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TIME–DOMAIN AND PARAMETRIC L^2-PROPERTIES CORRESPONDING TO POPOV INEQUALITY

MIHAIL VOICU AND OCTAVIAN PASTRAVANU

For Popov's frequency-domain inequality a general solution is constructed in L^2, which relies on the strict positive realness of a generating function. This solution allows revealing time-domain properties, equivalent to the fulfilment of Popov's inequality in the frequency-domain. Particular aspects occurring in the dynamics of the linear subsystem involved in Popov's inequality are further explored for step response, as representing a usual characterization in control system analysis. It is also shown that such behavioural particularities are directly related to the BIBO stability of the linear subsystem.

1. INTRODUCTION

The absolute stability is defined for a standard nonlinear closed-loop structure as shown in Figure 1.

The linear subsystem is modeled by the minimal state-space description:

$$\begin{cases} \dot{x} = Ax - bv, \quad t \in \mathbb{R}_+, \ x \in \mathbb{R}^r, \ v \in \mathbb{R}, \\ y = cx, \ y \in \mathbb{R}, \end{cases} \tag{1}$$

and the corresponding transfer function:

$$G(s) = c(I_r s - A)^{-1} b = \frac{1}{s^\nu} \cdot \frac{Q(s)}{P(s)} = \frac{1}{s^\nu} \cdot \frac{b_m s^m + \ldots + b_1 s + 1}{a_n s^n + \ldots + a_1 s + 1}, \tag{2}$$

with $r = n + \nu > m$. The gain factor of $G(s)$ is 1. The whole gain factor of the open-loop is included in the nonlinear subsystem, which is described by the function:

$$v = f(y). \tag{3}$$

This is a "sector function" belonging to the class of functions:

$$C_{[0,K]} = \left\{ f \in \overline{C}_0; \ 0 \le \frac{f(y)}{y} \le K, \ y \neq 0 \right\}, \tag{4}$$

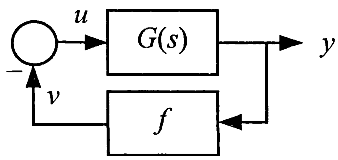

Fig. 1. Block diagram associated with the standard approach to absolute stability.

where $\overline{C}_0$ is the set of piecewise continuous functions. Class (4) may be replaced by one of the following classes: $C_{[\varepsilon,K]}$ ($\varepsilon > 0$, arbitrarily small), $C_{(0,K]}$, $C_{[0,K)}$, $C_{(0,K)}$ or $C_{(0,+\infty)}$ for which definitions similar to (4) (mutatis mutandis) may be formulated.

With respect to the equilibrium state:

$$x = 0; \quad v = 0, \quad y^{(k)}(t) = 0, \ k = 0, 1, \dots, r-1, \tag{5}$$

of nonlinear closed-loop system (1) and (3), the following definition is available.

Definition 1. Nonlinear closed-loop system (1) and (3) is called *absolutely stable* if for each $f \in C_{[0,K]}$ (or $C_{[\varepsilon,K]}$, $C_{(0,K]}$, $C_{[0,K)}$, $C_{(0,K)}, C_{(0,+\infty)}$, according to the case considered) equilibrium state (5) is global asymptotically stable.

According to the pole location of $G(s)$ in the complex plane, one may distinguish the following two cases:

a) Principal case: all the poles of $G(s)$ are located in the left complex semiplane, i.e. in $\{\Re e \ s < 0\}$.

b) Critical case: at least one pole of $G(s)$ is located on the imaginary axis of the complex plane, i.e. in $\{\Re e \ s = 0\}$, and the rest of the poles are all located in the left complex semiplane.

Regarding Definition 1, many results have been formulated. Among them, let us consider the following one [2].

Theorem 1. Nonlinear closed-loop system (1) and (3), where $G(s)$ is defined by (2) and:

a) $f \in C_{[0,K]}$, $0 < K \leq +\infty$, in the principal case,

or

b) $f \in C_{(0,K]}$, $0 < K < +\infty$, in the critical case,

is absolutely stable if there exists $q \in \mathbb{R}$ such that the following (Popov) inequality is met:

$$\Re e \left[(1 + jq\omega) \, G(j\omega) \right] > -\frac{1}{K} \quad \forall \omega \geq 0. \tag{6}$$

The purpose of the current paper consists in identifying time-domain properties for linear subsystem (1) that are equivalent to the fulfilment of inequality (6) in the frequency-domain. Our approach relies on deriving the general form of the L^2-solutions to Popov inequality (6), which can be expressed in terms of a generating function and allow exploiting the concept of strict positive realness. Particular aspects occurring in the dynamics of linear subsystem (1) are further explored for step response, as representing one of the most frequently discussed characterizations referred to in control engineering. It is also shown that such behavioural particularities are directly related to the BIBO stability of linear subsystem (1).

Within this context our results are primarily of mathematical interest in order to enlarge the theoretical knowledge and to create perspectives for alternative compensator-synthesis. These results can be also regarded as starting points for a future research on the development of testing procedures.

2. L^2-SOLUTIONS TO POPOV INEQUALITY

In the sequel we consider only the principal case, i.e. $\nu = 0$ and rational function (2) is BIBO stable. However the main lines of the method to be presented can be also applied for addressing the critical case ($\nu \geq 1$ and $P(s)$ is a Hurwitz polynomial in (2)) by an adequate treatment of $s^\nu = (j\omega)^\nu$ in inequality (6). As a matter of fact the usual cases are $\nu = 1, 2$ for which the condition of ε-stability, [2], is also requested.

Inequality (6), in which

$$G(j\omega) = R(\omega) + jI(\omega) \tag{7}$$

and $R(\omega)$ and $I(\omega)$ are conjugate functions, i.e. related by the Hilbert transformation [1], is met if

$$\begin{cases} R(\omega) + \dfrac{1}{K} \cdot \dfrac{1}{q^2\omega^2 + 1} > 0 & (8) \\[2ex] \left[I(\omega) - \dfrac{1}{K} \cdot \dfrac{q\omega}{q^2\omega^2 + 1} \right] \operatorname{sgn} q < 0. & (9) \end{cases}$$

The case $q = 0$ and/or $\omega = 0$ is obvious.

For $q > 0$ and $\omega > 0$ one multiplies (9) by $q\omega$, then the result is added with (8) and finally one obtains (6). The case $q < 0$ for inequality (6) may be reduced to the case $q > 0$ according to [3]. Therefore, in the remainder of the text, our discussion covers only the situation when $q > 0$.

Next we will examine in what extent Popov inequality (6) implies inequalities (8), (9).

a) First, we show that the solution on $L^2(-\infty, +\infty)$ of the equation:

$$R_0(\omega) + q(j\omega)jI_0(\omega) + \frac{1}{K} = 0, \tag{10}$$

associated with inequality (6), is:

$$\begin{cases} R_0(\omega) = -\dfrac{1}{K} \cdot \dfrac{1}{q^2\omega^2 + 1} & (11) \\[2ex] I_0(\omega) = \dfrac{1}{K} \cdot \dfrac{q\omega}{q^2\omega^2 + 1}. & (12) \end{cases}$$

Indeed, for

$$G_0(j\omega) = R_0(\omega) + jI_0(\omega) = \mathcal{F}_I^2\left\{g_0(t)\right\}, \tag{13}$$

$$g_0(t) = g_{0p}(t) + g_{0i}(t), \tag{14}$$

$$g_{0p}(t) = g_{0i}(t)\operatorname{sgn} t, \tag{15}$$

where $\mathcal{F}_I^2$ symbolizes the unilateral Fourier direct transformation (on $L^2(0, +\infty)$), $g_0(t)$ is the response to the Dirac impulse $\delta(t)$, and $g_{0p}(t)$ and $g_{0i}(t)$ are the even and odd parts of $g_0(t)$, respectively, it results:

$$g_{0p}(t) \overset{\mathcal{F}^2}{\circ\!\!-\!\!-\!\!\bullet} R_0(\omega), \quad g_{0i}(t) \overset{\mathcal{F}^2}{\circ\!\!-\!\!-\!\!\bullet} jI_0(\omega). \tag{16}$$

With (16), equation (10), taking into account (15), becomes:

$$qDg_{0i}(t) + g_{0i}(t)\operatorname{sgn} t + \frac{1}{K}\delta(t) = 0, \tag{17}$$

where D symbolizes the derivation in the distribution sense.

By integrating (17) between -0 and $+0$ one obtains:

$$q\left[g_{0i}(+0) - g_{0i}(-0)\right] + \frac{1}{K} = 0 \tag{18}$$

from which, with (15), it results:

$$g_{0i}(+0) = -\frac{1}{2qK}. \tag{19}$$

For $t > 0$, equation (17) has the form:

$$q\dot{g}_{0i}(t) + g_{0i}(t) = 0. \tag{20}$$

With the initial condition (19), the solution of equation (20) is:

$$g_{0i}(t) = -\frac{1}{2qK}e^{-\frac{t}{q}}, \quad t > 0. \tag{21}$$

With (14) and (15), from (21), it follows:

$$g_0(t) = -\frac{1}{qK}e^{-\frac{t}{q}}\sigma(t), \tag{22}$$

$$G_0(s) = \mathcal{L}\{g_0(t)\} = -\frac{1}{K}\cdot\frac{1}{qs+1}, \tag{23}$$

where $\sigma(t)$ it the unit step function and $\mathcal{L}$ symbolizes the Laplace transformation.

From (23), for $s = j\omega$, one obtains:

$$G_0(j\omega) = R_0(\omega) + jI_0(\omega) = -\frac{1}{K}\cdot\frac{1}{q^2\omega^2+1} + \frac{1}{K}\cdot\frac{jq\omega}{q^2\omega^2+1}, \tag{24}$$

from which it results (11) and (12).

b) To examine if (6) implies (8), (9), one associates to inequality (6) the following equation:

$$R(\omega) + q(j\omega)jI(\omega) + \frac{1}{K} = a + R_1(\omega), \tag{25}$$

from which, for

$$d = \lim_{\omega \to +\infty} \omega I(\omega), \quad d \in \mathbb{R}, \tag{26}$$

it results

$$-d + \frac{1}{K} = a. \tag{27}$$

On the other hand, according to (6), from (25) it follows:

$$a + R_1(\omega) > 0, \ \ \omega \in \mathbb{R}, \tag{28}$$

and $R_1(\omega)$ is an even function with $R_1 \in L^2(-\infty, +\infty)$.

Now let us introduce:

$$G(j\omega) = R(\omega) + jI(\omega) = \mathcal{F}_I^2\{g(t)\}, \tag{29}$$

$$g(t) = g_p(t) + g_i(t), \tag{30}$$

$$g_p(t) = g_i(t)\operatorname{sgn} t, \tag{31}$$

$$g_p(t) \circ\!\!\xrightarrow{\mathcal{F}^2}\!\!\bullet R(\omega), \quad g_i(t) \circ\!\!\xrightarrow{\mathcal{F}^2}\!\!\bullet jI(\omega), \tag{32}$$

and

$$G_1(j\omega) = R_1(\omega) + jI_1(\omega) = \mathcal{F}_I^2\{g_1(t)\}, \tag{33}$$

$$g_1(t) = g_{1p}(t) + g_{1i}(t), \tag{34}$$

$$g_{1p}(t) = g_{1i}(t)\operatorname{sgn} t, \tag{35}$$

$$g_{1p}(t) \circ\!\!\xrightarrow{\mathcal{F}^2}\!\!\bullet R_1(\omega), \quad g_{1i}(t) \circ\!\!\xrightarrow{\mathcal{F}^2}\!\!\bullet jI_1(\omega), \tag{36}$$

in which the subscripts p and i designate the even and odd parts of functions $g(t)$ and $g_1(t)$.

Under these circumstances, equation (25) becomes:

$$qDg_i(t) + g_i(t)\operatorname{sgn} t + \frac{1}{K}\delta(t) = a\delta(t) + g_{1p}(t). \tag{37}$$

By integrating (37) between -0 and $+0$, one obtains:

$$q\left[g_i(+0) - g_i(-0)\right] + \frac{1}{K} = a. \tag{38}$$

Using (31), from (38) it results:

$$g_i(+0) = \frac{1}{2q}\left(a - \frac{1}{K}\right). \tag{39}$$

For $t > 0$, equation (37) has the form:

$$q\dot{g}_i(t) + g_i(t) = g_{1p}(t), \tag{40}$$

to which, for $g(t)$ and $g_1(t)$ (with (30), (31) and (34), (35), respectively), it corresponds:

$$q\dot{g}(t) + g(t) = g_1(t), \quad t > 0. \tag{41}$$

Using the Laplace transformation in (41) with the initial condition:

$$g(+0) = 2g_i(+0) = \frac{1}{q}\left(a - \frac{1}{K}\right), \tag{42}$$

from (41) it follows:

$$G(s) = -\frac{1}{K} \cdot \frac{1}{qs+1} + \frac{1}{qs+1}\left[a + G_1(s)\right]. \tag{43}$$

One may easily ascertain from (43) (for $s = j\omega$) that, in general, inequality (6) does not imply (8), (9). Thus, the construction procedure developed in this section for the set of all L^2-solutions to Popov inequality (6) can be summarized as follows:

Theorem 2. All the L^2-solutions of Popov inequality (6) can be written in form (43), where $G_1(s)$ is strictly proper and $a + G_1(s)$ is a strictly positive real function.

Once the general form of L^2-solutions to Popov inequality has been found, the structure of transfer function $G(s)$ should be understood in terms of a parametrisation which allows a unified approach when dealing with the strict positive realness of the $a + G_1(s)$. These aspects are investigated in the next section.

3. COMMENTS ON THE PARAMETRISATION OF THE L^2-SOLUTIONS

The strictly positive real function $a + G_1(s)$ may be regarded as a *generatrix* of the transfer function $G(s)$ (43), which, according to Theorem 2, provides the general form of L^2-solutions to inequality (6). This fact and the properties of generatrix $a + G_1(s)$ will be exploited in the sequel, in order to derive some parametric aspects and time-domain properties of the linear subsystem ($G(s)$) involved in Popov inequality (6).

To (43) one has to add the condition $G(0) = 1$ (see (2) for $\nu = 0$), which leads to the following refinement of the generatrix:

$$a + G_1(s) = \alpha\left[\beta + G_2(s)\right], \tag{44}$$

where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $G_2(s)$ is a strictly proper rational function, with $G_2(j\omega) = R_2(\omega) + jI_2(\omega)$. For $\beta + G_2(s)$ one has to take into account the unit gain factor of $G(s)$, i.e.:

$$\beta + G_2(0) = 1. \tag{45}$$

According to inequality (28) one obtains:

$$\alpha\,[\beta + R_2\,(\omega)] > 0, \quad \omega \in I\!R, \tag{46}$$

which, together with (45), uniquely leads to:

$$\alpha > 0, \quad \beta \geq 0, \quad \beta + R_2\,(\omega) > 0, \quad \omega \in I\!R. \tag{47}$$

Using (44), equation (43) may be equivalently rewritten as:

$$G(s) = -\frac{1}{K} \cdot \frac{1}{qs+1} + \frac{1}{qs+1}\alpha\,[\beta + G_2(s)]\,, \tag{48}$$

from which, for $s = 0$, it results:

$$1 = G(0) = -\frac{1}{K} + \alpha\,[\beta + G_2(0)]\,. \tag{49}$$

Taking into account (45), relation (49) yields:

$$\alpha = 1 + \frac{1}{K}\cdot \tag{50}$$

On the other hand, from (48) one also obtains:

$$g\,(+0) = \lim_{s\to\infty} sG(s) = -\frac{1}{Kq} + \frac{1}{q}\alpha\beta, \tag{51}$$

which, together with (50), allows to express β as follows:

$$\beta = \frac{qKg\,(+0) + 1}{K+1}\cdot \tag{52}$$

It should be noticed that whenever $\beta = 0$, the strictly positive realness of $\beta + G_2(s)$ necessarily requires a difference of exactly one unit between the degrees of the numerator and denominator of $G_2(s)$.

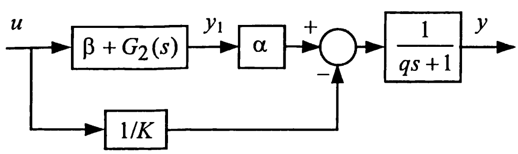

Fig. 2. Block diagram associated with parametrisation (48) of $G(s)$.

Finally, in order to illustrate, graphically, the above comments on the parametrisation of the L^2-solutions of Popov inequality (6), the block diagram in Figure 2 is associated with expression (48) of the transfer function $G(s)$.

4. TIME–DOMAIN INTERPRETATION OF POPOV INEQUALITY

Consider $G(s)$, given by (48), with the structure depicted in Figure 2, for which the following differential equation may be written:

$$q\dot{y} + y = \alpha y_1 - \frac{1}{K}u, \quad t \geq 0. \tag{53}$$

This equation (which renders evident the role played by $y_1(t)$ as the output of the subsystem with the strictly positive real function $\beta + G_2(s)$) allows deriving *time-domain properties* that are equivalent to Popov's frequency-domain inequality (6). According to parametrisation (48) of $G(s)$, the cases $\beta > 0$ and $\beta = 0$ are to be separately addressed.

Theorem 3. Popov's frequency-domain inequality (6) is met with $0 < K < +\infty$ and $q > 0$ if and only if:

(i) for $\beta > 0$, there exists $\eta > -\frac{1}{K}$ such that the time-domain inequality:

$$\int_0^T u(t)(q\dot{y}(t) + y(t))\mathrm{d}t \geq \eta \int_0^T u^2(t)\mathrm{d}t \tag{54}$$

holds for all $T > 0$ and all $u(t) \in L^2[0,T)$ ensuring finite values for integrals;

(ii) for $\beta = 0$, there exists $\gamma > 0$ such that the time-domain inequality:

$$\int_0^T e^{\gamma t}u(t)(q\dot{y}(t) + y(t))\mathrm{d}t \geq -\frac{1}{K}\int_0^T e^{\gamma t}u^2(t)\mathrm{d}t \tag{55}$$

holds for all $T > 0$ and all $u(t) \in L^2[0,T)$ ensuring finite values for integrals.

Proof. *Case* (i). The availability of $y_1(t)$ in the block-diagram given in Figure 2 permits exploiting the following necessary and sufficient condition [4] for the strictly positive realness of $\beta + G_2(s)$:

$$\exists\, \rho > 0 : \int_0^T u(t)y_1(t)\mathrm{d}t \geq \rho \int_0^T u^2(t)\mathrm{d}t \tag{56}$$

for all $T > 0$ and all $u(t) \in L^2[0,T)$.

On the other hand, by multiplying equation (53) by arbitrary $u(t) \in L^2[0,T)$, one gets:

$$u(t)\,(q\dot{y}(t) + y(t)) + \frac{1}{K}u^2(t) = \alpha u(t)y_1(t), \quad t \in [0,T), \tag{57}$$

which, by integration, yields:

$$\int_0^T u(t)\,(q\dot{y}(t) + y(t))\,\mathrm{d}t + \frac{1}{K}\int_0^T u^2(t)\mathrm{d}t = \alpha \int_0^T u(t)y_1(t)\mathrm{d}t. \tag{58}$$

Necessity. Function $\beta + G_2(s)$ is strictly positive real, since $G(s)$ in (48) defines all the L^2-solutions of the frequency-domain inequality (6). Thus, (56) is fulfilled and, from equality (58), one gets:

$$\int_0^T u(t)\left(q\dot{y}(t) + y(t)\right) \mathrm{d}t \geq \left(\alpha\rho - \frac{1}{K}\right) \int_0^T u^2(t)\mathrm{d}t, \tag{59}$$

which becomes (54), by denoting $\alpha\rho - \frac{1}{K} =: \eta$.

Sufficiency. According to equality (58), time-domain inequality (54) yields:

$$\alpha \int_0^T u(t)y_1(t)\mathrm{d}t \geq \left(\eta + \frac{1}{K}\right) \int_0^T u^2(t)\mathrm{d}t, \tag{60}$$

which becomes (56), by denoting $\frac{1}{\alpha}\left(\eta + \frac{1}{K}\right) =: \rho$. Hence, $\beta + G_2(s)$ is a strictly positive real function, and inequality (6) is satisfied for $G(s)$ given by (48).

Case (ii). The availability of $y_1(t)$ in the block-diagram given in Figure 2 permits exploiting the following necessary and sufficient condition [4] for the strictly positive realness of $G_2(s)$ (i. e. $\beta + G_2(s)$ with $\beta = 0$):

$$\exists\, \gamma > 0 : \int_0^T e^{\gamma t} u(t)y_1(t) \geq 0 \tag{61}$$

for all $T > 0$ and all $u(t) \in L^2[0, T)$ ensuring a finite value for the integral.

On the other hand, by multiplying equation (53) by $e^{\gamma t}$ and arbitrary $u(t) \in L^2[0, T)$, one gets:

$$e^{\gamma t}u(t)\left(q\dot{y}(t) + y(t)\right) + \frac{1}{K}e^{\gamma t}u^2(t) = \alpha e^{\gamma t}u(t)y_1(t), \quad t \in [0, T), \tag{62}$$

which, by integration, yields:

$$\int_0^T e^{\gamma t}u(t)\left(q\dot{y}(t) + y(t)\right) \mathrm{d}t + \frac{1}{K}\int_0^T e^{\gamma t}u^2(t)\mathrm{d}t = \alpha \int_0^T e^{\gamma t}u(t)y_1(t)\mathrm{d}t. \tag{63}$$

Necessity. Function $G_2(s)$ is strictly positive real, since $G(s)$ in (48) defines all L^2-solutions of the frequency-domain inequality (6). Thus, (61) is fulfilled and, from equality (63), one gets (55).

Sufficiency. According to equality (63), time-domain inequality (55) yields (61). Hence $G_2(s)$ is a strictly positive real function, and inequality (6) is satisfied for $G(s)$ given by (48) with $\beta = 0$. □

It is worth noticing that Theorem 3 highlights a qualitative-type equivalence between time- and frequency-domain in interpreting Popov inequality (6). Obviously, time-domain inequalities (54) and (55) cannot be applied directly to practical studies, as *sufficient conditions*, because they involve input signals $u(t)$ for linear subsystem (1) belonging to the whole class $L^2[0,T)$. An experimental approach based on these inequalities would require a drastic limitation for the class of signals $u(t)$, fact which means (along the lines of the above proof) to formulate a much stronger time-domain characterization for the strict positive realness, than given in [4]. Although such a result was not found in literature by the authors of the present paper, in their opinion, the possibility to restrict the input signal class appears quite natural, and, therefore, their future efforts will focus on this problem. However, inequalities (54) and (55), regarded as *necessary conditions* for the fulfilment of Popov inequality (6), can be used with "standard" input signal (e. g. step, ramp, etc.), for which the corresponding output signals $y(t)$ of linear subsystem (1) are expected to exhibit some particular features. This idea is illustrated in the next section for step input signals.

5. PARTICULARITIES OF THE STEP RESPONSE INDUCED BY POPOV INEQUALITY

The result proved in the previous section shows that inequality (6), operating in the *frequency-domain* as a constraint for the *transfer function* of linear subsystem (1), means *time-domain* particular properties, expressed by (54) or (55) in terms of *input* and *output* signals. Such particular behavioural aspects are visible in the time-domain for all "standard" input signals applied to linear subsystem (1) and their corresponding responses. The case of step response is extremely relevant in this sense and requires a reasonable mathematical effort to point out the existence of some specific characteristics induced by Popov inequality.

For $u(t) = \sigma(t)$ (unit step function) and by denoting with $h_1(t)$ and $h(t)$ the unit step responses corresponding to y_1 and y (Figure 2), equation (53) becomes:

$$q\dot{h}(t) + h(t) = \alpha h_1(t) - \frac{1}{K}, \quad t \geq 0. \tag{64}$$

Because $h(0) = 0$, it follows that (64) may be equivalently written as:

$$q\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t h(\theta)\mathrm{d}\theta\right] + \int_0^t h(\theta)\mathrm{d}\theta = \alpha\int_0^t h_1(\theta)\mathrm{d}\theta - \frac{1}{K}t. \tag{65}$$

Theorem 4. Let $K > 0$ with the signification issued from Theorem 1 and let $g(+0) = h'(+0)$ be the slope of $h(t)$ in $t = +0$. If

(i) there exists $q > 0$ such that inequality (6) is satisfied

and

(ii) for $q > 0$ at (i), the following inequality is met:

$$qKh'(+0) + 1 > 0, \tag{66}$$

then there exists $\rho > 0$ such that

$$\int_0^t \frac{1}{t}h(\theta)\mathrm{d}\theta \geq \left[\rho\left(1+\frac{1}{K}\right)-\frac{1}{K}\right]\left[1-\frac{q}{t}\left(1-e^{-\frac{t}{q}}\right)\right], \quad t>0. \tag{67}$$

P r o o f. For $\beta > 0$ (see (52) and (66)), one may use in (65) the following result from [4]:
There exists $\rho > 0$ such that

$$\int_0^t h_1(\theta)\mathrm{d}\theta \geq \rho t \quad \text{for each } t \geq 0. \tag{68}$$

With (68), from (65), it follows:

$$\int_0^t h(\theta)\mathrm{d}\theta \geq \left(\alpha\rho - \frac{1}{K}\right)\frac{1}{q}\int_0^t e^{-\frac{t-\theta}{q}}\theta\mathrm{d}\theta = \left[\rho\left(1+\frac{1}{K}\right)-\frac{1}{K}\right]\left[t-q\left(1-e^{-\frac{t}{q}}\right)\right]. \tag{69}$$

By multiplying this result by $1/t$, $t > 0$, one obtains (67). □

Because in (47) we have $\beta \geq 0$, it remains to derive a result for the case $\beta = 0$ when (68) is no more valid, i.e. (67) is not valid too.

Theorem 5. Let $K > 0$ with the signification issued from Theorem 1 and let $g(+0) = h'(+0)$ be the slope of $h(t)$ in $t = +0$. If

(i) there exists $q > 0$ such that equality (6) is satisfied

and

(ii) for $q > 0$ at (i), the following equality is met:

$$qKh'(+0) + 1 = 0, \tag{70}$$

then there exists $\gamma > 0$ such that:

$$\int_0^t e^{-\gamma(t-\theta)}h(\theta)\mathrm{d}\theta \geq -\frac{1}{\gamma K}\left[1+\frac{1}{1-\gamma q}\left(\gamma q e^{-\frac{t}{q}} - e^{-\gamma t}\right)\right], \quad t \geq 0. \tag{71}$$

P r o o f. For $\beta = 0$ (see (52) and (70)), necessarily implying a difference of exactly one unit between the degrees of the two polynomials defining $G_2(s)$, the following result from [4] may be used in (65):

There exists $\gamma > 0$ such that

$$\int_0^t e^{\gamma\theta} h_1(\theta)\mathrm{d}\theta \geq 0 \quad \text{for each} \quad t \geq 0. \tag{72}$$

By multiplying (64) by $e^{\gamma\theta}$, after some simple calculation, it results:

$$q\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t e^{\gamma\theta} h(\theta)\mathrm{d}\theta\right] + (1-q\gamma)\int_0^t e^{\gamma\theta} h(\theta)\mathrm{d}\theta$$

$$= \alpha \int_0^t e^{\gamma\theta} h_1(\theta)\mathrm{d}\theta - \frac{1}{\gamma K}(e^{\gamma t} - 1). \tag{73}$$

Now, by using (72), one obtains:

$$\int_0^t e^{\gamma\theta} h(\theta)\mathrm{d}\theta \geq -\frac{1}{q\gamma K}\int_0^t e^{-(\frac{1}{q}-\gamma)(t-\theta)}(e^{\gamma\theta} - 1)\mathrm{d}\theta$$

$$= -\frac{e^{\gamma t}}{\gamma K}\left[1 + \frac{1}{1-\gamma q}\left(\gamma q e^{-\frac{t}{q}} - e^{-\gamma t}\right)\right]. \tag{74}$$

Finally, by multiplying this result by $e^{-\gamma t}$, $t \geq 0$, one obtains (71). □

In order to evaluate the full signification of the integrals got in (67) and (71), we also present two BIBO-stability results referring to arbitrary linear time-invariant systems, which are based exactly on these integrals.

Theorem 6. A linear time-invariant dynamical system characterized by unit step response $h(t)$ is BIBO-stable if and only if one of the following two equivalent conditions is met:

a) $\lim_{t\to\infty} \int_0^t \frac{1}{t} h(\theta)\mathrm{d}\theta$ exists and is finite;

b) for any $\gamma > 0$ $\lim_{t\to\infty} \int_0^t e^{-\gamma(t-\theta)} h(\theta)\mathrm{d}\theta$ exists and is finite.

Proof. Is based on the absolute integrability of the impulse response $g(t)$ and on the relation $h(t) = \int_0^t g(\theta)\mathrm{d}\theta$. □

It is eminently clear that the time-domain inequalities stated in Theorems 4 and 5 can be considered for future studies on the development of experimental procedures, by applying $h(t)$ as the input signal of an integrator (Theorem 4), or of a linear, first-order filter with the time-constant $1/\gamma > 0$ (Theorem 5). The practical meaning refers to the fact that, whenever time-domain inequality (67) or (71), respectively, is violated, Popov's frequency-domain inequality cannot be fulfilled.

Finally, it is worth mentioning that results similar to Theorems 4, 5 and 6, can be formulated for *any polynomial-type input signal* applied to linear subsystem (1), in order to emphasize particular time-domain properties induced by Popov's frequency-domain inequality (6).

6. CONCLUSIONS

The analysis of time-domain properties corresponding to Popov inequality relies on the general form of the L^2-solutions built for this inequality (Theorem 2). Such a general form reveals the role played by the generatrix and its strict positive realness in the fulfilment of Popov inequality. A key result (Theorem 3) shows that the frequency-domain inequality formulated for the transfer function is equivalent to time-domain constraints expressed in terms of input and output signals for the linear subsystem. A deeper investigation is devoted to the particular behavioural aspects occurring in the case of step response (Theorems 4, 5) and the link to BIBO-stability (Theorem 6). It is expected that further researches will be able to replace, in the sufficiency part of Theorem 3, the usage of the whole class of signals $u(t) \in L^2[0,T)$ by a considerably smaller subset of signals, appropriately selected.

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