Volkmar Henschel

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EXACT DISTRIBUTIONS IN THE MODEL OF A REGRESSION LINE FOR THE THRESHOLD PARAMETER WITH EXPONENTIAL DISTRIBUTION OF ERRORS

VOLKMAR HENSCHEL

In this paper it is shown how one can work out exact distributions of estimators and test statistics in the model of a regression line for the threshold parameter with exponential distribution of errors. This is done on a test statistics which is related to a problem of Zvára [6].

1. INTRODUCTION

Zvára [6] describes as an biological application where one should look for a "boundary line" that separates real from non-real situations an example of Nátrová and Nátr [4] who search for the dependence of maximal possible grain yield of winter wheat on the size of the phloem cross-sectional area.

For this situation the model

$$y_i = \theta_i + \sigma x_i, \quad \theta_i = \alpha + \beta s_i, \quad i = 1, \dots, n \tag{1}$$

is considered where α, β and σ are unknown parameters and the x_i are independent, identically, standard exponentially distributed random variables.
The maximum likelihood estimators for α, β and σ are derived. Let be $\mathbf{Y}$ an exponentially distributed random vector $\mathbf{Y} \sim \mathrm{E}_{\boldsymbol{\theta},\sigma \mathrm{I}_n}$, where $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$, the θ_i fulfill equation (1) and I_n denotes the identity matrix. $\mathbf{Y}$ has the density

$$f_{\mathbf{Y}}(\mathbf{y}) = \begin{cases} \frac{1}{\sigma^n} \exp\left\{ -\frac{1}{\sigma} \left(\sum\limits_{i=1}^{n} y_i - n\alpha - \sum\limits_{i=1}^{n} s_i \beta \right) \right\}, \alpha + \beta s_i < y_i, & i = 1, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

With the notation

$$M_n = \left\{ (\alpha, \beta)' \in \mathbb{R}^2 : \alpha + \beta s_i < y_i, \ i = 1, \dots, n \right\}$$

one gets for the logarithm of the likelihood function

$$\ln l(\alpha, \beta, \sigma) = \begin{cases} -n \ln \sigma - \frac{n}{\sigma}\left(\overline{y}_n - \alpha - \overline{s}_n \beta\right), & (\alpha, \beta)' \in M_n, \sigma > 0 \\ -\infty, & \text{otherwise.} \end{cases}$$

This implies that the maximum likelihood estimators $\hat{\alpha}$, $\hat{\beta}$ are solutions of the linear optimization problem

$$\begin{aligned} &\alpha + \beta\, \overline{s}_n \longrightarrow \max \\ &\alpha + \beta s_i \leq y_i, \quad i = 1, \ldots, n \end{aligned}$$

and it holds

$$\hat{\sigma} = \overline{y}_n - \hat{\alpha} - \hat{\beta}\, \overline{s}_n.$$

Zvára gives a confidence band for the regression line with covers at least $1 - p$, $0 < p < 1$. Therefore he derives from the likelihood ratio for the hypothesis

$$H_0 : \alpha = \alpha_0, \beta = \beta_0$$

the test statistic

$$T_{\mathbf{Y}} = \frac{\hat{\sigma}}{\hat{\sigma}_0} = \frac{\overline{Y}_n - \hat{\alpha} - \hat{\beta}\, \overline{s}_n}{\overline{Y}_n - \alpha_0 - \beta_0 \overline{s}_n}.$$

In this paper the exact distribution of a test statistic closely related to T' is worked out. Therefore a more heuristic but more visual description of the estimators $\hat{\alpha}$ and $\hat{\beta}$ is given by translating the linear optimization problem from the (β, α)-plane to the (s, y)-plane.

Given the likelihood function with fixed (s_i, y_i), $i = 1, \ldots, n$, one looks for a line described by β and α which maximizes the likelihood. Every line can be chosen for which all points (s_i, y_i), $i = 1, \ldots, n$, lie on or above this line. Such a line is called permissible. A point is called permissible, if there is a permissible line which goes through it. The likelihood is maximized, if $\overline{y}_n - \alpha - \beta\, \overline{s}_n = \frac{1}{n} \sum_{i=1}^{n} (y_i - \alpha - \beta s_i)$, i.e. the sum of the differences in y-direction between the points and the line, is minimized. This is realized by the line which goes through the permissible points (s_l, y_l) and (s_u, y_u) for which holds that $s_l \leq \overline{s}_n < s_u$ and there is no permissible point (s_i, y_i) with $s_l < s_i < s_u$, i.e.

$$\hat{\beta} = \frac{Y_u - Y_l}{s_u - s_l}, \qquad \hat{\alpha} = Y_l - \frac{Y_u - Y_l}{s_u - s_l}\, s_l.$$

Example 1.

i	1	2	3	4	arith. mean
s_i	-1.5	-0.5	1.0	2.0	0.25
y_i	2.0	1.5	3.0	3.5	2.50

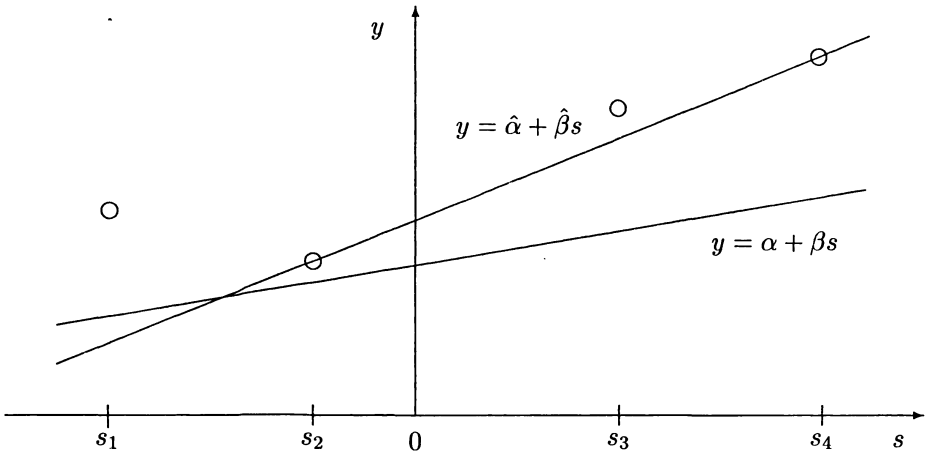

Fig. 1. (s, y)-plane.

Figure 1 shows the (s, y)-plane belonging to the example. The point (s_3, y_3) is not permissible, because there is no line through it such that all other points lie on or above it.

For reducing the dependence of the estimator for the additive parameter of the choice of the measuring points the model is reparameterized.

$$\theta_i = \alpha' + \beta(s_i - \overline{s}_n), \quad i = 1, \ldots, n$$

with

$$\alpha' := \alpha + \beta \, \overline{s}_n.$$

The considered estimator is now

$$\hat{\alpha}' = Y_l + \frac{Y_u - Y_l}{s_u - s_l} (\overline{s}_n - s_l)$$

and it holds

$$\hat{\sigma} = \overline{Y}_n - \hat{\alpha}'.$$

For testing the hypothesis H_0 the test statistic

$$T_{\alpha'} = \frac{\hat{\alpha}' - \alpha'_0}{\hat{\sigma}}$$

is used where $\alpha'_0 = \alpha_0 + \beta_0 \overline{s}_n$ and which is related to $T_{\mathbf{Y}}$ by

$$T_{\mathbf{Y}} = \frac{1}{1 + T_{\alpha'}}.$$

The considerations are restricted to the case of different measuring points s_i, $i = 1, \ldots, n$, $s_1 < \cdots < s_n$.

The class of handled distributions shall be enlarged. Let g be a function, $g|\mathbb{R}_+ \to \mathbb{R}_+$, which fulfills the condition $0 < \int_0^\infty r^{n-1}g(r)\mathrm{d}r < \infty$. With the constant $c(n,g) = \frac{\Gamma(n)}{\int_0^\infty r^{n-1}g(r)\mathrm{d}r}$ one gets the density of a random vector $\mathbf{X} \sim \mathrm{E}_{n;g}$ as $p_{\mathbf{X}}(\mathbf{x}) = c(n,g)g(\|\mathbf{x}\|_1)\mathrm{I}_{\mathbb{R}^n_+}(\mathbf{x})$ where I denotes the indicator function and $\|\cdot\|_1$ the ℓ_1-norm of the $\mathbb{R}^n$, g is called the density generating function. This allows to influence the behaviour of the densities in the tails.

For example one has the class of density generating functions of Kotz-type, $g_K(r) = r^{k-1}\exp\{-tr^s\}I_{\mathbb{R}_+}(r)$, where $s,t > 0$, $n+k > 1$, holds. This class includes the density generating function for the n-dimensional exponential law, $k = s = t = 1$, the only multivariate ℓ_1-norm symmetric distribution with independent marginals.

In Henschel and Richter [3] it is proposed to call this distributions regular simplicial distribution, because the densities are constant on the simplicial spheres $S^1_n(r) = \{\mathbf{x} \in \mathbb{R}^n_+ : \|\mathbf{x}\|_1 = r\}$ and in analogy to the spherical distributions which are generalizations of the normal law with constant densities on the Euclidean spheres $S^2_n(r) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 = r\}$.

Introducing threshold and dispersion parameters leads to a more general class of distributions. The random vector $\mathbf{W} = \boldsymbol{\theta} + \Sigma_n\mathbf{X}$, where $\boldsymbol{\theta} \in \mathbb{R}^n$, $\Sigma_n = \mathrm{diag}(\sigma_1,\ldots,\sigma_n)$, $\sigma_i > 0$, $i = 1,\ldots,n$ and $\mathbf{X}$ regular simplicial distributed, $\mathbf{X} \sim \mathrm{E}_{n;g}$, is called simplicially contoured distributed, $\mathbf{W} \sim \mathrm{E}_{\boldsymbol{\theta},\Sigma_n;g}$.

In this paper simplicially contoured distributed random vectors $\mathbf{W} \sim \mathrm{E}_{\boldsymbol{\theta},\sigma\mathrm{I}_n;g}$ are considered, where the θ_i fulfil equation (1).

2. DISTRIBUTION OF THE TEST STATISTIC

Theorem 2. Let be $\mathbf{W} \sim \mathrm{E}_{\boldsymbol{\theta},\sigma\mathrm{I}_n;g}$ with θ_i fulfil equation (1) and $s_m \le \bar{s}_n < s_{m+1}$. The test statistic $T_{\alpha'}$ has under the hypothesis $H_0 : \alpha' = \alpha'_0$ the cumulative distribution function:

$$F_{T_{\alpha'}(\mathbf{W})}(t) = 1 - \Bigg\{ \sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\sum_{u=m+1}^{n}(s_u - s_l)$$

$$\begin{cases} \frac{1}{\sum_{i=1}^k(\bar{s}_n - s_i)}\Bigg[\frac{\bar{s}_n - s_k}{\sum_{i=k+1}^n(s_i - s_k)}\left(1 - \frac{t}{1+t}\frac{\sum_{i=k+1}^n(s_i-s_k)}{n(\bar{s}_n - s_k)}\right)^{n-1} \\ \quad - \frac{\bar{s}_n - s_{k+1}}{\sum_{i=k+1}^n(s_i - s_{k+1})}\left(1 - \frac{t}{1+t}\frac{\sum_{i=k+1}^n(s_i-s_{k+1})}{n(\bar{s}_n - s_{k+1})}\right)^{n-1}\Bigg], & 0 \le t < -\frac{\sum_{i=1}^n(s_i - s_{k+1})}{\sum_{i=1}^k(s_i - s_{k+1})} \\ \frac{1}{\sum_{i=1}^k(\bar{s}_n - s_i)}\frac{\bar{s}_n - s_k}{\sum_{i=k+1}^n(s_i - s_k)}\left(1 - \frac{t}{1+t}\frac{\sum_{i=k+1}^n(s_i-s_k)}{n(\bar{s}_n - s_k)}\right)^{n-1}, & -\frac{\sum_{i=1}^n(s_i - s_{k+1})}{\sum_{i=1}^k(s_i - s_{k+1})} \le t < -\frac{\sum_{i=1}^n(s_i - s_k)}{\sum_{i=1}^k(s_i - s_k)} \\ 0, & -\frac{\sum_{i=1}^n(s_i - s_k)}{\sum_{i=1}^k(s_i - s_k)} \le t \end{cases}$$

$$+ \sum_{k=m+1}^{n-1}\sum_{l=1}^{m}\sum_{u=m+1}^{k}(s_u - s_l)$$

$$
\begin{cases}
\frac{1}{\sum_{i=k+1}^{n}(\bar{s}_n-s_i)}\Bigg[\frac{\bar{s}_n-s_k}{\sum_{i=1}^{k}(s_i-s_k)}\left(1-\frac{t}{1+t}\frac{\sum_{i=1}^{k}(s_i-s_k)}{n(\bar{s}_n-s_k)}\right)^{n-1} \\
\quad -\frac{\bar{s}_n-s_{k+1}}{\sum_{i=1}^{k}(s_i-s_{k+1})}\left(1-\frac{t}{1+t}\frac{\sum_{i=1}^{k}(s_i-s_{k+1})}{n(\bar{s}_n-s_{k+1})}\right)^{n-1}\Bigg], & 0\le t<-\frac{\sum_{i=1}^{n}(s_i-s_k)}{\sum_{i=k+1}^{n}(s_i-s_k)} \\
-\frac{1}{\sum_{i=k+1}^{n}(\bar{s}_n-s_i)}\frac{\bar{s}_n-s_{k+1}}{\sum_{i=1}^{k}(s_i-s_{k+1})}\left(1-\frac{t}{1+t}\frac{\sum_{i=1}^{k}(s_i-s_{k+1})}{n(\bar{s}_n-s_{k+1})}\right)^{n-1}, & \\
 & -\frac{\sum_{i=1}^{n}(s_i-s_k)}{\sum_{i=k+1}^{n}(s_i-s_k)}\le t<-\frac{\sum_{i=1}^{n}(s_i-s_{k+1})}{\sum_{i=k+1}^{n}(s_i-s_{k+1})} \\
0, & -\frac{\sum_{i=1}^{n}(s_i-s_{k+1})}{\sum_{i=k+1}^{n}(s_i-s_{k+1})}\le t
\end{cases}
$$

$$
+\sum_{l=1}^{m}\sum_{u=m+1}^{n}(s_u-s_l)\frac{1}{n}\left[\frac{1}{\sum_{i=1}^{n}(s_i-s_1)}-\frac{1}{\sum_{i=1}^{n}(s_i-s_n)}\right]
$$

$$
\left.\left[(1+t(n-1))(\frac{1}{1+t})^{n-1}\right]\right\}, \text{for} \quad t>0,
$$

$$
F_{T_{\alpha'}}(t)=0, \quad \text{for} \quad t\le 0.
$$

Sketch of the proof. The proof orients on the working with the normalized spacings for the order statistics in exponential sample distributions, see Sukhatme [5]. The parts of the positive orthant $\mathbb{R}_+^n$ are determined in which the indices l and u describing the estimators do not change. Then it is standardized. Next the transformed parts are put up again to the whole positive orthant. Therefore the vectors describing the edges of the parts are transformed to the standard base vectors describing the edges of the positive orthant. It is changed to a coordinate system suitable to regular simplicial distributions. Variables which do not appear in the indicator function are integrated out. The inequation of the indicator function is applied to the limits of the integrals and the integrals are calculated. The proof is given in the appendix. □

The distribution functions of the estimators $\hat{\alpha}$, $\hat{\beta}$, $\hat{\sigma}$ and of the test statistic $T_\beta = \frac{\hat{\beta}-\beta_0}{\hat{\sigma}}$ can be derived similarly, see Henschel [2]. The distribution function of the estimator $\hat{\beta}$ simplifies to

$$
F_{\hat{\beta}}(t)=
$$

$$
\begin{cases}
m\sum_{k=m+1}^{n}\left(\frac{1}{k-1}-\frac{1}{k}\right)\frac{c(n,g)}{\Gamma(n)}\int_0^\infty r^{n-1}g\big(r+\sum_{i=1}^{k-1}(s_i-s_k)\frac{t-\beta}{\sigma}\big)\mathrm{d}r, & t<\beta, \\
1-\frac{m}{n}, & t=\beta, \\
1-(n-m)\sum_{k=1}^{m}\left(\frac{1}{n-k}-\frac{1}{n-k+1}\right) & \\
\qquad \frac{c(n,g)}{\Gamma(n)}\int_0^\infty r^{n-1}g\big(r+\sum_{i=k+1}^{n}(s_i-s_k)\frac{t-\beta}{\sigma}\big)\mathrm{d}r, & t>\beta
\end{cases}
$$

and in the case of the exponential distribution to

$$
F_{\hat{\beta}}(t)=
$$

$$
\begin{cases}
m\sum_{k=m+1}^{n}\left(\frac{1}{k-1}-\frac{1}{k}\right)\exp\{-\sum_{i=1}^{k-1}(s_i-s_k)\frac{t-\beta}{\sigma}\}, & t<\beta, \\
1-\frac{m}{n}, & t=\beta, \\
1-(n-m)\sum_{k=1}^{m}\left(\frac{1}{n-k}-\frac{1}{n-k+1}\right)\exp\{-\sum_{i=k+1}^{n}(s_i-s_k)\frac{t-\beta}{\sigma}\}, & t>\beta.
\end{cases}
$$

The distribution function of the test statistic T_β is under the hypothesis $H_0 : \beta = \beta_0$

$$F_{T_\beta(\mathbf{W})}(t) = \sum_{k=m+1}^{n-1} \sum_{l=1}^{m} \sum_{u=m+1}^{k} (s_u - s_l)$$

$$\begin{cases} \frac{1+t(\bar{s}_n - s_k)}{\sum_{i=1}^{k}(s_i - s_k)\left[t\sum_{i=k+1}^{n}(\bar{s}_n - s_i) - k\right]} \left(\frac{1+\frac{t}{n}\sum_{i=k+1}^{n}(s_i - s_k)}{1+t(\bar{s}_n - s_k)}\right)^{n-1} \\ \quad - \frac{1+t(\bar{s}_n - s_{k+1})}{\sum_{i=1}^{k}(s_i - s_{k+1})\left[t\sum_{i=k+1}^{n}(\bar{s}_n - s_i) - k\right]} \left(\frac{1+\frac{t}{n}\sum_{i=k+1}^{n}(s_i - s_{k+1})}{1+t(\bar{s}_n - s_{k+1})}\right)^{n-1}, \\ \qquad\qquad -\frac{n}{\sum_{i=k+1}^{n}(s_i - s_k)} < t < 0, t \neq \frac{k}{\sum_{i=k+1}^{n}(\bar{s}_n - s_i)} \\ \frac{1}{k}\left[\frac{1}{\sum_{i=1}^{k}(s_i - s_{k+1})} - \frac{1}{\sum_{i=1}^{k}(s_i - s_k)}\right]\left[(n-1)\left(1-\frac{k}{n}\right)^{n-2}\frac{k}{n} + \left(1-\frac{k}{n}\right)^{n-1}\right], \\ \qquad\qquad t = \frac{k}{\sum_{i=k+1}^{n}(\bar{s}_n - s_i)} \\ -\frac{1+t(\bar{s}_n - s_{k+1})}{\sum_{i=1}^{k}(s_i - s_{k+1})\left[t\sum_{i=k+1}^{n}(\bar{s}_n - s_i) - k\right]} \left(\frac{1+\frac{t}{n}\sum_{i=k+1}^{n}(s_i - s_{k+1})}{1+t(\bar{s}_n - s_{k+1})}\right)^{n-1}, \\ \qquad\qquad -\frac{n}{\sum_{i=k+1}^{n}(s_i - s_{k+1})} < t \leq -\frac{n}{\sum_{i=k+1}^{n}(s_i - s_k)} \\ 0, \qquad\qquad t \leq -\frac{n}{\sum_{i=k+1}^{n}(s_i - s_{k+1})} \end{cases}$$

$$+ \frac{\sum_{l=1}^{m}\sum_{u=m+1}^{n}(s_l - s_u)}{\sum_{i=1}^{n}(s_i - s_n)} \frac{1}{n} \left(\frac{1}{1+\frac{t}{n}\sum_{i=1}^{n}(s_i - s_n)}\right)^{n-2}, \quad \text{for } t < 0,$$

$$F_{T_\beta(\mathbf{W})}(t) = 1 - \frac{m}{n}, \quad \text{for } t = 0,$$

$$F_{T_\beta(\mathbf{W})}(t) = 1 - \left\{ \sum_{k=1}^{m-1} \sum_{l=k+1}^{m} \sum_{u=m+1}^{n} (s_u - s_l) \right.$$

$$\begin{cases} \frac{1+t(\bar{s}_n - s_k)}{\sum_{i=k+1}^{n}(s_i - s_k)\left[t\sum_{i=1}^{k}(\bar{s}_n - s_i) - (n-k)\right]} \left(\frac{1+\frac{t}{n}\sum_{i=1}^{k}(s_i - s_k)}{1+t(\bar{s}_n - s_k)}\right)^{n-1} \\ \quad - \frac{1+t(\bar{s}_n - s_{k+1})}{\sum_{i=k+1}^{n}(s_i - s_{k+1})\left[t\sum_{i=1}^{k}(\bar{s}_n - s_i) - (n-k)\right]} \left(\frac{1+\frac{t}{n}\sum_{i=1}^{k}(s_i - s_{k+1})}{1+t(\bar{s}_n - s_{k+1})}\right)^{n-1}, \\ \qquad\qquad 0 < t < -\frac{n}{\sum_{i=1}^{k}(s_i - s_{k+1})}, t \neq \frac{n-k}{\sum_{i=1}^{k}(\bar{s}_n - s_i)} \\ \frac{1}{n-k}\left[\frac{1}{\sum_{i=k+1}^{n}(s_i - s_{k+1})} - \frac{1}{\sum_{i=k+1}^{n}(s_i - s_k)}\right]\left[(n-1)\left(\frac{k}{n}\right)^{n-2}\left(1-\frac{k}{n}\right) + \left(\frac{k}{n}\right)^{n-1}\right], \\ \qquad\qquad t = \frac{n-k}{\sum_{i=1}^{k}(\bar{s}_n - s_i)} \\ \frac{1+t(\bar{s}_n - s_k)}{\sum_{i=k+1}^{n}(s_i - s_k)\left[t\sum_{i=1}^{k}(\bar{s}_n - s_i) - (n-k)\right]} \left(\frac{1+\frac{t}{n}\sum_{i=1}^{k}(s_i - s_k)}{1+t(\bar{s}_n - s_k)}\right)^{n-1}, \\ \qquad\qquad -\frac{n}{\sum_{i=1}^{k}(s_i - s_{k+1})} \leq t < -\frac{n}{\sum_{i=1}^{k}(s_i - s_k)} \\ 0, \qquad\qquad -\frac{n}{\sum_{i=1}^{k}(s_i - s_k)} \leq t < \infty \end{cases}$$

$$\left. + \frac{1}{n}\frac{\sum_{l=1}^{m}\sum_{u=m+1}^{n}(s_u - s_l)}{\sum_{i=1}^{n}(s_i - s_1)} \left(\frac{1}{1+\frac{t}{n}\sum_{i=1}^{n}(s_i - s_1)}\right)^{n-2} \right\}, \quad \text{for } t > 0.$$

Remark 3. A statistic is called scale invariant, if $T(\mathrm{X}) \stackrel{d}{=} T(a\mathrm{X})$ for all $a > 0$. A statistic is called robust, if its distribution does not depend on the choice of the density generating function. A statistic is robust, iff it is scale invariant, see Fang et al [1]. The test statistics $T_{\alpha'}$ and T_β are under the null hypotheses scale invariant and hence robust.

APPENDIX

Proof of Theorem 2.

$$F_{T_{\alpha'}(\mathrm{W})}(t) = \int_{\theta_1}^{\infty} \cdots \int_{\theta_n}^{\infty} \mathrm{I}\left(T_{\alpha'}(\mathrm{W}) < t\right) \frac{c(n,g)}{\sigma^n} g\left(\frac{\|\mathrm{w} - \boldsymbol{\theta}\|_1}{\sigma}\right) \mathrm{d}w_1 \ldots \mathrm{d}w_n.$$

The domain of integration is partitioned into such parts that the estimators $\hat{\alpha}'$ and $\hat{\sigma}$ can be described by concrete indices l and u.

Therefore every $l = 1, \ldots, m$ can be combined with every $u = m+1, \ldots, n$, because then $s_l \leq \bar{s}_n < s_u$ holds.

The variables w_l or w_u, respectively, integrate from $\theta_i = \alpha + \beta s_i$ to ∞, $i = l, u$. A line is determined by the points (s_l, w_l) and (s_u, w_u). All further points must lie above this line, i. e. it holds

$$w_l - \frac{w_u - w_l}{s_u - s_l} s_l + \frac{w_u - w_l}{s_u - s_l} s_i < w_i < \infty, \qquad i = 1, \ldots, n, i \neq l, u.$$

Furthermore it must hold

$$\theta_i = \alpha + \beta s_i < w_i, \qquad i = 1, \ldots, n, i \neq l, u,$$

where this condition is trivially fulfilled for $i = l+1, \ldots, u-1$. Hence it holds

$$F_{T_{\alpha'}(\mathrm{W})}(t) = \sum_{l=1}^{m} \sum_{u=m+1}^{n} \int_{\theta_l}^{\infty} \int_{\theta_u}^{\infty}$$

$$\int_{\max\{\theta_1, w_l - \frac{w_u - w_l}{s_u - s_l} s_l + \frac{w_u - w_l}{s_u - s_l} s_1\}}^{\infty} \cdots \int_{\max\{\theta_{l-1}, w_l - \frac{w_u - w_l}{s_u - s_l} s_l + \frac{w_u - w_l}{s_u - s_l} s_{l-1}\}}^{\infty}$$

$$\int_{w_l - \frac{w_u - w_l}{s_u - s_l} s_l + \frac{w_u - w_l}{s_u - s_l} s_{l+1}}^{\infty} \cdots \int_{w_l - \frac{w_u - w_l}{s_u - s_l} s_l + \frac{w_u - w_l}{s_u - s_l} s_{u-1}}^{\infty}$$

$$\int_{\max\{\theta_{u+1}, w_l - \frac{w_u - w_l}{s_u - s_l} s_l + \frac{w_u - w_l}{s_u - s_l} s_{u+1}\}}^{\infty} \cdots \int_{\max\{\theta_n, w_l - \frac{w_u - w_l}{s_u - s_l} s_l + \frac{w_u - w_l}{s_u - s_l} s_n\}}^{\infty}$$

$$\mathrm{I}\left(\frac{w_l + \frac{w_u - w_l}{s_u - s_l}(\bar{s}_n - s_l) - \alpha'_0}{\overline{w}_n - w_l - \frac{w_u - w_l}{s_u - s_l}(\bar{s}_n - s_l)} < t\right)$$

$$\frac{c(n,g)}{\sigma^n} g\left(\frac{\|\mathrm{w} - \boldsymbol{\theta}\|_1}{\sigma}\right) \mathrm{d}w_n \ldots \mathrm{d}w_{u+1} \mathrm{d}w_{u-1} \ldots \mathrm{d}w_{l+1} \mathrm{d}w_{l-1} \ldots \mathrm{d}w_1 \mathrm{d}w_u \mathrm{d}w_l.$$

Now it is standardized.

$$
\begin{aligned}
x_i &:= \frac{w_i - \theta_i}{\sigma} = \frac{w_i - (\alpha + \beta s_i)}{\sigma}, \quad i = 1, \ldots, n \\
w_i &= \sigma x_i + \theta_i = \sigma x_i + \alpha + \beta s_i \\
\mathrm{d}w_i &= \sigma \mathrm{d}x_i.
\end{aligned}
$$

With that one has

$$
\begin{aligned}
w_l + \frac{w_u - w_l}{s_u - s_l}(\overline{s}_n - s_l) &= \sigma \left(\frac{s_u - \overline{s}_n}{s_u - s_l} x_l + \frac{\overline{s}_n - s_l}{s_u - s_l} x_u \right) + \alpha' \\
\overline{w}_n - w_l - \frac{w_u - w_l}{s_u - s_l}(\overline{s}_n - s_l) &= \sigma \left(\overline{x}_n - \frac{s_u - \overline{s}_n}{s_u - s_l} x_l - \frac{\overline{s}_n - s_l}{s_u - s_l} x_u \right) \\
w_l - \frac{w_u - w_l}{s_u - s_l} s_l + \frac{w_u - w_l}{s_u - s_l} s_i &= \sigma \left(x_i - \frac{x_u - x_l}{s_u - s_l} s_l + \frac{x_u - x_l}{s_u - s_l} s_i \right) + \alpha + \beta s_i
\end{aligned}
$$

and it holds

$$
\begin{aligned}
&F_{T_{\alpha'}(\mathrm{w})}(t) = \sum_{l=1}^{m} \sum_{u=m+1}^{n} \int_0^\infty \int_0^\infty \\
&\int_{\max\{0, x_l - \frac{x_u - x_l}{s_u - s_l} s_l + \frac{x_u - x_l}{s_u - s_l} s_1\}}^\infty \cdots \int_{\max\{0, x_l - \frac{x_u - x_l}{s_u - s_l} s_l + \frac{x_u - x_l}{s_u - s_l} s_{l-1}\}}^\infty \\
&\int_{x_l - \frac{x_u - x_l}{s_u - s_l} s_l + \frac{x_u - x_l}{s_u - s_l} s_{l+1}}^\infty \cdots \int_{x_l - \frac{x_u - x_l}{s_u - s_l} s_l + \frac{x_u - x_l}{s_u - s_l} s_{u-1}}^\infty \\
&\int_{\max\{0, x_l - \frac{x_u - x_l}{s_u - s_l} s_l + \frac{x_u - x_l}{s_u - s_l} s_{u+1}\}}^\infty \cdots \int_{\max\{0, x_l - \frac{x_u - x_l}{s_u - s_l} s_l + \frac{x_u - x_l}{s_u - s_l} s_n\}}^\infty \\
&\mathrm{I} \left(\frac{\sigma \left(\frac{s_u - \overline{s}_n}{s_u - s_l} x_l + \frac{\overline{s}_n - s_l}{s_u - s_l} x_u \right) + \alpha' - \alpha'_0}{\sigma \left(\overline{x}_n - \frac{s_u - \overline{s}_n}{s_u - s_l} x_l - \frac{\overline{s}_n - s_l}{s_u - s_l} x_u \right)} < t \right) \\
&c(n, g) g(\|\mathrm{x}\|_1) \mathrm{d}x_n \ldots \mathrm{d}x_{u+1} \mathrm{d}x_{u-1} \ldots \mathrm{d}x_{l+1} \mathrm{d}x_{l-1} \ldots \mathrm{d}x_1 \mathrm{d}x_u \mathrm{d}x_l.
\end{aligned}
$$

The maxima are resolved. It holds

$$
\begin{aligned}
0 = x_l - \frac{x_u - x_l}{s_u - s_l} s_l + \frac{x_u - x_l}{s_u - s_l} s_k &= \frac{s_u - s_k}{s_u - s_l} x_l + \frac{s_k - s_l}{s_u - s_l} x_u \\
\Longleftrightarrow x_u = \frac{s_u - s_k}{s_l - s_k} x_l, &\qquad k = 1, \ldots, l-1, u+1, \ldots, n.
\end{aligned}
$$

With

$$
s_0 := -\infty, \qquad s_{n+1} := \infty
$$

holds

$$
0 = \frac{s_u - s_u}{s_l - s_u} < \frac{s_u - s_{u+1}}{s_l - s_{u+1}} < \cdots < \frac{s_u - s_n}{s_l - s_n} < \frac{s_u - s_{n+1}}{s_l - s_{n+1}} = 1
$$

and

$$1 = \frac{s_u - s_0}{s_l - s_0} < \frac{s_u - s_1}{s_l - s_1} < \cdots < \frac{s_u - s_{l-1}}{s_l - s_{l-1}} < \frac{s_u - s_l}{s_l - s_l} = \infty,$$

i. e. the domain of integration of x_u can be partitioned in dependence of x_l into the parts $\frac{s_u - s_k}{s_l - s_k} x_l \leq x_u < \frac{s_u - s_{k+1}}{s_l - s_{k+1}} x_l$, $k = u, \ldots, n, 0, \ldots, l-1$, such that within these parts the lower borders of integration of $x_{u+1}, \ldots, x_n$ or $x_1, \ldots, x_{l-1}$, respectively, are 0 or greater than 0.

Let l and u be fixed.

For x_i with $i \in \{1, \ldots, l-1\}$ holds for $k \in \{u, \ldots, n, 0, \ldots, i-1\}$ that $\frac{s_u - s_{k+1}}{s_l - s_{k+1}} \leq \frac{s_u - s_i}{s_l - s_i}$ and hence that $x_u < \frac{s_u - s_{k+1}}{s_l - s_{k+1}} x_l \leq \frac{s_u - s_i}{s_l - s_i} x_l$ and from that with $s_i - s_l < 0$

$$\frac{s_u - s_i}{s_u - s_l} x_l + \frac{s_i - s_l}{s_u - s_l} x_u > \frac{s_u - s_i}{s_u - s_l} x_l + \frac{s_i - s_l}{s_u - s_l} \frac{s_u - s_i}{s_l - s_i} x_l = 0.$$

For x_i with $i \in \{u+1, \ldots, n\}$ holds for $k \in \{i, \ldots, n, 0, \ldots, l-1\}$ that $\frac{s_u - s_k}{s_l - s_k} \geq \frac{s_u - s_i}{s_l - s_i}$ and hence that $x_u \geq \frac{s_u - s_k}{s_l - s_k} x_l \geq \frac{s_u - s_i}{s_l - s_i} x_l$ and from that with $s_i - s_l > 0$

$$\frac{s_u - s_i}{s_u - s_l} x_l + \frac{s_i - s_l}{s_u - s_l} x_u \geq \frac{s_u - s_i}{s_u - s_l} x_l + \frac{s_i - s_l}{s_u - s_l} \frac{s_u - s_i}{s_l - s_i} x_l = 0.$$

$$F_{T_{\alpha'}(\mathbf{w})}(t) = \sum_{l=1}^{m} \sum_{u=m+1}^{n} \int_0^\infty \sum_{k=u,\ldots,n,0,\ldots,l-1} \int_{\frac{s_u - s_k}{s_l - s_k} x_l}^{\frac{s_u - s_{k+1}}{s_l - s_{k+1}} x_l}$$

$$\int_{\{\frac{s_u - s_1}{s_u - s_l} x_l + \frac{s_1 - s_l}{s_u - s_l} x_u\} \mathrm{I}_{\{u,\ldots,n,0\}}(k)}^\infty \cdots \int_{\{\frac{s_u - s_{l-1}}{s_u - s_l} x_l + \frac{s_{l-1} - s_l}{s_u - s_l} x_u\} \mathrm{I}_{\{u,\ldots,n,0,\ldots,l-2\}}(k)}^\infty$$

$$\int_{\frac{s_u - s_{l+1}}{s_u - s_l} x_l + \frac{s_{l+1} - s_l}{s_u - s_l} x_u}^\infty \cdots \int_{\frac{s_u - s_{u-1}}{s_u - s_l} x_l + \frac{s_{u-1} - s_l}{s_u - s_l} x_u}^\infty$$

$$\int_{\{\frac{s_u - s_{u+1}}{s_u - s_l} x_l + \frac{s_{u+1} - s_l}{s_u - s_l} x_u\} \mathrm{I}_{\{u+1,\ldots,n,0,\ldots,l-1\}}(k)}^\infty \cdots \int_{\{\frac{s_u - s_n}{s_u - s_l} x_l + \frac{s_n - s_l}{s_u - s_l} x_u\} \mathrm{I}_{\{n,0,\ldots,l-1\}}(k)}^\infty$$

$$\mathrm{I}\left(\frac{\sigma\left(\frac{s_u - \overline{s}_n}{s_u - s_l} x_l + \frac{\overline{s}_n - s_l}{s_u - s_l} x_u\right) + \alpha' - \alpha'_0}{\sigma\left(\overline{x}_n - \frac{s_u - \overline{s}_n}{s_u - s_l} x_l - \frac{\overline{s}_n - s_l}{s_u - s_l} x_u\right)} < t \right)$$

$$c(n, g) g(\|\mathbf{x}\|_1) \mathrm{d}x_n \ldots \mathrm{d}x_{u+1} \mathrm{d}x_{u-1} \ldots \mathrm{d}x_{l+1} \mathrm{d}x_{l-1} \ldots \mathrm{d}x_1 \mathrm{d}x_u \mathrm{d}x_l$$

The domain of integration is for each k a part of the positive orthant $\mathbb{R}^n_+$. For each k the whole positive orthant is put up again now.

The transformations are organized in such a way that the ℓ_1-norm of the vectors does not change, $\overline{x}_n = \overline{v}_n$.

The cases $k = 0$, $k = 1, \ldots, l-1$, $k = u, \ldots, n-1$ and $k = n$ are treated separately.

For $k = 1, \ldots, l-1$ one has with the notations

$$\sum_{k+1}^{n}(k) := \sum_{i=k+1}^{n} (s_i - s_k) \quad \text{and} \quad \sum_{k+1}^{n}(k+1) := \sum_{i=k+1}^{n} (s_i - s_{k+1}),$$

where

$$0 < \sum_{k+1}^{n}(k+1) < \sum_{k+1}^{n}(k), \tag{2}$$

$$\begin{aligned}
v_i &= x_i, \qquad i = 1, \ldots, k \\
v_i &= x_i - \frac{s_u - s_i}{s_u - s_l}x_l - \frac{s_i - s_l}{s_u - s_l}x_u, \qquad i = k+1, \ldots, n, i \neq l, u \\
v_l &= \frac{(s_u - s_{k+1})\sum_{k+1}^{n}(k)}{(s_{k+1} - s_k)(s_u - s_l)}x_l - \frac{(s_l - s_{k+1})\sum_{k+1}^{n}(k)}{(s_{k+1} - s_k)(s_u - s_l)}x_u, \\
v_u &= \frac{(s_l - s_k)\sum_{k+1}^{n}(k+1)}{(s_{k+1} - s_k)(s_u - s_l)}x_u - \frac{(s_u - s_k)\sum_{k+1}^{n}(k+1)}{(s_{k+1} - s_k)(s_u - s_l)}x_l.
\end{aligned}$$

For the inverse of the Jacobian one gets

$$\begin{aligned}
|J_k^{-1}| &= \frac{(s_u - s_l)(s_{k+1} - s_k)}{\sum_{k+1}^{n}(k)\sum_{k+1}^{n}(k+1)} = \frac{s_u - s_l}{n-k}\left(\frac{1}{\sum_{k+1}^{n}(k+1)} - \frac{1}{\sum_{k+1}^{n}(k)}\right) \\
&= \frac{s_u - s_l}{\sum_{i=1}^{k}(\bar{s}_n - s_i)}\left(\frac{\bar{s}_n - s_k}{\sum_{k+1}^{n}(k)} - \frac{\bar{s}_n - s_{k+1}}{\sum_{k+1}^{n}(k+1)}\right)
\end{aligned} \tag{3}$$

and for the estimators

$$\frac{s_u - \bar{s}_n}{s_u - s_l}x_l + \frac{\bar{s}_n - s_l}{s_u - s_l}x_u = \frac{\bar{s}_n - s_k}{\sum_{k+1}^{n}(k)}v_l + \frac{\bar{s}_n - s_{k+1}}{\sum_{k+1}^{n}(k+1)}v_u.$$

For $k = 0$ one has

$$\begin{aligned}
v_i &= x_i - \frac{s_u - s_i}{s_u - s_l}x_l - \frac{s_i - s_l}{s_u - s_l}x_u, \qquad i = 1, \ldots, n, i \neq l, u \\
v_l &= n\frac{s_u - s_1}{s_u - s_l}x_l - n\frac{s_l - s_1}{s_u - s_l}x_u, \\
v_u &= \frac{\sum_1^n(1)}{s_u - s_l}(x_u - x_l).
\end{aligned}$$

For the inverse of the Jacobian one gets

$$|J_0^{-1}| = \frac{1}{n}\frac{s_u - s_l}{\sum_1^n(1)}$$

and for the estimators

$$\frac{s_u - \bar{s}_n}{s_u - s_l}x_l + \frac{\bar{s}_n - s_l}{s_u - s_l}x_u = \frac{1}{n}(v_l + v_u).$$

For $k = u, \ldots, n-1$ one has with the notations

$$\sum_1^k(k) := \sum_{i=1}^{k}(s_i - s_k) \quad \text{and} \quad \sum_1^k(k+1) := \sum_{i=1}^{k}(s_i - s_{k+1}),$$

where

$$0 > \sum_1^k(k) > \sum_1^k(k+1), \tag{4}$$

$$\begin{aligned}
v_i &= x_i, \qquad i = k+1, \dots, n \\
v_i &= x_i - \frac{s_u - s_i}{s_u - s_l} x_l - \frac{s_i - s_l}{s_u - s_l} x_u, \qquad i = 1, \dots, k, i \neq l, u \\
v_l &= \frac{(s_u - s_{k+1}) \sum_1^k(k)}{(s_{k+1} - s_k)(s_u - s_l)} x_l - \frac{(s_l - s_{k+1}) \sum_1^k(k)}{(s_{k+1} - s_k)(s_u - s_l)} x_u, \\
v_u &= \frac{(s_l - s_k) \sum_1^k(k+1)}{(s_{k+1} - s_k)(s_u - s_l)} x_u - \frac{(s_u - s_k) \sum_1^k(k+1)}{(s_{k+1} - s_k)(s_u - s_l)} x_l.
\end{aligned}$$

For the inverse of the Jacobian one gets

$$\begin{aligned}
|J_k^{-1}| &= \frac{(s_u - s_l)(s_{k+1} - s_k)}{\sum_1^k(k) \sum_1^k(k+1)} = \frac{s_u - s_l}{k} \left(\frac{1}{\sum_1^k(k+1)} - \frac{1}{\sum_1^k(k)} \right) \\
&= \frac{s_u - s_l}{\sum_{i=k+1}^n (\bar{s}_n - s_i)} \left(\frac{\bar{s}_n - s_k}{\sum_1^k(k)} - \frac{\bar{s}_n - s_{k+1}}{\sum_1^k(k+1)} \right)
\end{aligned} \tag{5}$$

and for the estimators

$$\frac{s_u - \bar{s}_n}{s_u - s_l} x_l + \frac{\bar{s}_n - s_l}{s_u - s_l} x_u = \frac{\bar{s}_n - s_k}{\sum_1^k(k)} v_l + \frac{\bar{s}_n - s_{k+1}}{\sum_1^k(k+1)} v_u.$$

For $k = n$ one has

$$\begin{aligned}
v_i &= x_i - \frac{s_u - s_i}{s_u - s_l} x_l - \frac{s_i - s_l}{s_u - s_l} x_u, \qquad i = 1, \dots, n, i \neq l, u, \\
v_l &= \frac{\sum_1^n(n)}{s_u - s_l} (x_u - x_l) \\
v_u &= n \frac{s_u - s_n}{s_u - s_l} x_l - n \frac{s_l - s_n}{s_u - s_l} x_u.
\end{aligned}$$

For the inverse of the Jacobian one gets

$$|J_n^{-1}| = \frac{s_l - s_u}{\sum_1^n(n)} \frac{1}{n}$$

and for the estimators

$$\frac{s_u - \bar{s}_n}{s_u - s_l} x_l + \frac{\bar{s}_n - s_l}{s_u - s_l} x_u = \frac{1}{n} (v_l + v_u).$$

For the cdf one gets on this way

$$F_{T_{\alpha'}(\mathbf{w})}(t) = \sum_{l=1}^{m} \sum_{u=m+1}^{n}$$

$$\Bigg\{ \int_0^\infty \cdots \int_0^\infty \frac{1}{n} \frac{s_u - s_l}{\sum_1^n (1)} \mathrm{I}\left(\frac{\sigma \frac{1}{n}(v_l + v_u) + \alpha' - \alpha_0'}{\sigma\left(\bar{v}_n - \frac{1}{n}(v_l + v_u)\right)} < t \right) c(n,g) g\left(\|\mathbf{v}\|_1\right) \mathrm{d}v_n \ldots \mathrm{d}v_1$$

$$+ \sum_{k=1}^{l-1} \int_0^\infty \cdots \int_0^\infty \frac{(s_u - s_l)(s_{k+1} - s_k)}{\sum_{k+1}^n (k) \sum_{k+1}^n (k+1)}$$

$$\mathrm{I}\left(\frac{\sigma\left(\frac{\bar{s}_n - s_k}{\sum_{k+1}^n (k)} v_l + \frac{\bar{s}_n - s_{k+1}}{\sum_{k+1}^n (k+1)} v_u \right) + \alpha' - \alpha_0'}{\sigma\left(\bar{v}_n - \frac{\bar{s}_n - s_k}{\sum_{k+1}^n (k)} v_l - \frac{\bar{s}_n - s_{k+1}}{\sum_{k+1}^n (k+1)} v_u \right)} < t \right) c(n,g) g\left(\|\mathbf{v}\|_1\right) \mathrm{d}v_n \ldots \mathrm{d}v_1$$

$$+ \sum_{k=u}^{n-1} \int_0^\infty \cdots \int_0^\infty \frac{(s_u - s_l)(s_{k+1} - s_k)}{\sum_1^k (k) \sum_1^k (k+1)}$$

$$\mathrm{I}\left(\frac{\sigma\left(\frac{\bar{s}_n - s_k}{\sum_1^k (k)} v_l + \frac{\bar{s}_n - s_{k+1}}{\sum_1^k (k+1)} v_u \right) + \alpha' - \alpha_0'}{\sigma\left(\bar{v}_n - \frac{\bar{s}_n - s_k}{\sum_1^k (k)} v_l - \frac{\bar{s}_n - s_{k+1}}{\sum_1^k (k+1)} v_u \right)} < t \right) c(n,g) g\left(\|\mathbf{v}\|_1\right) \mathrm{d}v_n \ldots \mathrm{d}v_1$$

$$+ \int_0^\infty \cdots \int_0^\infty \frac{s_l - s_u}{\sum_1^n (n)} \frac{1}{n} \mathrm{I}\left(\frac{\sigma \frac{1}{n}(v_l + v_u) + \alpha' - \alpha_0'}{\sigma\left(\bar{v}_n - \frac{1}{n}(v_l + v_u)\right)} < t \right) c(n,g) g\left(\|\mathbf{v}\|_1\right) \mathrm{d}v_n \ldots \mathrm{d}v_1 \Bigg\}.$$

It is changed to simplex coordinates, analogues to polar coordinates, see Henschel and Richter [3].

$$\begin{array}{llll} Z_1: & r_2 & := & v_l + v_u, & v_l = r_2\psi_l, \\ & \psi_l & := & \frac{v_l}{v_l + v_u}, & v_u = r_2(1 - \psi_l) \\ & r_{n-2} & := & \sum_{\substack{i=1 \\ i \neq l,u}}^n v_i, & \\ & \psi_i & := & \frac{v_i}{\sum_{\substack{j=i,\ldots,n \\ j \neq l,u}} v_i}, & i = 1, \ldots, n-1, i \neq l, u \end{array}$$

$$\|J_{Z_1^{-1}}\| = r_2 r_{n-2}^{n-3} \prod_{\substack{i=1 \\ i \neq l,u}}^{n-1} (1 - \psi_i)^{n-1-i-\mathrm{I}_{\{1,\ldots,l-1\}}(i) - \mathrm{I}_{\{1,\ldots,u-1\}}(i)}$$

$$\begin{array}{llll} Z_2: & r & := & r_2 + r_{n-2}, & r_2 = r\psi_u, \quad \|J_{Z_2^{-1}}\| = r \\ & \psi_u & := & \frac{r_2}{r_2 + r_{n-2}}, & r_{n-2} = r(1 - \psi_u) \\ & \psi_i, & & i = 1, \ldots, n-1, i \neq l, u \text{ unchanged} & \end{array}$$

For Z, the composition of Z_1 and Z_2, one gets $r = \sum_{i=1}^n v_i$ and

$$\|J_{Z^{-1}}\| = r^{n-1} \psi_u (1 - \psi_u)^{n-3} \prod_{\substack{i=1 \\ i \neq l,u}}^{n-1} (1 - \psi_i)^{n-1-i-\mathrm{I}_{\{1,\ldots,l-1\}}(i) - \mathrm{I}_{\{1,\ldots,u-1\}}(i)}.$$

For the estimators one has for $k = 0, n$

$$\frac{1}{n}(v_l + v_u) = \frac{1}{n} r\psi_u$$

for $k = 1, \dots, l-1$

$$\frac{\overline{s}_n - s_k}{\sum_{k+1}^n (k)} v_l + \frac{\overline{s}_n - s_{k+1}}{\sum_{k+1}^n (k+1)} v_u = r\psi_u \left(\frac{\overline{s}_n - s_k}{\sum_{k+1}^n (k)} \psi_l + \frac{\overline{s}_n - s_{k+1}}{\sum_{k+1}^n (k+1)} (1 - \psi_l) \right)$$

for $k = u, \dots, n-1$

$$\frac{\overline{s}_n - s_k}{\sum_1^k (k)} v_l + \frac{\overline{s}_n - s_{k+1}}{\sum_1^k (k+1)} v_u = r\psi_u \left(\frac{\overline{s}_n - s_k}{\sum_1^k (k)} \psi_l + \frac{\overline{s}_n - s_{k+1}}{\sum_1^k (k+1)} (1 - \psi_l) \right),$$

Taking into account, that

$$\int_0^1 \cdots \int_0^1 \int_0^1 \cdots \int_0^1 \int_0^1 \cdots \int_0^1 \prod_{\substack{i=1 \\ i \neq l,u}}^{n-1} (1 - \psi_i)^{n-1-i-\mathrm{I}_{\{1,\dots,l-1\}}(i) - \mathrm{I}_{\{1,\dots,u-1\}}(i)}$$
$$\mathrm{d}\psi_{n-1} \dots \mathrm{d}\psi_{u+1} \mathrm{d}\psi_{u-1} \dots \mathrm{d}\psi_{l+1} \mathrm{d}\psi_{l-1} \dots \mathrm{d}\psi_1 = \frac{1}{\Gamma(n-2)}$$

one has

$$F_{T_{\alpha'}(\mathbf{W})}(t) = \frac{1}{\Gamma(n-2)} \sum_{l=1}^{m} \sum_{u=m+1}^{n}$$
$$\left\{ \int_0^\infty \int_0^1 \int_0^1 \frac{1}{n} \frac{s_u - s_l}{\sum_1^n (1)} \mathrm{I} \left(\frac{\frac{1}{n} r\psi_u + \frac{\alpha' - \alpha'_0}{\sigma}}{\frac{1}{n} r(1 - \psi_u)} < t \right) \right.$$
$$r^{n-1} \psi_u (1 - \psi_u)^{n-3} c(n, g) g(r) \mathrm{d}\psi_l \mathrm{d}\psi_u \mathrm{d}r$$
$$+ \sum_{k=1}^{l-1} \int_0^\infty \int_0^1 \int_0^1 \frac{(s_u - s_l)(s_{k+1} - s_k)}{\sum_{k+1}^n (k) \sum_{k+1}^n (k+1)}$$
$$\mathrm{I} \left(\frac{r\psi_u \left(\frac{\overline{s}_n - s_k}{\sum_{k+1}^n (k)} \psi_l + \frac{\overline{s}_n - s_{k+1}}{\sum_{k+1}^n (k+1)} (1 - \psi_l) \right) + \frac{\alpha' - \alpha'_0}{\sigma}}{r \left(\frac{1}{n} - \psi_u \left(\frac{\overline{s}_n - s_k}{\sum_{k+1}^n (k)} \psi_l + \frac{\overline{s}_n - s_{k+1}}{\sum_{k+1}^n (k+1)} (1 - \psi_l) \right) \right)} < t \right)$$
$$r^{n-1} \psi_u (1 - \psi_u)^{n-3} c(n, g) g(r) \mathrm{d}\psi_l \mathrm{d}\psi_u \mathrm{d}r$$

$$+\sum_{k=u}^{n-1}\int_0^\infty\int_0^1\int_0^1 \frac{(s_u-s_l)(s_{k+1}-s_k)}{\sum_1^k(k)\sum_1^k(k+1)}$$

$$\mathrm{I}\left(\frac{r\psi_u\left(\frac{\bar{s}_n-s_k}{\sum_1^k(k)}\psi_l+\frac{\bar{s}_n-s_{k+1}}{\sum_1^k(k+1)}(1-\psi_l)\right)+\frac{\alpha'-\alpha_0'}{\sigma}}{r\left(\frac{1}{n}-\psi_u\left(\frac{\bar{s}_n-s_k}{\sum_1^k(k)}\psi_l+\frac{\bar{s}_n-s_{k+1}}{\sum_1^k(k+1)}(1-\psi_l)\right)\right)}<t\right)$$

$$r^{n-1}\psi_u(1-\psi_u)^{n-3}c(n,g)g(r)\mathrm{d}\psi_l\mathrm{d}\psi_u\mathrm{d}r$$

$$+\int_0^\infty\int_0^1\int_0^1 \frac{s_l-s_u}{\sum_1^n(n)}\frac{1}{n}\mathrm{I}\left(\frac{\frac{1}{n}r\psi_u+\frac{\alpha'-\alpha_0'}{\sigma}}{\frac{1}{n}r(1-\psi_u)}<t\right)$$

$$\left.r^{n-1}\psi_u(1-\psi_u)^{n-3}c(n,g)g(r)\mathrm{d}\psi_l\mathrm{d}\psi_u\mathrm{d}r\right\},$$

where

$$\frac{\frac{1}{n}r\psi_u+\frac{\alpha'-\alpha_0'}{\sigma}}{\frac{1}{n}r(1-\psi_u)}\overset{H_0}{=}\frac{\psi_u}{1-\psi_u}$$

$$\frac{r\psi_u\left(\frac{\bar{s}_n-s_k}{\sum_{k+1}^n(k)}\psi_l+\frac{\bar{s}_n-s_{k+1}}{\sum_{k+1}^n(k+1)}(1-\psi_l)\right)+\frac{\alpha'-\alpha_0'}{\sigma}}{r\left(\frac{1}{n}-\psi_u\left(\frac{\bar{s}_n-s_k}{\sum_{k+1}^n(k)}\psi_l+\frac{\bar{s}_n-s_{k+1}}{\sum_{k+1}^n(k+1)}(1-\psi_l)\right)\right)}$$

$$\overset{H_0}{=}\frac{\psi_u\left(\frac{n(\bar{s}_n-s_k)}{\sum_{k+1}^n(k)}\psi_l+\frac{n(\bar{s}_n-s_{k+1})}{\sum_{k+1}^n(k+1)}(1-\psi_l)\right)}{1-\psi_u\left(\frac{n(\bar{s}_n-s_k)}{\sum_{k+1}^n(k)}\psi_l+\frac{n(\bar{s}_n-s_{k+1})}{\sum_{k+1}^n(k+1)}(1-\psi_l)\right)}$$

$$\frac{r\psi_u\left(\frac{\bar{s}_n-s_k}{\sum_1^k(k)}\psi_l+\frac{\bar{s}_n-s_{k+1}}{\sum_1^k(k+1)}(1-\psi_l)\right)+\frac{\alpha'-\alpha_0'}{\sigma}}{r\left(\frac{1}{n}-\psi_u\left(\frac{\bar{s}_n-s_k}{\sum_1^k(k)}\psi_l+\frac{\bar{s}_n-s_{k+1}}{\sum_1^k(k+1)}(1-\psi_l)\right)\right)}$$

$$\overset{H_0}{=}\frac{\psi_u\left(\frac{n(\bar{s}_n-s_k)}{\sum_1^k(k)}\psi_l+\frac{n(\bar{s}_n-s_{k+1})}{\sum_1^k(k+1)}(1-\psi_l)\right)}{1-\psi_u\left(\frac{n(\bar{s}_n-s_k)}{\sum_1^k(k)}\psi_l+\frac{n(\bar{s}_n-s_{k+1})}{\sum_1^k(k+1)}(1-\psi_l)\right)},$$

i. e. under H_0 the indicator function does not depend on r, i. e. the test statistic does not depend on the concrete choice of g under H_0. The variable r can be integrated out.

$$F_{T_{\alpha'}(\mathrm{w})}(t)=(n-1)(n-2)\sum_{l=1}^{m}\sum_{u=m+1}^{n}$$

$$\left\{\int_0^1\int_0^1\frac{s_u-s_l}{n}\left[\frac{1}{\sum_1^n(1)}-\frac{1}{\sum_1^n(n)}\right]\mathrm{I}\left(\frac{\psi_u}{1-\psi_u}<t\right)\psi_u(1-\psi_u)^{n-3}\mathrm{d}\psi_l\mathrm{d}\psi_u\right.$$

$$+\sum_{k=1}^{l-1}\int_0^1\int_0^1 \frac{(s_u-s_l)(s_{k+1}-s_k)}{\sum_{k+1}^n(k)\sum_{k+1}^n(k+1)}$$

$$\mathrm{I}\left(\frac{\psi_u\left(\frac{n(\bar{s}_n-s_k)}{\sum_{k+1}^n(k)}\psi_l+\frac{n(\bar{s}_n-s_{k+1})}{\sum_{k+1}^n(k+1)}(1-\psi_l)\right)}{1-\psi_u\left(\frac{n(\bar{s}_n-s_k)}{\sum_{k+1}^n(k)}\psi_l+\frac{n(\bar{s}_n-s_{k+1})}{\sum_{k+1}^n(k+1)}(1-\psi_l)\right)}<t\right)\psi_u(1-\psi_u)^{n-3}\mathrm{d}\psi_l\mathrm{d}\psi_u$$

$$+\sum_{k=u}^{n-1}\int_0^1\int_0^1 \frac{(s_u-s_l)(s_{k+1}-s_k)}{\sum_1^k(k)\sum_1^k(k+1)}$$

$$\mathrm{I}\left(\frac{\psi_u\left(\frac{n(\bar{s}_n-s_k)}{\sum_1^k(k)}\psi_l+\frac{n(\bar{s}_n-s_{k+1})}{\sum_1^k(k+1)}(1-\psi_l)\right)}{1-\psi_u\left(\frac{n(\bar{s}_n-s_k)}{\sum_1^k(k)}\psi_l+\frac{n(\bar{s}_n-s_{k+1})}{\sum_1^k(k+1)}(1-\psi_l)\right)}<t\right)\psi_u(1-\psi_u)^{n-3}\mathrm{d}\psi_l\mathrm{d}\psi_u\Bigg\}.$$

The indicator functions are now applied to the domains of integration. Only $t>0$ are considered, because $T_{\alpha'}$ is nonnegative by the construction of the estimators.
$k=0,n$:

$$\begin{aligned}
&\int_0^1\int_0^1 \mathrm{I}\left(\frac{\psi_u}{1-\psi_u}<t\right)\psi_u(1-\psi_u)^{n-3}\mathrm{d}\psi_l\mathrm{d}\psi \\
=\;&\int_0^1 \mathrm{I}\left(\frac{\psi_u}{1-\psi_u}<t\right)\psi_u(1-\psi_u)^{n-3}\mathrm{d}\psi_u \\
=\;&\int_0^{\frac{t}{1+t}}\psi_u(1-\psi_u)^{n-3}\mathrm{d}\psi_u \\
=\;&\frac{1}{(n-1)(n-2)}\left[1-(1+t(n-1))(\frac{1}{1+t})^{n-1}\right]
\end{aligned}$$

$k=1,\ldots,l-1$:
Let $0\le t<-\frac{\sum_{i=1}^n(s_i-s_{k+1})}{\sum_{i=1}^k(s_i-s_{k+1})}$.

$$\int_0^1\int_0^1 \mathrm{I}\left(\frac{\psi_u\left(\frac{n(\bar{s}_n-s_k)}{\sum_{k+1}^n(k)}\psi_l+\frac{n(\bar{s}_n-s_{k+1})}{\sum_{k+1}^n(k+1)}(1-\psi_l)\right)}{1-\psi_u\left(\frac{n(\bar{s}_n-s_k)}{\sum_{k+1}^n(k)}\psi_l+\frac{n(\bar{s}_n-s_{k+1})}{\sum_{k+1}^n(k+1)}(1-\psi_l)\right)}<t\right)$$
$$\psi_u(1-\psi_u)^{n-3}\mathrm{d}\psi_l\mathrm{d}\psi_u$$

$$=\int_0^{\frac{t}{1+t}\frac{\sum_{k+1}^n(k)}{n(\bar{s}_n-s_k)}}\int_0^1\psi_u(1-\psi_u)^{n-3}\mathrm{d}\psi_l\mathrm{d}\psi_u$$

$$+\int_{\frac{t}{1+t}\frac{\sum_{k+1}^n(k)}{n(\bar{s}_n-s_k)}}^{\frac{t}{1+t}\frac{\sum_{k+1}^n(k+1)}{n(\bar{s}_n-s_{k+1})}}\int_0^{\frac{\frac{t}{1+t}\frac{1}{\psi_u}-\frac{n(\bar{s}_n-s_{k+1})}{\sum_{k+1}^n(k+1)}}{\frac{n(\bar{s}_n-s_k)}{\sum_{k+1}^n(k)}-\frac{n(\bar{s}_n-s_{k+1})}{\sum_{k+1}^n(k+1)}}}\psi_u(1-\psi_u)^{n-3}\mathrm{d}\psi_l\mathrm{d}\psi_u$$

$$= \frac{1}{(n-1)(n-2)}\left[1 - \frac{\frac{n(\bar{s}_n - s_k)}{\sum_{k+1}^n (k)}}{\frac{n(\bar{s}_n - s_k)}{\sum_{k+1}^n (k)} - \frac{n(\bar{s}_n - s_{k+1})}{\sum_{k+1}^n (k+1)}}\left(1 - \frac{t}{1+t}\frac{\sum_{k+1}^n (k)}{n(\bar{s}_n - s_k)}\right)^{n-1}\right.$$

$$\left. + \frac{\frac{n(\bar{s}_n - s_{k+1})}{\sum_{k+1}^n (k+1)}}{\frac{n(\bar{s}_n - s_k)}{\sum_{k+1}^n (k)} - \frac{n(\bar{s}_n - s_{k+1})}{\sum_{k+1}^n (k+1)}}\left(1 - \frac{t}{1+t}\frac{\sum_{k+1}^n (k+1)}{n(\bar{s}_n - s_{k+1})}\right)^{n-1}\right]$$

Let $-\frac{\sum_{i=1}^n (s_i - s_{k+1})}{\sum_{i=1}^k (s_i - s_{k+1})} \leq t < -\frac{\sum_{i=1}^n (s_i - s_k)}{\sum_{i=1}^k (s_i - s_k)}$,

$$\int_0^1 \int_0^1 \mathrm{I}\left(\frac{\psi_u\left(\frac{n(\bar{s}_n - s_k)}{\sum_{k+1}^n (k)}\psi_l + \frac{n(\bar{s}_n - s_{k+1})}{\sum_{k+1}^n (k+1)}(1-\psi_l)\right)}{1 - \psi_u\left(\frac{n(\bar{s}_n - s_k)}{\sum_{k+1}^n (k)}\psi_l + \frac{n(\bar{s}_n - s_{k+1})}{\sum_{k+1}^n (k+1)}(1-\psi_l)\right)} < t\right)$$
$$\psi_u(1-\psi_u)^{n-3}\mathrm{d}\psi_l \mathrm{d}\psi_u$$

$$= \int_0^{\frac{t}{1+t}\frac{\sum_{k+1}^n (k)}{n(\bar{s}_n - s_k)}} \int_0^1 \psi_u(1-\psi_u)^{n-3}\mathrm{d}\psi_l \mathrm{d}\psi_u$$

$$+ \int_{\frac{t}{1+t}\frac{\sum_{k+1}^n (k)}{n(\bar{s}_n - s_k)}}^1 \int_0^{\frac{\frac{t}{1+t}\frac{1}{\psi_u} - \frac{n(\bar{s}_n - s_{k+1})}{\sum_{k+1}^n (k+1)}}{\frac{n(\bar{s}_n - s_k)}{\sum_{k+1}^n (k)} - \frac{n(\bar{s}_n - s_{k+1})}{\sum_{k+1}^n (k+1)}}} \psi_u(1-\psi_u)^{n-3}\mathrm{d}\psi_l \mathrm{d}\psi_u,$$

i.e. the integrals one gets from the case $0 \leq t < -\frac{\sum_{i=1}^n (s_i - s_{k+1})}{\sum_{i=1}^k (s_i - s_{k+1})}$, by replacing $\frac{t}{1+t}\frac{\sum_{k+1}^n (k+1)}{n(\bar{s}_n - s_{k+1})} = 1$.

$$= \frac{1}{(n-1)(n-2)}\left[1 - \frac{\frac{n(\bar{s}_n - s_k)}{\sum_{k+1}^n (k)}}{\frac{n(\bar{s}_n - s_k)}{\sum_{k+1}^n (k)} - \frac{n(\bar{s}_n - s_{k+1})}{\sum_{k+1}^n (k+1)}}\left(1 - \frac{t}{1+t}\frac{\sum_{k+1}^n (k)}{n(\bar{s}_n - s_k)}\right)^{n-1}\right]$$

Let $-\frac{\sum_{i=1}^n (s_i - s_k)}{\sum_{i=1}^k (s_i - s_k)} \leq t$. For the domains of integration one gets no restrictions.

$$\int_0^1 \int_0^1 \mathrm{I}\left(\frac{\psi_u\left(\frac{n(\bar{s}_n - s_k)}{\sum_{k+1}^n (k)}\psi_l + \frac{n(\bar{s}_n - s_{k+1})}{\sum_{k+1}^n (k+1)}(1-\psi_l)\right)}{1 - \psi_u\left(\frac{n(\bar{s}_n - s_k)}{\sum_{k+1}^n (k)}\psi_l + \frac{n(\bar{s}_n - s_{k+1})}{\sum_{k+1}^n (k+1)}(1-\psi_l)\right)} < t\right)$$
$$\psi_u(1-\psi_u)^{n-3}\mathrm{d}\psi_l \mathrm{d}\psi_u$$

$$= \int_0^1 \int_0^1 \psi_u(1-\psi_u)^{n-3}\mathrm{d}\psi_l \mathrm{d}\psi_u = \int_0^1 \psi_u(1-\psi_u)^{n-3}\mathrm{d}\psi_u = \frac{1}{(n-1)(n-2)}$$

$k = u, \dots, n-1$:
Let $0 \le t < -\frac{\sum_{i=1}^n (s_i - s_k)}{\sum_{i=k+1}^n (s_i - s_k)}$.

$$\int_0^1 \int_0^1 \mathrm{I}\left(\frac{\psi_u \left(\frac{n(\bar{s}_n - s_k)}{\sum_1^k (k)} \psi_l + \frac{n(\bar{s}_n - s_{k+1})}{\sum_1^k (k+1)} (1-\psi_l) \right)}{1 - \psi_u \left(\frac{n(\bar{s}_n - s_k)}{\sum_1^k (k)} \psi_l + \frac{n(\bar{s}_n - s_{k+1})}{\sum_1^k (k+1)} (1-\psi_l) \right)} < t \right) \psi_u (1-\psi_u)^{n-3} \mathrm{d}\psi_l \mathrm{d}\psi_u$$

$$= \int_0^{\frac{t}{1+t} \frac{\sum_1^k (k+1)}{n(\bar{s}_n - s_{k+1})}} \int_0^1 \psi_u (1-\psi_u)^{n-3} \mathrm{d}\psi_l \mathrm{d}\psi_u$$

$$+ \int_{\frac{t}{1+t} \frac{\sum_1^k (k+1)}{n(\bar{s}_n - s_{k+1})}}^{\frac{t}{1+t} \frac{\sum_1^k (k)}{n(\bar{s}_n - s_k)}} \int^{\frac{\frac{t}{1+t}\frac{1}{\psi_u} - \frac{n(\bar{s}_n - s_{k+1})}{\sum_1^k (k+1)}}{\frac{n(\bar{s}_n - s_k)}{\sum_1^k (k)} - \frac{n(\bar{s}_n - s_{k+1})}{\sum_1^k (k+1)}}} \psi_u (1-\psi_u)^{n-3} \mathrm{d}\psi_l \mathrm{d}\psi_u$$

$$= \frac{1}{(n-1)(n-2)} \left[1 + \frac{\frac{n(\bar{s}_n - s_{k+1})}{\sum_1^k (k+1)}}{\frac{n(\bar{s}_n - s_k)}{\sum_1^k (k)} - \frac{n(\bar{s}_n - s_{k+1})}{\sum_1^k (k+1)}} \left(1 - \frac{t}{1+t} \frac{\sum_1^k (k+1)}{n(\bar{s}_n - s_{k+1})} \right)^{n-1} \right.$$
$$\left. - \frac{\frac{n(\bar{s}_n - s_k)}{\sum_1^k (k)}}{\frac{n(\bar{s}_n - s_k)}{\sum_1^k (k)} - \frac{n(\bar{s}_n - s_{k+1})}{\sum_1^k (k+1)}} \left(1 - \frac{t}{1+t} \frac{\sum_1^k (k)}{n(\bar{s}_n - s_k)} \right)^{n-1} \right]$$

Let $-\frac{\sum_{i=1}^n (s_i - s_k)}{\sum_{i=k+1}^n (s_i - s_k)} \le t < -\frac{\sum_{i=1}^n (s_i - s_{k+1})}{\sum_{i=k+1}^n (s_i - s_{k+1})}$,

$$\int_0^1 \int_0^1 \mathrm{I}\left(\frac{\psi_u \left(\frac{n(\bar{s}_n - s_k)}{\sum_1^k (k)} \psi_l + \frac{n(\bar{s}_n - s_{k+1})}{\sum_1^k (k+1)} (1-\psi_l) \right)}{1 - \psi_u \left(\frac{n(\bar{s}_n - s_k)}{\sum_1^k (k)} \psi_l + \frac{n(\bar{s}_n - s_{k+1})}{\sum_1^k (k+1)} (1-\psi_l) \right)} < t \right) \psi_u (1-\psi_u)^{n-3} \mathrm{d}\psi_l \mathrm{d}\psi_u$$

$$= \int_0^{\frac{t}{1+t} \frac{\sum_1^k (k+1)}{n(\bar{s}_n - s_{k+1})}} \int_0^1 \psi_u (1-\psi_u)^{n-3} \mathrm{d}\psi_l \mathrm{d}\psi_u$$

$$+ \int_{\frac{t}{1+t} \frac{\sum_1^k (k+1)}{n(\bar{s}_n - s_{k+1})}}^1 \int^{\frac{\frac{t}{1+t}\frac{1}{\psi_u} - \frac{n(\bar{s}_n - s_{k+1})}{\sum_1^k (k+1)}}{\frac{n(\bar{s}_n - s_k)}{\sum_1^k (k)} - \frac{n(\bar{s}_n - s_{k+1})}{\sum_1^k (k+1)}}} \psi_u (1-\psi_u)^{n-3} \mathrm{d}\psi_l \mathrm{d}\psi_u,$$

i.e. the integrals one gets from the case $0 \le t < -\frac{\sum_{i=1}^{n}(s_i-s_k)}{\sum_{i=k+1}^{n}(s_i-s_k)}$, by replacing $\frac{t}{1+t}\frac{\sum_{k+1}^{n}(k)}{n(\bar{s}_n-s_k)} = 1$.

$$= \frac{1}{(n-1)(n-2)}\left[1 + \frac{\frac{n(\bar{s}_n-s_{k+1})}{\sum_1^k(k+1)}}{\frac{n(\bar{s}_n-s_k)}{\sum_1^k(k)} - \frac{n(\bar{s}_n-s_{k+1})}{\sum_1^k(k+1)}}\left(1 - \frac{t}{1+t}\frac{\sum_1^k(k+1)}{n(\bar{s}_n-s_{k+1})}\right)^{n-1}\right]$$

Let $-\frac{\sum_{i=1}^{n}(s_i-s_{k+1})}{\sum_{i=k+1}^{n}(s_i-s_{k+1})} \le t$. For the domains of integration one gets no restrictions.

$$\int_0^1\int_0^1 \mathrm{I}\left(\frac{\psi_u\left(\frac{n(\bar{s}_n-s_k)}{\sum_1^k(k)}\psi_l + \frac{n(\bar{s}_n-s_{k+1})}{\sum_1^k(k+1)}(1-\psi_l)\right)}{1-\psi_u\left(\frac{n(\bar{s}_n-s_k)}{\sum_1^k(k)}\psi_l + \frac{n(\bar{s}_n-s_{k+1})}{\sum_1^k(k+1)}(1-\psi_l)\right)} < t\right)$$
$$\psi_u(1-\psi_u)^{n-3}\mathrm{d}\psi_l\mathrm{d}\psi_u$$
$$= \int_0^1\int_0^1 \psi_u(1-\psi_u)^{n-3}\mathrm{d}\psi_l\mathrm{d}\psi_u = \int_0^1 \psi_u(1-\psi_u)^{n-3}\mathrm{d}\psi_u = \frac{1}{(n-1)(n-2)}.$$

For the cdf one gets hence

$$F_{T_{\alpha'}(\mathbf{w})}(t) = \sum_{l=1}^{m}\sum_{u=m+1}^{n}$$
$$\left\{\sum_{k=1}^{l-1}\frac{(s_u-s_l)(s_{k+1}-s_k)}{\sum_{k+1}^{n}(k)\sum_{k+1}^{n}(k+1)}\right.$$
$$\begin{cases} 1 - \frac{\frac{\bar{s}_n-s_k}{\sum_{k+1}^{n}(k)}}{\frac{\bar{s}_n-s_k}{\sum_{k+1}^{n}(k)} - \frac{\bar{s}_n-s_{k+1}}{\sum_{k+1}^{n}(k+1)}}\left(1-\frac{t}{1+t}\frac{\sum_{k+1}^{n}(k)}{n(\bar{s}_n-s_k)}\right)^{n-1} \\ \quad + \frac{\frac{\bar{s}_n-s_{k+1}}{\sum_{k+1}^{n}(k+1)}}{\frac{\bar{s}_n-s_k}{\sum_{k+1}^{n}(k)} - \frac{\bar{s}_n-s_{k+1}}{\sum_{k+1}^{n}(k+1)}}\left(1-\frac{t}{1+t}\frac{\sum_{k+1}^{n}(k+1)}{n(\bar{s}_n-s_{k+1})}\right)^{n-1}, & 0 \le t < -\frac{\sum_{i=1}^{n}(s_i-s_{k+1})}{\sum_{i=1}^{k}(s_i-s_{k+1})} \\ 1 - \frac{\frac{\bar{s}_n-s_k}{\sum_{k+1}^{n}(k)}}{\frac{\bar{s}_n-s_k}{\sum_{k+1}^{n}(k)} - \frac{\bar{s}_n-s_{k+1}}{\sum_{k+1}^{n}(k+1)}}\left(1-\frac{t}{1+t}\frac{\sum_{k+1}^{n}(k)}{n(\bar{s}_n-s_k)}\right)^{n-1}, & \\ & -\frac{\sum_{i=1}^{n}(s_i-s_{k+1})}{\sum_{i=1}^{k}(s_i-s_{k+1})} \le t < -\frac{\sum_{i=1}^{n}(s_i-s_k)}{\sum_{i=1}^{k}(s_i-s_k)} \\ 1, & -\frac{\sum_{i=1}^{n}(s_i-s_k)}{\sum_{i=1}^{k}(s_i-s_k)} \le t \end{cases}$$

$$+\sum_{k=u}^{n-1}\frac{(s_u-s_l)(s_{k+1}-s_k)}{\sum_1^k(k)\sum_1^k(k+1)}$$

$$\begin{cases}
1+\dfrac{\frac{\bar{s}_n-s_{k+1}}{\sum_1^k(k+1)}}{\frac{\bar{s}_n-s_k}{\sum_1^k(k)}-\frac{\bar{s}_n-s_{k+1}}{\sum_1^k(k+1)}}\left(1-\frac{t}{1+t}\frac{\sum_1^k(k+1)}{n(\bar{s}_n-s_{k+1})}\right)^{n-1} \\
\qquad -\dfrac{\frac{\bar{s}_n-s_k}{\sum_1^k(k)}}{\frac{\bar{s}_n-s_k}{\sum_1^k(k)}-\frac{\bar{s}_n-s_{k+1}}{\sum_1^k(k+1)}}\left(1-\frac{t}{1+t}\frac{\sum_1^k(k)}{n(\bar{s}_n-s_k)}\right)^{n-1}, & 0\le t<-\frac{\sum_{i=1}^n(s_i-s_k)}{\sum_{i=k+1}^n(s_i-s_k)} \\
1+\dfrac{\frac{\bar{s}_n-s_{k+1}}{\sum_1^k(k+1)}}{\frac{\bar{s}_n-s_k}{\sum_1^k(k)}-\frac{\bar{s}_n-s_{k+1}}{\sum_1^k(k+1)}}\left(1-\frac{t}{1+t}\frac{\sum_1^k(k+1)}{n(\bar{s}_n-s_{k+1})}\right)^{n-1}, & -\frac{\sum_{i=1}^n(s_i-s_k)}{\sum_{i=k+1}^n(s_i-s_k)}\le t<-\frac{\sum_{i=1}^n(s_i-s_{k+1})}{\sum_{i=k+1}^n(s_i-s_{k+1})} \\
1, & -\frac{\sum_{i=1}^n(s_i-s_{k+1})}{\sum_{i=k+1}^n(s_i-s_{k+1})}\le t
\end{cases}$$

$$\left.+\frac{s_u-s_l}{n}\left[\frac{1}{\sum_1^n(1)}-\frac{1}{\sum_1^n(n)}\right]\left[1-(1+t(n-1))(\frac{1}{1+t})^{n-1}\right]\right\}$$

$$=1-\sum_{l=1}^{m}\sum_{u=m+1}^{n}$$

$$\left\{\sum_{k=1}^{l-1}(s_u-s_l)\right.$$

$$\begin{cases}
\frac{1}{\sum_{i=1}^k(\bar{s}_n-s_i)}\left[\frac{\bar{s}_n-s_k}{\sum_{k+1}^n(k)}\left(1-\frac{t}{1+t}\frac{\sum_{k+1}^n(k)}{n(\bar{s}_n-s_k)}\right)^{n-1}\right. \\
\qquad \left.-\frac{\bar{s}_n-s_{k+1}}{\sum_{k+1}^n(k+1)}\left(1-\frac{t}{1+t}\frac{\sum_{k+1}^n(k+1)}{n(\bar{s}_n-s_{k+1})}\right)^{n-1}\right], & 0\le t<-\frac{\sum_{i=1}^n(s_i-s_{k+1})}{\sum_{i=1}^k(s_i-s_{k+1})} \\
\frac{1}{\sum_{i=1}^k(\bar{s}_n-s_i)}\frac{\bar{s}_n-s_k}{\sum_{k+1}^n(k)}\left(1-\frac{t}{1+t}\frac{\sum_{k+1}^n(k)}{n(\bar{s}_n-s_k)}\right)^{n-1}, & -\frac{\sum_{i=1}^n(s_i-s_{k+1})}{\sum_{i=1}^k(s_i-s_{k+1})}\le t<-\frac{\sum_{i=1}^n(s_i-s_k)}{\sum_{i=1}^k(s_i-s_k)} \\
0, & -\frac{\sum_{i=1}^n(s_i-s_k)}{\sum_{i=1}^k(s_i-s_k)}\le t
\end{cases}$$

$$+\sum_{k=u}^{n-1}(s_u-s_l)$$

$$\begin{cases}
\frac{1}{\sum_{i=k+1}^n(\bar{s}_n-s_i)}\left[\frac{\bar{s}_n-s_k}{\sum_1^k(k)}\left(1-\frac{t}{1+t}\frac{\sum_1^k(k)}{n(\bar{s}_n-s_k)}\right)^{n-1}\right. \\
\qquad \left.-\frac{\bar{s}_n-s_{k+1}}{\sum_1^k(k+1)}\left(1-\frac{t}{1+t}\frac{\sum_1^k(k+1)}{n(\bar{s}_n-s_{k+1})}\right)^{n-1}\right], & 0\le t<-\frac{\sum_{i=1}^n(s_i-s_k)}{\sum_{i=k+1}^n(s_i-s_k)} \\
-\frac{1}{\sum_{i=k+1}^n(\bar{s}_n-s_i)}\frac{\bar{s}_n-s_{k+1}}{\sum_1^k(k+1)}\left(1-\frac{t}{1+t}\frac{\sum_1^k(k+1)}{n(\bar{s}_n-s_{k+1})}\right)^{n-1}, & -\frac{\sum_{i=1}^n(s_i-s_k)}{\sum_{i=k+1}^n(s_i-s_k)}\le t<-\frac{\sum_{i=1}^n(s_i-s_{k+1})}{\sum_{i=k+1}^n(s_i-s_{k+1})} \\
0, & -\frac{\sum_{i=1}^n(s_i-s_{k+1})}{\sum_{i=k+1}^n(s_i-s_{k+1})}\le t
\end{cases}$$

$$+ \frac{s_u - s_l}{n} \left[\frac{1}{\sum_1^n(1)} - \frac{1}{\sum_1^n(n)} \right] \left[(1 + t(n-1))(\frac{1}{1+t})^{n-1} \right] \Bigg\}$$

and with rearrangements

$$\sum_{l=1}^{m} \sum_{u=m+1}^{n} \sum_{k=1}^{l-1} (s_u - s_l) f(k) = \sum_{k=1}^{m-1} f(k) \sum_{l=k+1}^{m} \sum_{u=m+1}^{n} (s_u - s_l)$$

and

$$\sum_{l=1}^{m} \sum_{u=m+1}^{n} \sum_{k=u}^{n-1} (s_u - s_l) f(k) = \sum_{k=m+1}^{n-1} f(k) \sum_{l=1}^{m} \sum_{u=m+1}^{k} (s_u - s_l)$$

one gets

$$= 1 - \Bigg\{ \sum_{k=1}^{m-1} \sum_{l=k+1}^{m} \sum_{u=m+1}^{n} (s_u - s_l)$$

$$\begin{cases} \frac{1}{\sum_{i=1}^{k}(\bar{s}_n - s_i)} \left[\frac{\bar{s}_n - s_k}{\sum_{k+1}^{n}(k)} \left(1 - \frac{t}{1+t} \frac{\sum_{k+1}^{n}(k)}{n(\bar{s}_n - s_k)} \right)^{n-1} \right. \\ \qquad \left. - \frac{\bar{s}_n - s_{k+1}}{\sum_{k+1}^{n}(k+1)} \left(1 - \frac{t}{1+t} \frac{\sum_{k+1}^{n}(k+1)}{n(\bar{s}_n - s_{k+1})} \right)^{n-1} \right], & 0 \le t < -\frac{\sum_{i=1}^{n}(s_i - s_{k+1})}{\sum_{i=1}^{k}(s_i - s_{k+1})} \\ \frac{1}{\sum_{i=1}^{k}(\bar{s}_n - s_i)} \frac{\bar{s}_n - s_k}{\sum_{k+1}^{n}(k)} \left(1 - \frac{t}{1+t} \frac{\sum_{k+1}^{n}(k)}{n(\bar{s}_n - s_k)} \right)^{n-1}, & -\frac{\sum_{i=1}^{n}(s_i - s_{k+1})}{\sum_{i=1}^{k}(s_i - s_{k+1})} \le t < -\frac{\sum_{i=1}^{n}(s_i - s_k)}{\sum_{i=1}^{k}(s_i - s_k)} \\ 0, & -\frac{\sum_{i=1}^{n}(s_i - s_k)}{\sum_{i=1}^{k}(s_i - s_k)} \le t \end{cases}$$

$$+ \sum_{k=m+1}^{n-1} \sum_{l=1}^{m} \sum_{u=m+1}^{k} (s_u - s_l)$$

$$\begin{cases} \frac{1}{\sum_{i=k+1}^{n}(\bar{s}_n - s_i)} \left[\frac{\bar{s}_n - s_k}{\sum_{1}^{k}(k)} \left(1 - \frac{t}{1+t} \frac{\sum_{1}^{k}(k)}{n(\bar{s}_n - s_k)} \right)^{n-1} \right. \\ \qquad \left. - \frac{\bar{s}_n - s_{k+1}}{\sum_{1}^{k}(k+1)} \left(1 - \frac{t}{1+t} \frac{\sum_{1}^{k}(k+1)}{n(\bar{s}_n - s_{k+1})} \right)^{n-1} \right], & 0 \le t < -\frac{\sum_{i=1}^{n}(s_i - s_k)}{\sum_{i=k+1}^{n}(s_i - s_k)} \\ -\frac{1}{\sum_{i=k+1}^{n}(\bar{s}_n - s_i)} \frac{\bar{s}_n - s_{k+1}}{\sum_{1}^{k}(k+1)} \left(1 - \frac{t}{1+t} \frac{\sum_{1}^{k}(k+1)}{n(\bar{s}_n - s_{k+1})} \right)^{n-1}, & -\frac{\sum_{i=1}^{n}(s_i - s_k)}{\sum_{i=k+1}^{n}(s_i - s_k)} \le t < -\frac{\sum_{i=1}^{n}(s_i - s_{k+1})}{\sum_{i=k+1}^{n}(s_i - s_{k+1})} \\ 0, & -\frac{\sum_{i=1}^{n}(s_i - s_{k+1})}{\sum_{i=k+1}^{n}(s_i - s_{k+1})} \le t \end{cases}$$

$$+ \sum_{l=1}^{m} \sum_{u=m+1}^{n} (s_u - s_l) \frac{1}{n} \left[\frac{1}{\sum_1^n(1)} - \frac{1}{\sum_1^n(n)} \right] \left[(1 + t(n-1)) \left(\frac{1}{1+t} \right)^{n-1} \right] \Bigg\}.$$

□

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Dr. Volkmar Henschel, Institute for Medical Biometry and Informatics, Department of Medical Biometry, University of Heidelberg, Im Neuenheimer Feld 305, D-69120 Heidelberg. Germany.
e-mail: henschel@imbi.uni-heidelberg.de