & 0\\ 0, & 0\end{pmatrix}}\right]'$$

(cf. (2)). It implies

$$r\left(\mathrm{Var}_\vartheta\left[\widehat{\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}\beta(\vartheta)}\right]\right) = r\left[\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}\mathbf{L}'\begin{pmatrix}\boldsymbol{\Sigma}_\vartheta, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}\right]$$
$$\geq r\left[(\mathbf{X}',\mathbf{B}')\begin{pmatrix}\boldsymbol{\Sigma}_\vartheta + \mathbf{X}\mathbf{X}', & \mathbf{X}\mathbf{B}'\\ \mathbf{B}\mathbf{X}', & \mathbf{B}\mathbf{B}'\end{pmatrix}^-\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}\mathbf{L}'\begin{pmatrix}\boldsymbol{\Sigma}_\vartheta, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}\right]$$
$$= r\left[(\mathbf{X}',\mathbf{B}')\begin{pmatrix}\boldsymbol{\Sigma}_\vartheta + \mathbf{X}\mathbf{X}', & \mathbf{X}\mathbf{B}'\\ \mathbf{B}\mathbf{X}', & \mathbf{B}\mathbf{B}'\end{pmatrix}^-\begin{pmatrix}\boldsymbol{\Sigma}_\vartheta, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}\right]$$
$$\geq r\left[\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}\mathbf{L}'\begin{pmatrix}\boldsymbol{\Sigma}_\vartheta, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}\right].$$

Thus

$$r\left(\mathrm{Var}_\vartheta\left[\widehat{\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}\beta(\vartheta)}\right]\right)$$
$$= r\left[(\mathbf{X}',\mathbf{B}')\begin{pmatrix}\boldsymbol{\Sigma}_\vartheta + \mathbf{X}\mathbf{X}', & \mathbf{X}\mathbf{B}'\\ \mathbf{B}\mathbf{X}', & \mathbf{B}\mathbf{B}'\end{pmatrix}^-\begin{pmatrix}\boldsymbol{\Sigma}_\vartheta, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}\right]$$

and the proof is finished by using the relation (3). □

Corollary 4.3.

$$r\left(\mathrm{Var}_\vartheta\left[\widehat{\mathbf{X}_1\boldsymbol{\beta}(\vartheta)}\right]\right) = r\left(\mathbf{M}_{B'}\mathbf{X}'[\boldsymbol{\Sigma}_\vartheta + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}']^-\boldsymbol{\Sigma}_\vartheta\right).$$

Proof. The statement is an obvious consequence of Lemma 4.2 since

$$\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix} = \begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\\ \mathbf{B}_1\end{pmatrix}, \quad \mathrm{Var}_\vartheta\left[\widehat{\mathbf{X}_2\boldsymbol{\beta}(\vartheta)}\right] = \mathbf{0}, \quad \mathrm{Var}_\vartheta\left[\widehat{\mathbf{B}_1\boldsymbol{\beta}(\vartheta)}\right] = \mathbf{0}. \qquad \square$$

The vector function $\mathbf{X}_1\boldsymbol{\beta}$ represents the class of all unbiasedly estimable functions $\boldsymbol{h}'\boldsymbol{\beta}$ in the subspace $\mathcal{M}(\mathbf{X}_1')$. The $(1-\alpha)$-confidence region for the function $\mathbf{X}_1\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{B}$, is given by the expression

$$\begin{aligned}\mathcal{E}_{1-\alpha}(\mathbf{X}_1\boldsymbol{\beta}) &= \Big\{\boldsymbol{u} : \ \boldsymbol{u} \in \mathcal{M}(\mathbf{X}_1), \ \chi_s^2(0, 1-\alpha) \geq \left(\widehat{\mathbf{X}_1\boldsymbol{\beta}(\vartheta)} - \boldsymbol{u}\right)' \\ &\times \ \left(\mathbf{X}_1\left[(\mathbf{M}_{B'}\mathbf{W}_\vartheta\mathbf{M}_{B'})^+ - \mathbf{M}_{B'}\right]\mathbf{X}_1'\right)^- \left(\widehat{\mathbf{X}_1\boldsymbol{\beta}(\vartheta)} - \boldsymbol{u}\right)\Big\},\end{aligned}$$

where

$$s = r\left(\mathbf{M}_{B'}\mathbf{X}'[\boldsymbol{\Sigma}_\vartheta + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}']^-\boldsymbol{\Sigma}_\vartheta\right)$$

and $\chi_s^2(0, 1-\alpha)$ means the $(1-\alpha)$-quantile of chi-square distribution with s degrees of freedom.

From Lemma 4.1 and Corollary 4.3 it follows $k \geq \dim\mathcal{M}(\mathbf{X}_1') \geq s$ and the rank s depends on $\boldsymbol{\Sigma}_\vartheta$. Hence, a confidence region for the function $\mathbf{X}_1\boldsymbol{\beta}$ is an s-dimensional domain in the space $\mathcal{M}(\mathbf{X}_1')$. It cannot be characterized as an regular ellipsoid. There exist directions $\boldsymbol{f}_i \in \mathcal{M}(\mathbf{X}_1')$, $i = 1, \dots, \dim\mathcal{M}(\mathbf{X}_1') - s$, such that a confidence interval for the function $\boldsymbol{f}_i'\boldsymbol{\beta}$ is degenerated into a point and it holds $\mathrm{Var}_\vartheta\left[\widehat{\boldsymbol{f}_i'\boldsymbol{\beta}(\vartheta)}\right] = 0$, $i = 1, \dots, \dim\mathcal{M}(\mathbf{X}_1') - s$. These vectors $\boldsymbol{f}_i$ generate the subspace $\mathcal{N} \subset \mathcal{M}(\mathbf{X}_1')$ and $\mathcal{M}(\mathbf{X}_1') = \mathcal{N} \oplus \mathcal{F}$. In the subspace $\mathcal{N}$ the estimator of $\mathbf{P}_\mathcal{N}\boldsymbol{\beta}$, where $\mathbf{P}_\mathcal{N}$ is the Euclidean projector matrix on the subspace $\mathcal{N}$, is the vector $\widehat{\mathbf{P}_\mathcal{N}\boldsymbol{\beta}}$ with the property $P\left\{\widehat{\mathbf{P}_\mathcal{N}\boldsymbol{\beta}} = \mathbf{P}_\mathcal{N}\boldsymbol{\beta}\right\} = 1$, i.e., $\mathrm{Var}_\vartheta\left(\widehat{\mathbf{P}_\mathcal{N}\boldsymbol{\beta}}\right) = \mathbf{0}$. Here a vector $\mathbf{P}_\mathcal{N}\boldsymbol{\beta}$ represents all unbiasedly estimable linear functions of the parameter $\boldsymbol{\beta}$ in the subspace $\mathcal{N}$. In the subspace $\mathcal{F}$ the confidence region of $\mathbf{P}_\mathcal{F}\boldsymbol{\beta}$ is a regular s-dimensional ellipsoid.

Summarizing all results from this section, the confidence region for the function $\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}\boldsymbol{\beta}$ is a cylinder $\mathcal{Z} \oplus \{Ker(\mathbf{X}) \cap Ker(\mathbf{B})\}$ with a basis $\mathcal{Z}$ in the subspace $\mathcal{M}(\mathbf{X}', \mathbf{B}')$. The basis is an ellipsoid in the subspace $\mathcal{F}$ with the center given by the vector

$$\widehat{\mathbf{P}_{B'}\boldsymbol{\beta}} + \widehat{\mathbf{P}_\mathcal{N}\boldsymbol{\beta}} = -\mathbf{B}'(\mathbf{B}\mathbf{B}')^-\boldsymbol{b} + \widehat{\mathbf{P}_\mathcal{N}\boldsymbol{\beta}}.$$

Sometimes, it is necessary to determine confidence intervals for several functions of the parameter $\boldsymbol{\beta}$ simultaneously. The following theorem, due to Scheffé is useful in such cases.

Lemma 4.4 (Scheffé theorem) Let $\mathbf{V}$ be any $n \times n$ symmetric positive semidefinite matrix, $k > 0$ and $\boldsymbol{x} \in \mathcal{M}(\mathbf{V})$. Then

$$\forall \boldsymbol{y} \in \mathbb{R}^n : \ |\boldsymbol{y}'\boldsymbol{x}| \leq k\sqrt{\boldsymbol{y}'\mathbf{V}\boldsymbol{y}} \ \Leftrightarrow \ \boldsymbol{x}'\mathbf{V}^-\boldsymbol{x} \leq k^2.$$

Proof. It is an obvious generalization of the statement for a symmetric positive definite matrix $\mathbf{V}$ which can be found in [8], p. 69. □

Theorem 4.5. Consider a subclass $\mathcal{K}$ of estimable functions of $\boldsymbol{\beta}$ characterized by a $k \times l$ matrix $\mathbf{K}$ with the properties

$$\boldsymbol{h}'\boldsymbol{\beta} \in \mathcal{K} \ \Leftrightarrow \ \boldsymbol{h} \in \mathcal{M}(\mathbf{K}) \subset \mathcal{M}(\mathbf{X}', \mathbf{B}').$$

Then for all $\boldsymbol{h} \in \mathcal{M}(\mathbf{K})$ confidence intervals of functions $\boldsymbol{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{B}$, are given by

$$1 - \alpha = P\Big\{\forall\, \boldsymbol{h}'\boldsymbol{\beta} \in \mathcal{K} : \Big|\boldsymbol{h}'\boldsymbol{\beta} - \widehat{\boldsymbol{h}'\boldsymbol{\beta}}(\boldsymbol{\vartheta})\Big| \leq \sqrt{\chi^2_f(0, 1-\alpha)}\sqrt{\boldsymbol{h}'\left[(\mathbf{M}_{B'}\mathbf{W}_\vartheta\mathbf{M}_{B'})^+ - \mathbf{M}_{B'}\right]\boldsymbol{h}}\Big\},$$

where degrees of freedom are

$$f = r\left\{\mathbf{K}'(\mathbf{M}_{B'}\mathbf{W}_\vartheta\mathbf{M}_{B'})^+\mathbf{X}'[\boldsymbol{\Sigma}_\vartheta + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}']^+\boldsymbol{\Sigma}_\vartheta\right\}.$$

Proof. Let $\boldsymbol{h} \in \mathcal{K}$, i.e., $\exists \boldsymbol{f} \in \mathbb{R}^l : \ \boldsymbol{h} = \mathbf{K}\boldsymbol{f}$. Thus

$$\Big|\boldsymbol{h}'\boldsymbol{\beta} - \widehat{\boldsymbol{h}'\boldsymbol{\beta}}(\boldsymbol{\vartheta})\Big| = \Big|\boldsymbol{f}'\left(\mathbf{K}'\boldsymbol{\beta} - \widehat{\mathbf{K}'\boldsymbol{\beta}}(\boldsymbol{\vartheta})\right)\Big|.$$

Since

$$P\left\{\left(\mathbf{K}'\boldsymbol{\beta} - \widehat{\mathbf{K}'\boldsymbol{\beta}}(\boldsymbol{\vartheta})\right) \in \mathcal{M}\left(\mathrm{Var}_\vartheta[\widehat{\mathbf{K}'\boldsymbol{\beta}}(\boldsymbol{\vartheta})]\right)\right\} = 1,$$

according to Lemma 4.4 it holds

$$\forall \boldsymbol{f} \in \mathbb{R}^l : \ \Big|\boldsymbol{f}'\left(\mathbf{K}'\boldsymbol{\beta} - \widehat{\mathbf{K}'\boldsymbol{\beta}}(\boldsymbol{\vartheta})\right)\Big| \leq k\sqrt{\boldsymbol{f}'\mathrm{Var}_\vartheta[\widehat{\mathbf{K}'\boldsymbol{\beta}}(\boldsymbol{\vartheta})]\boldsymbol{f}} \ \Leftrightarrow$$
$$\left(\mathbf{K}'\boldsymbol{\beta} - \widehat{\mathbf{K}'\boldsymbol{\beta}}(\boldsymbol{\vartheta})\right)'\left(\mathrm{Var}_\vartheta[\widehat{\mathbf{K}'\boldsymbol{\beta}}(\boldsymbol{\vartheta})]\right)^-\left(\mathbf{K}'\boldsymbol{\beta} - \widehat{\mathbf{K}'\boldsymbol{\beta}}(\boldsymbol{\vartheta})\right) \leq k^2.$$

Hence

$$1 - \alpha = P\Big\{\forall\, \boldsymbol{h}'\boldsymbol{\beta} \in \mathcal{K} : \Big|\boldsymbol{h}'\boldsymbol{\beta} - \widehat{\boldsymbol{h}'\boldsymbol{\beta}}(\boldsymbol{\vartheta})\Big| \leq \sqrt{\chi^2_f(0, 1-\alpha)}\sqrt{\boldsymbol{h}'\left[(\mathbf{M}_{B'}\mathbf{W}_\vartheta\mathbf{M}_{B'})^+ - \mathbf{M}_{B'}\right]\boldsymbol{h}}\Big\}$$

where

$$f = r\left\{\mathrm{Var}_\vartheta[\widehat{\mathbf{K}'\boldsymbol{\beta}}(\boldsymbol{\vartheta})]\right\}.$$

Further, since $\mathcal{M}(\mathbf{K}) \subset \mathcal{M}(\mathbf{X}', \mathbf{B}')$, there exist matrices $\mathbf{U}_1$, $\mathbf{U}_2$ such that $\mathbf{K} = \mathbf{X}'\mathbf{U}_1 + \mathbf{B}'\mathbf{U}_2$. Now, using Lemma 3.2 we obtain

$$
\begin{aligned}
& r\left\{\mathrm{Var}_{\vartheta}[\widehat{\mathbf{K}'\boldsymbol{\beta}}(\vartheta)]\right\} \\
&= r\left\{(\mathbf{U}_1', \mathbf{U}_2')\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}\mathbf{L}'\begin{pmatrix}\boldsymbol{\Sigma}_{\vartheta}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}\right\} \\
&= r\left\{(\mathbf{U}_1', \mathbf{U}_2')\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}\left[(\mathbf{M}_{B'}\mathbf{W}_{\vartheta}\mathbf{M}_{B'})^{+}\mathbf{X}'(\boldsymbol{\Sigma}_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^{+}\boldsymbol{\Sigma}_{\vartheta}, \mathbf{0}\right]\right\} \\
&= r\left\{\mathbf{U}_1'\mathbf{X}(\mathbf{M}_{B'}\mathbf{W}_{\vartheta}\mathbf{M}_{B'})^{+}\mathbf{X}'(\boldsymbol{\Sigma}_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^{+}\boldsymbol{\Sigma}_{\vartheta}\right\} \\
&= r\left\{\mathbf{K}'(\mathbf{M}_{B'}\mathbf{W}_{\vartheta}\mathbf{M}_{B'})^{+}\mathbf{X}'(\boldsymbol{\Sigma}_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^{+}\boldsymbol{\Sigma}_{\vartheta}\right\},
\end{aligned}
$$

since

$$
\mathbf{M}_{B'}(\mathbf{M}_{B'}\mathbf{W}_{\vartheta}\mathbf{M}_{B'})^{+} = (\mathbf{M}_{B'}\mathbf{W}_{\vartheta}\mathbf{M}_{B'})^{+}, \quad \mathbf{K}'\mathbf{M}_{B'} = \mathbf{U}_1'\mathbf{X}\mathbf{M}_{B'}. \qquad \square
$$

5. INSENSITIVITY REGION

In this section, let the mixed linear model with constraints (1) be under consideration, i.e., $\mathbf{V}_i$, $i = 1, \dots, p$, are symmetric positive semidefinite and ϑ_i are positive. Authors have not been able to solve the problem in the model with variance components, i.e., in the case that $\mathbf{V}_i$, $i = 1, \dots, p$, is symmetric only and ϑ_i can be negative. The problem is to derive g-inverse of matrices depending on parameters $\vartheta_1, \dots, \vartheta_p$. In mixed linear model it is valid the relationship (6) from the proof of Lemma 5.1, i.e.,

$$
\mathcal{M}\left(\frac{\partial}{\partial\vartheta_i}\sum_{i=1}^{p}\vartheta_i\mathbf{V}_i\right) = \mathcal{M}(\mathbf{V}_i) \subset \mathcal{M}\left(\sum_{i=1}^{p}\vartheta_i\mathbf{V}_i\right),
$$

what is not true in the case of models with variance components.

Let the observation vector $\boldsymbol{Y}$ be normally distributed. Let $\boldsymbol{\vartheta}^*$ be true value of the parameter $\boldsymbol{\vartheta}$. The small change of $\boldsymbol{\vartheta}^*$ into $\boldsymbol{\vartheta}^* + \boldsymbol{\Delta}$ causes a change of the $\boldsymbol{\vartheta}^*$-LBLUE $\widehat{\boldsymbol{h}'\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)$ of the function $\boldsymbol{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{B}$. Analogously, this change influences the confidence level of the confidence region, the risk of the test, etc. In the following, the problem with the confidence level will be studied. Here an analogous procedure is used as in the regular mixed model with constraints (cf. [5]).

According to the previous section, it is sufficient to study the problem of the sensitivity for functions $\mathbf{X}_1\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{B}$, only.

Denote by $\eta(\boldsymbol{\vartheta}^* + \boldsymbol{\Delta})$ the random variable

$$
\begin{aligned}
& \eta(\boldsymbol{\vartheta}^* + \boldsymbol{\Delta}) \\
&= \left(\widehat{\mathbf{X}_1\boldsymbol{\beta}}(\boldsymbol{\vartheta}^* + \boldsymbol{\Delta}) - \mathbf{X}_1\boldsymbol{\beta}\right)' \left(\mathbf{X}_1\left[(\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*+\Delta}\mathbf{M}_{B'})^{+} - \mathbf{M}_{B'}\right]\mathbf{X}_1'\right)^{-} \\
&\quad \times \left(\widehat{\mathbf{X}_1\boldsymbol{\beta}}(\boldsymbol{\vartheta}^* + \boldsymbol{\Delta}) - \mathbf{X}_1\boldsymbol{\beta}\right).
\end{aligned}
$$

Then $\eta(\vartheta^*) \sim \chi_s^2(0)$, $s = r\left(\mathbf{M}_{B'}\mathbf{X}'[\boldsymbol{\Sigma}_{\vartheta^*} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}']^- \boldsymbol{\Sigma}_{\vartheta^*}\right)$ and the random variable

$$\xi = \sum_{i=1}^{p} \Delta_i \left.\frac{\partial \eta(\vartheta)}{\partial \vartheta_i}\right|_{\vartheta=\vartheta^*}$$

describes the change of $\eta(\vartheta^*)$ caused by the shift $\boldsymbol{\Delta}$ of the parameter $\boldsymbol{\vartheta}$ around $\boldsymbol{\vartheta}^*$. Omitting the second and higher derivatives in the Taylor series, the variable $\eta(\boldsymbol{\vartheta}^* + \boldsymbol{\Delta})$ can be linearly approximated by

$$\eta(\boldsymbol{\vartheta}^* + \boldsymbol{\Delta}) \approx \eta(\boldsymbol{\vartheta}^*) + \xi.$$

Lemma 5.1. Let the mixed model (1) be under consideration. Then the mean value of ξ is

$$E[\xi] = -\sum_{i=1}^{p} \Delta_i \mathrm{Tr}\,(\mathbf{U}\mathbf{V}_i)\,,$$

where $\mathrm{Tr}\,(\mathbf{U}\mathbf{V}_i)$ means the trace of the matrix $\mathbf{U}\mathbf{V}_i$ and

$$\begin{aligned}
\mathbf{U} &= (\boldsymbol{\Sigma}_{\vartheta^*} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+ \mathbf{X}\,[\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}]^+ \mathbf{X}_1' \\
&\quad \times \left[\mathbf{X}_1\left([\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}]^+ - \mathbf{M}_{B'}\right)\mathbf{X}_1'\right]^+ \\
&\quad \times \mathbf{X}_1\,[\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}]^+ \mathbf{X}'\,(\boldsymbol{\Sigma}_{\vartheta^*} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+ .
\end{aligned}$$

The variance of ξ is

$$\mathrm{Var}_{\vartheta^*}[\xi] = \boldsymbol{\Delta}'\mathbf{A}\boldsymbol{\Delta},$$

where

$$\begin{aligned}
\mathbf{A} &= 2\mathbf{S}_U + 4\mathbf{C}_{U,T}, \qquad (5)\\
&\quad \{\mathbf{S}_U\}_{i,j} = \mathrm{Tr}\,(\mathbf{U}\mathbf{V}_i\mathbf{U}\mathbf{V}_j)\,,\ i,j = 1,\dots,p,\\
\{\mathbf{C}_{U,T}\}_{i,j} &= \mathrm{Tr}\,(\mathbf{U}\mathbf{V}_i\mathbf{T}\mathbf{V}_j)\,,\\
\mathbf{T} &= -(\boldsymbol{\Sigma}_{\vartheta^*} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+ \mathbf{X}\,[\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}]^+ \mathbf{X}' \\
&\quad \times (\boldsymbol{\Sigma}_{\vartheta^*} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+ + (\boldsymbol{\Sigma}_{\vartheta^*} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+ .
\end{aligned}$$

Further, the explicit expression of the term $\frac{\partial \eta(\vartheta^*)}{\partial \vartheta_i}$, $i = 1,\dots,p$, is

$$\begin{aligned}
\frac{\partial \eta(\vartheta^*)}{\partial \vartheta_i} &= -2\left(\widehat{\mathbf{X}_1\boldsymbol{\beta}}(\vartheta^*) - \mathbf{X}_1\boldsymbol{\beta}\right)' \left[\mathbf{X}_1\left([\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}]^+ - \mathbf{M}_{B'}\right)\mathbf{X}_1'\right]^+ \\
&\quad \times \mathbf{X}_1\,[\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}]^+ \mathbf{X}'\,(\boldsymbol{\Sigma}_{\vartheta^*} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+ \mathbf{V}_i \\
&\quad \times (\boldsymbol{\Sigma}_{\vartheta^*} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+ v_1(\vartheta^*) \\
&\quad - \left(\widehat{\mathbf{X}_1\boldsymbol{\beta}}(\vartheta^*) - \mathbf{X}_1\boldsymbol{\beta}\right)' \left[\mathbf{X}_1\left([\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}]^+ - \mathbf{M}_{B'}\right)\mathbf{X}_1'\right]^+ \\
&\quad \times \mathbf{X}_1\,[\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}]^+ \mathbf{X}'\,(\boldsymbol{\Sigma}_{\vartheta^*} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+ \mathbf{V}_i \\
&\quad \times (\boldsymbol{\Sigma}_{\vartheta^*} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+ \mathbf{X}\,[\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}]^+ \mathbf{X}_1' \\
&\quad \times \left[\mathbf{X}_1\left([\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}]^+ - \mathbf{M}_{B'}\right)\mathbf{X}_1'\right]^+ \left(\widehat{\mathbf{X}_1\boldsymbol{\beta}}(\vartheta^*) - \mathbf{X}_1\boldsymbol{\beta}\right),
\end{aligned}$$

where

$$\boldsymbol{v}_1(\boldsymbol{\vartheta}^*) = \boldsymbol{Y}_1 - \widehat{\mathbf{X}_1\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*).$$

Proof. The explicit expression of $\frac{\partial\eta(\boldsymbol{\vartheta}^*)}{\partial\vartheta_i}$ can be derived by using the relation

$$\mathcal{M}\left(\frac{\partial\mathbf{A}^+(t)}{\partial t}\right) \subset \mathcal{M}(\mathbf{A}(t)) \Rightarrow \frac{\partial\mathbf{A}^+(t)}{\partial t} = -\mathbf{A}^+(t)\frac{\partial\mathbf{A}(t)}{\partial t}\mathbf{A}^+(t), \tag{6}$$

which can be easily proved. Then

$$\begin{aligned}
\frac{\partial\widehat{\mathbf{X}_1\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)}{\partial\vartheta_i} &= -\mathbf{X}_1\left[\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}\right]^+\mathbf{X}'\left(\boldsymbol{\Sigma}_{\vartheta^*}+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}'\right)^+\mathbf{V}_i \\
&\quad \times\left(\boldsymbol{\Sigma}_{\vartheta^*}+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}'\right)^+\boldsymbol{v}_1(\boldsymbol{\vartheta}^*), \\
\frac{\partial\mathbf{W}_{\vartheta^*}}{\partial\vartheta_i} &= -\mathbf{M}_{B'}\mathbf{X}'\left(\boldsymbol{\Sigma}_{\vartheta^*}+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}'\right)^+\mathbf{V}_i \\
&\quad \times\left(\boldsymbol{\Sigma}_{\vartheta^*}+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}'\right)^+\mathbf{X}\mathbf{M}_{B'}, \\
&\frac{\partial\left[\mathbf{X}_1\left(\left[\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}\right]^+-\mathbf{M}_{B'}\right)\mathbf{X}_1'\right]}{\partial\vartheta_i} \\
&= \mathbf{X}_1\left[\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}\right]^+\mathbf{X}'\left(\boldsymbol{\Sigma}_{\vartheta^*}+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}'\right)^+\mathbf{V}_i \\
&\quad \times\left(\boldsymbol{\Sigma}_{\vartheta^*}+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}'\right)^+\mathbf{X}\left[\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}\right]^+\mathbf{X}_1'.
\end{aligned}$$

In the next step we use the notation

$$\begin{aligned}
\mathbf{F}_i &= \left[\mathbf{X}_1\left(\left[\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}\right]^+-\mathbf{M}_{B'}\right)\mathbf{X}_1'\right]^+\mathbf{X}_1\left[\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}\right]^+ \\
&\quad \times\mathbf{X}'\left(\boldsymbol{\Sigma}_{\vartheta^*}+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}'\right)^+\mathbf{V}_i\left(\boldsymbol{\Sigma}_{\vartheta^*}+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}'\right)^+, \\
\mathbf{D}_i &= \left[\mathbf{X}_1\left(\left[\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}\right]^+-\mathbf{M}_{B'}\right)\mathbf{X}_1'\right]^+\mathbf{X}_1\left[\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}\right]^+ \\
&\quad \times\mathbf{X}'\left(\boldsymbol{\Sigma}_{\vartheta^*}+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}'\right)^+\mathbf{V}_i\left(\boldsymbol{\Sigma}_{\vartheta^*}+\mathbf{X}\mathbf{M}_{B'}\mathbf{X}'\right)^+ \\
&\quad \times\mathbf{X}\left[\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}\right]^+\mathbf{X}_1'\left[\mathbf{X}_1\left(\left[\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}\right]^+-\mathbf{M}_{B'}\right)\mathbf{X}_1'\right]^+
\end{aligned}$$

and

$$\boldsymbol{\zeta}(\boldsymbol{\vartheta}^*) = \widehat{\mathbf{X}_1\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*) - \mathbf{X}_1\boldsymbol{\beta}.$$

Then

$$\begin{aligned}
\boldsymbol{\zeta}(\boldsymbol{\vartheta}^*) &\sim N_{n_1}\left[\mathbf{0},\mathbf{X}_1\left(\left[\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}\right]^+-\mathbf{M}_{B'}\right)\mathbf{X}_1'\right], \\
\boldsymbol{v}_1(\boldsymbol{\vartheta}^*) &\sim N_{n_1}\left[\mathbf{0},\boldsymbol{\Sigma}_{\vartheta^*,11}-\mathbf{X}_1\left(\left[\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'}\right]^+-\mathbf{M}_{B'}\right)\mathbf{X}_1'\right],
\end{aligned}$$

where $\boldsymbol{\Sigma}_{\vartheta^*,11}$ is the submatrix from (4) and $\boldsymbol{\zeta}$, $\boldsymbol{v}_1$ are stochastically independent. The random variable $\frac{\partial\eta(\boldsymbol{\vartheta}^*)}{\partial\vartheta_i}$ can be rewritten in the form

$$\frac{\partial\eta(\boldsymbol{\vartheta}^*)}{\partial\vartheta_i} = -2\boldsymbol{\zeta}'(\boldsymbol{\vartheta}^*)\mathbf{F}_i\boldsymbol{v}_1(\boldsymbol{\vartheta}^*) - \boldsymbol{\zeta}'(\boldsymbol{\vartheta}^*)\mathbf{D}_i\boldsymbol{\zeta}(\boldsymbol{\vartheta}^*).$$

Now the proof can be finished by using the following relations

$$E\left[\zeta'(\vartheta^*)\mathbf{D}_i\zeta(\vartheta^*)\right] = \mathrm{Tr}\left(\mathbf{D}_i \mathrm{Var}_{\vartheta^*}\left[\zeta(\vartheta^*)\right]\right),$$
$$E\left[\zeta'(\vartheta^*)\mathbf{F}_i v_1(\vartheta^*)\right] = 0$$

and

$$\mathrm{Var}_{\vartheta^*}\left[\zeta'(\vartheta^*)\mathbf{D}_i\zeta(\vartheta^*)\right] = 2\mathrm{Tr}\left(\mathbf{D}_i \mathrm{Var}_{\vartheta^*}\left[\zeta(\vartheta^*)\right]\mathbf{D}_i \mathrm{Var}_{\vartheta^*}\left[\zeta(\vartheta^*)\right]\right),$$
$$E\left[v_1'(\vartheta^*)\mathbf{F}_i'\zeta(\vartheta^*)\zeta'(\vartheta^*)\mathbf{F}_i v_1(\vartheta^*)\right]$$
$$= \mathrm{Tr}\left(\mathbf{F}_i' \mathrm{Var}_{\vartheta^*}\left[\zeta(\vartheta^*)\right]\mathbf{F}_i \mathrm{Var}_{\vartheta^*}\left[v_1(\vartheta^*)\right]\right),$$
$$E\left[v_i'(\vartheta^*)\mathbf{F}_i'\zeta(\vartheta^*)\zeta'(\vartheta^*)\mathbf{D}_i\zeta(\vartheta^*)\right] = 0. \qquad \square$$

Let ε be a chosen probability expressing the maximum tolerable decrease of the confidence level caused by the fact that the true value ϑ^* of the parameter ϑ is unknown. The notation

$$\boldsymbol{a} = \left[\mathrm{Tr}(\mathbf{U}\mathbf{V}_1), \dots, \mathrm{Tr}(\mathbf{U}\mathbf{V}_p)\right]' \tag{7}$$

will be used.

Definition 5.2. Let

$$\mathcal{K}_\varepsilon = \left\{\boldsymbol{\Delta}:\ \boldsymbol{\Delta} \in \mathbb{R}^p,\ \Delta_i + \vartheta_i^* > 0,\ i = 1, \dots, p,\ \Phi(\boldsymbol{\Delta}) \le \delta_\varepsilon\right\},$$

where

$$\begin{aligned} \Phi(\boldsymbol{\Delta}) &= -\boldsymbol{\Delta}'\boldsymbol{a} + t\sqrt{\boldsymbol{\Delta}'\mathbf{A}\boldsymbol{\Delta}}, \\ \delta_\varepsilon &= \chi_s^2(0, 1-\alpha) - \chi_s^2(0, 1-\alpha-\varepsilon). \end{aligned}$$

The set $\mathcal{K}_\varepsilon$ is called *the insensitivity region for the confidence domain.*

Theorem 5.3. Let $\mathbf{A}$ and $\boldsymbol{a}$ be given by (5) and (7). The boundary of the insensitivity region for the $(1-\alpha)$-confidence domain of the function $\mathbf{X}_1\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{B}$, is the set

$$\overline{\mathcal{K}}_\varepsilon = \Bigg\{\boldsymbol{\Delta}:\ \boldsymbol{\Delta} \in \mathbb{R}^p,\ \Delta_i + \vartheta_i^* > 0,\ i = 1, \dots, p,$$
$$(\boldsymbol{\Delta} - \boldsymbol{u}_0)'(t^2\mathbf{A} - \boldsymbol{a}\boldsymbol{a}')(\boldsymbol{\Delta} - \boldsymbol{u}_0) = \frac{\delta_\varepsilon^2 t^2}{t^2 - \boldsymbol{a}'\mathbf{A}^-\boldsymbol{a}}\Bigg\},$$

where ε, t are chosen positive numbers and

$$\begin{aligned} \boldsymbol{u}_0 &= \frac{\delta_\varepsilon}{t^2 - \boldsymbol{a}'\mathbf{A}^-\boldsymbol{a}}\mathbf{A}^-\boldsymbol{a}, \\ \delta_\varepsilon &= \chi_s^2(0, 1-\alpha) - \chi_s^2(0, 1-\alpha-\varepsilon), \\ s &= r\left(\mathbf{M}_{B'}\mathbf{X}'(\boldsymbol{\Sigma}_{\vartheta^*} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^+\boldsymbol{\Sigma}_{\vartheta^*}\right). \end{aligned}$$

P r o o f. The statement can be proved in the following steps

1. matrices $\mathbf{S}_U$ and $\mathbf{C}_{U,T}$ from Lemma 5.1 are positive semidefinite,
2. $\boldsymbol{a} \in \mathcal{M}(\mathbf{S}_U) \subset \mathcal{M}(\mathbf{A})$,
3. solving the equation $\Phi(\boldsymbol{\Delta}) = \delta_\varepsilon$.

(in more detail cf. [2, 5]). □

6. NUMERICAL DEMONSTRATION

Example 6.1. The problem is to determine two straight lines p_1 and p_2 in a plane. Straight line p_1 has to intersect the point $[2, T]$ and analogously p_2 has to intersect the point $[2, T+1]$, where T is unknown. Moreover, p_1 intersects the point $[0, 0]$. We have only one measurement for each straight line at our disposal; $y_1 = 2.00$ at the point $x = 1$ with the accuracy (standard deviation) $\sigma_1 = 0.01$ for the straight line p_1 and $y_2 = 1.50$ at the point $x = 3$ with the accuracy $\sigma_2 = 0.02$ for the p_2. Both measurements are linearly dependent (correlation coefficient $\rho = 1$).

Let us denote

$$\begin{aligned} p_1: &\quad y = \beta_1 x, \\ p_2: &\quad y = \beta_2 + \beta_3 x, \end{aligned}$$

i.e., the vector of unknown parameters is $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)'$. Constraints on the model are

$$\begin{aligned} 2\beta_1 &= T, \\ \beta_2 + 2\beta_3 &= T + 1, \end{aligned}$$

hence

$$2\beta_1 - \beta_2 - 2\beta_3 + 1 = 0, \tag{8}$$

i.e., $\mathbf{B} = (2, -1, -2)$ and $b = 1$.

Let us put the true value ϑ_i^* equal to the certificate value σ_i^2, $i = 1, 2$. The acceptability of this equality we verify by using sensitivity region. Then

$$\vartheta_1^* = 0.0001,\ \vartheta_2^* = 0.0004,$$
$$\boldsymbol{\Sigma}_{\vartheta^*} = \begin{pmatrix} 0.0001, & 0.0002 \\ 0.0002, & 0.0004 \end{pmatrix},$$
$$\mathbf{V}_1 = \begin{pmatrix} 0.5, & 1.0 \\ 1.0, & 2.0 \end{pmatrix}, \quad \mathbf{V}_2 = \begin{pmatrix} 0.125, & 0.250 \\ 0.250, & 0.500 \end{pmatrix}.$$

Stochastic model describing the process of measurement is given by

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N_2[\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}_{\vartheta^*}],\ \boldsymbol{\beta} \in \{\boldsymbol{u}: \ \boldsymbol{u} \in \mathbb{R}^3,\ \mathbf{B}\boldsymbol{u} + b = 0\} = \mathcal{B},$$
$$\mathbf{X} = \begin{pmatrix} 1, & 0, & 0 \\ 0, & 1, & 3 \end{pmatrix}.$$

Then

$$\mathcal{M}(\mathbf{X}') = \begin{pmatrix} 1, & 0 \\ 0, & 1 \\ 0, & 3 \end{pmatrix}, \quad Ker(\mathbf{X}) = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix},$$

$$\mathcal{M}(\mathbf{B}') = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}, \quad Ker(\mathbf{B}) = \begin{pmatrix} 4, & 5 \\ -2, & 2 \\ 5, & 4 \end{pmatrix}.$$

Since $\mathcal{M}(\mathbf{X}', \mathbf{B}') = \mathbb{R}^3$, all functions $\boldsymbol{h}'\boldsymbol{\beta}$, $\boldsymbol{h} \in \mathbb{R}^3$, are unbiasedly estimable and thus the vector $\boldsymbol{\beta}$ is also unbiasedly estimable. The $\boldsymbol{\vartheta}^*$-LBLUE of the whole parameter $\boldsymbol{\beta}$ is $\widehat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*) = (2.00, 12.00, -3.50)'$ (cf. Lemma 3.2). The $(1-\alpha)$-confidence region of $\boldsymbol{\beta}$ is given as

$$\mathcal{E}_{1-\alpha}(\boldsymbol{\beta}) = \Big\{\boldsymbol{\beta} : \ \boldsymbol{\beta} \in Ker(\mathbf{B}) + \boldsymbol{\beta}_0,$$
$$\big[\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\big]' \Big[\mathrm{Var}_{\vartheta^*}\big(\widehat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\big)\Big]^- \big[\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\big] \le \chi^2_s(1-\alpha)\Big\},$$

where $\boldsymbol{\beta}_0$ is a vector satisfying the relation (8), e.g. $\boldsymbol{\beta}_0 = (0.5, 6.0, -2.0)'$, and

$$s = r\Big(\mathrm{Var}_{\vartheta^*}\big(\widehat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\big)\Big) = r\left(\mathbf{M}_{B'}\mathbf{X}'[\boldsymbol{\Sigma}_{\vartheta^*} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}']^-\boldsymbol{\Sigma}_{\vartheta^*}\right) = 1.$$

(cf. Corollary 4.3).

The covariance matrix $\mathrm{Var}_{\vartheta^*}\big(\widehat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\big)$ is

$$\mathrm{Var}_{\vartheta^*}\big(\widehat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\big) = (\mathbf{M}_{B'}\mathbf{W}_{\vartheta^*}\mathbf{M}_{B'})^+ - \mathbf{M}_{B'} = \begin{pmatrix} 0.0001, & 0.0002, & 0 \\ 0.0002, & 0.0004, & 0 \\ 0, & 0, & 0 \end{pmatrix}.$$

Since its spectral decomposition is given by

$$\mathrm{Var}_{\vartheta^*}\big(\widehat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\big) = 0.0005 \begin{pmatrix} 0.4472 \\ 0.8944 \\ 0 \end{pmatrix} (0.4472, 0.8944, 0),$$

the subspace $Ker(\mathbf{B})$ can be expressed, in our case, in a more suitable form as

$$Ker(\mathbf{B}) = \begin{pmatrix} 0.4472, & -0.5963 \\ 0.8944, & 0.2981 \\ 0, & 0.7454 \end{pmatrix}.$$

Thus 0.95-confidence region of $\boldsymbol{\beta}$ is the set characterized by the abscissa $[-0.0438, 0.0438]$ ($0.0438 = \sqrt{0.0005\chi^2_1(0.95)}$, $\chi^2_1(0.95) = 3.84$), which center is shifted to the point $\widehat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*) = (2.00, 12.00, -3.50)'$ and its direction is given by the vector $(0.4472, 0.8944, 0)'$ (cf. Figure 1).

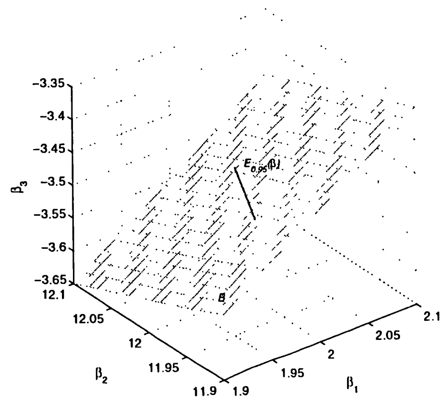

Fig. 1. 0.95-confidence region $\mathcal{E}_{0.95}(\boldsymbol{\beta})$ in the linear manifold $\mathcal{B}$.

Now, let us consider the problem of the sensitivity. Let the tolerable decrease of the 0.95-confidence level be 10 %, i. e., $\varepsilon = 0.1$. Let $t = 4$. The center of the insensitivity region $\mathcal{K}_{0.1}$ for the confidence domain $\mathcal{E}_{0.95}(\boldsymbol{\beta})$ is $\boldsymbol{u}_0 = (-0.3328\cdot 10^{-3}, -0.0832\cdot 10^{-3})'$. Eigenvalues and eigenvectors of the matrix $t^2\mathbf{A} - \boldsymbol{a}\boldsymbol{a}'$ are

$$\lambda_1 = 0, \quad \boldsymbol{v}_1 = (0.2425, -0.9701)',$$
$$\lambda_2 = 2.6603 \cdot 10^7, \quad \boldsymbol{v}_2 = (-0.9701, -0.2425)'.$$

Hence, the insensitivity region is the band in the direction $\boldsymbol{v}_1$ and its width γ is equal to

$$\gamma = 2\sqrt{\frac{\delta_\varepsilon^2 t^2}{(t^2 - \boldsymbol{a}'\mathbf{A}^-\boldsymbol{a})\lambda_2}} = 0.9702 \cdot 10^{-3}.$$

Moreover, tolerable shifts $(\delta\vartheta_1, \delta\vartheta_2)'$ must satisfy $\delta\vartheta_1 > -0.0001$ and $\delta\vartheta_2 > -0.0004$. Thus, the insensitivity region is not the whole band but the triangle $P_1P_2P_3$ only (see Figure 2). Here $P_1 = (-0.0001, -0.0004)'$, $P_2 = (0.0011, -0.0004)'$ and $P_3 = (-0.0001, 0.0042)'$.

From the practical viewpoint it is more important to determine tolerable shifts $\delta\sigma = \sqrt{\delta\boldsymbol{\vartheta}}$ of the parameter $\boldsymbol{\sigma}$ around its true value $\boldsymbol{\sigma}^*$. Evidently, the shift $\delta\boldsymbol{\sigma}$ is tolerable iff the shift $\delta\boldsymbol{\vartheta}$ is tolerable. The tolerable shifts $\delta\boldsymbol{\sigma}$ are given by the equation

$$\delta\sigma_i = \sqrt{\vartheta_i^* + \delta\vartheta_i} - \sigma_i^*, \ i = 1, 2.$$

The region $Q_1Q_2Q_3$ of all tolerable shifts $(\delta\sigma_1, \delta\sigma_2)'$ is shown in Figure 2. Here $Q_1 = (-0.01, -0.02)'$, $Q_2 = (0.0246, -0.02)'$ and $Q_3 = (-0.01, 0.0478)'$.

The maximum tolerable shift $\delta\sigma_1$ is 0.0246 if the measurement Y_2 is exact. Analogously, the maximum tolerable shift $\delta\sigma_2$ is 0.0478 if the measurement Y_1 is exact. In both cases, standard deviation σ_i can increase approximately by 240 %. Tolerable

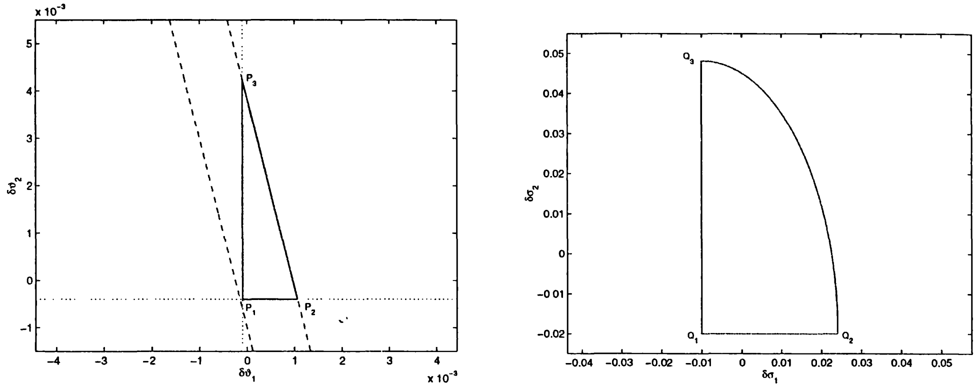

Fig. 2. Insensitivity regions $\mathcal{K}_{0.1}$. Left: tolerable shifts $\delta\boldsymbol{\vartheta}$ around $\boldsymbol{\vartheta}^*$, right: tolerable shifts $\delta\boldsymbol{\sigma}$ around $\boldsymbol{\sigma}^*$.

shifts $\delta\boldsymbol{\sigma}$ are large enough therefore we can put the true values $\boldsymbol{\sigma}^*$ equal to the certificate values. Further, both measurements are equally important from the aspect of the accuracy of the estimator $\widehat{\boldsymbol{\beta}}$.

(Received November 29, 2002.)

REFERENCES

[1] L. Kubáček, L. Kubáčková, and J. Volaufová: Statistical Models with Linear Structures. Veda, Bratislava 1995.

[2] L. Kubáček: Linear model with inaccurate variance components. Appl. Math. *41* (1996), 433–445.

[3] L. Kubáček and L. Kubáčková: Unified approach to determining nonsesitiveness regions. Tatra Mt. Math. Publ. *17* (1999), 121–128.

[4] L. Kubáček and L. Kubáčková: Nonsensitiveness regions in universal models. Math. Slovaca *50* (2000), 2, 219–240.

[5] E. Lešanská: Insensitivity regions for estimators of mean value parameters in mixed models with constraints. Tatra Mt. Math. Publ. *22* (2001), 37–49.

[6] E. Lešanská: Effect of inaccurate components in mixed models with constraints. In: Proc. Datastat'01, Folia Fac. Sci. Nat. Univ. Masaryk. Brun. Math., Mathematica *11* 2002, pp. 163–172.

[7] C. R. Rao and S. K. Mitra: Generalized Inverse of Matrices and its Applications. Wiley, New York – London – Sydney – Toronto 1971.

[8] H. Scheffé: The Analysis of Variance. Wiley, New York – London – Sydney 1967.

RNDr. Eva Fišerová, Ph.D. and Prof. Ing. RNDr. Lubomír Kubáček, DrSc., Department of Mathematical Analysis and Applied Mathematics, Faculty of Science, Palacký University, Tomkova 40, 779 00 Olomouc. Czech Republic.
e-mails: fiserova@inf.upol.cz, kubacekl@aix.upol.cz.