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CONTROLLABILITY OF SEMILINEAR FUNCTIONAL INTEGRODIFFERENTIAL SYSTEMS IN BANACH SPACES

KRISHNAN BALACHANDRAN AND RATHINASAMY SAKTHIVEL

Sufficient conditions for controllability of semilinear functional integrodifferential systems in a Banach space are established. The results are obtained by using the Schaefer fixed-point theorem.

1. INTRODUCTION

Controllability of nonlinear systems represented by ordinary differential equations in infinite-dimensional spaces has been extensively studied by several authors. Naito [12, 13] has studied the controllability of semilinear systems whereas Yamamoto and Park [19] discussed the same problem for parabolic equation with uniformly bounded nonlinear term. Chukwu and Lenhart [3] have studied the controllability of nonlinear systems in abstract spaces. Do [4] and Zhou [20] investigated the approximate controllability for a class of semilinear abstract equations. Kwun et al [7] established the approximate controllability for delay Volterra systems with bounded linear operators. Controllability for nonlinear Volterra integrodifferential systems has been studied by Naito [14]. Recently Balachandran et al [1, 2] studied the controllability and local null controllability of Sobolev-type integrodifferential systems and functional differential systems in Banach spaces by using Schauder's fixed-point theorem. The purpose of this paper is to study the controllability of semilinear functional integrodifferential systems in Banach spaces by using the Schaefer fixed-point theorem. The semilinear functional integrodifferential equation considered here serves as an abstract formulation of partial functional integrodifferential equations which arise in heat flow in material with memory [5, 6, 8, 9, 18].

2. PRELIMINARIES

Consider the semilinear functional integrodifferential system of the form

$$(Ex(t))' + Ax(t) = (Bu)(t) + \int_0^t f(s, x_s) ds, \quad t \in J = [0, b], \quad (1)$$

$$x(t) = \phi(t), \quad t \in [-r, 0],$$

where E and A are linear operators with domains contained in a Banach space X and ranges contained in a Banach space Y , the state $x(\cdot)$ takes values in X and the control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions with U as a Banach space, B is a bounded linear operator from U into Y and the nonlinear operator $f : J \times C \rightarrow Y$ is a given function. Here $C = C([-r, 0], X)$ is the Banach space of all continuous functions $\phi : [-r, 0] \rightarrow X$ endowed with the norm $\|\phi\| = \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}$. Also for $x \in C([-r, b], X)$ we have $x_t \in C$ for $t \in [0, b]$, $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$. The norm of X is denoted by $\|\cdot\|$ and Y by $|\cdot|$.

The operators $A : D(A) \subset X \rightarrow Y$ and $E : D(E) \subset X \rightarrow Y$ satisfy the hypotheses $[C_i]$ for $i = 1, \dots, 4$:

$[C_1]$ A and E are closed linear operators

$[C_2]$ $D(E) \subset D(A)$ and E is bijective

$[C_3]$ $E^{-1} : Y \rightarrow D(E)$ is continuous

$[C_4]$ For each $t \in [0, b]$ and for some $\lambda \in \rho(-AE^{-1})$, the resolvent set of $-AE^{-1}$, the resolvent $R(\lambda, -AE^{-1})$ is a compact operator.

The hypotheses $[C_1], [C_2]$ and the closed graph theorem imply the boundedness of the linear operator $AE^{-1} : Y \rightarrow Y$.

Lemma. [15] Let $S(t)$ be a uniformly continuous semigroup and let A be its infinitesimal generator. If the resolvent set $R(\lambda : A)$ of A is compact for every $\lambda \in \rho(A)$, then $S(t)$ is a compact semigroup.

From the above fact, $-AE^{-1}$ generates a compact semigroup $T(t)$, $t \geq 0$, on Y .

Definition. The system (1) is said to be controllable on the interval J if for every continuous initial function $\phi \in C$ and $x_1 \in X$, there exists a control $u \in L^2(J, U)$ such that the solution $x(t)$ of (1) satisfies $x(b) = x_1$.

We further assume the following hypotheses:

$[C_5]$ $-AE^{-1}$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in Y satisfying

$$|T(t)| \leq M_1 e^{\omega t}, \quad t \geq 0 \text{ for some } M_1 \geq 1 \text{ and } \omega \geq 0.$$

$[C_6]$ The linear operator $W : L^2(J, U) \rightarrow X$ defined by

$$Wu = \int_0^b E^{-1}T(b-s)Bu(s) ds$$

has an inverse operator $\widetilde{W}^{-1} : X \rightarrow L^2(J, U)/\ker W$ and there exist positive constants M_2, M_3 such that $|B| \leq M_2$ and $|\widetilde{W}^{-1}| \leq M_3$ (See the remark for the construction of \widetilde{W}^{-1}).

[C₇] For each $t \in J$, the function $f(t, \cdot) : C \rightarrow Y$ is continuous and for each $x \in C$, the function $f(\cdot, x) : J \rightarrow Y$ is strongly measurable.

[C₈] There exists an integrable function $m : [0, b] \rightarrow [0, \infty)$ such that

$$|f(t, \phi)| \leq m(t)\Omega(\|\phi\|), \quad 0 \leq t \leq b, \quad \phi \in C,$$

where $\Omega : [0, \infty) \rightarrow (0, \infty)$ is a continuous nondecreasing function.

[C₉]

$$\int_0^b \hat{m}(s) ds < \int_c^\infty \frac{ds}{1 + s + \Omega(s)},$$

where

$$c = \|E^{-1}\|M_1|E\phi(0)|, \quad \hat{m}(t) = \max \left\{ \omega, \|E^{-1}\|M_1N, \|E^{-1}\|M_1 \int_0^t m(s) ds \right\}$$

and

$$N = M_2M_3[\|x_1\| + \|E^{-1}\|M_1e^{\omega b}\|\phi\| + \|E^{-1}\|M_1 \int_0^b e^{\omega(b-s)} \int_0^s m(\tau)\Omega(\|x_\tau\|) d\tau ds].$$

We need the following fixed point theorem due to Schaefer [17].

Theorem 1. Let E be a normed space. Let $F : E \rightarrow E$ be a completely continuous operator, i. e., it is continuous and the image of any bounded set is contained in a compact set, and let

$$\zeta(F) = \{x \in E; x = \lambda Fx \text{ for some } 0 < \lambda < 1\}$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

Then the system (1) has a mild solution of the following form

$$\begin{aligned} x(t) &= E^{-1}T(t)E\phi(0) + \int_0^t E^{-1}T(t-s) \left[(Bu)(s) \right. \\ &\quad \left. + \int_0^s f(\tau, x_\tau) d\tau \right] ds, \quad t \in J, \\ x(t) &= \phi(t), \quad t \in [-r, 0] \end{aligned}$$

and $Ex(t) \in C([0, b]; Y) \cap C'((0, b); Y)$.

3. MAIN RESULT

Theorem 2. If the hypotheses $[C_1]$ - $[C_9]$ are satisfied, then the system (1) is controllable on J .

Proof. Using the hypothesis $[C_6]$ for an arbitrary function $x(\cdot)$, define the control

$$u(t) = \widetilde{W}^{-1} \left[x_1 - E^{-1}T(b)E\phi(0) - \int_0^b E^{-1}T(b-s) \int_0^s f(\tau, x_\tau) d\tau ds \right] (t).$$

For $\phi \in C$, define $\hat{\phi} \in C_b$, $C_b = C([-r, b], X)$ by

$$\hat{\phi}(t) = \begin{cases} \phi(t), & -r \leq t \leq 0, \\ E^{-1}T(t)E\phi(0), & 0 \leq t \leq b. \end{cases}$$

If $x(t) = y(t) + \hat{\phi}(t)$, $t \in [-r, b]$, it is easy to see that y satisfies

$$\begin{aligned} y_0 &= 0, \\ y(t) &= \int_0^t E^{-1}T(t-\eta)B\widetilde{W}^{-1}[x_1 - E^{-1}T(b)E\phi(0) \\ &\quad - \int_0^b E^{-1}T(b-s) \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau) d\tau ds](\eta) d\eta \\ &\quad + \int_0^t E^{-1}T(t-s) \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau) d\tau ds, \quad 0 \leq t \leq b \end{aligned} \quad (2)$$

if and only if x satisfies

$$\begin{aligned} x(t) &= E^{-1}T(t)E\phi(0) + \int_0^t E^{-1}T(t-\eta)B\widetilde{W}^{-1}[x_1 - E^{-1}T(b)E\phi(0) \\ &\quad - \int_0^b E^{-1}T(b-s) \int_0^s f(\tau, x_\tau) d\tau ds](\eta) d\eta \\ &\quad + \int_0^t E^{-1}T(t-s) \int_0^s f(\tau, x_\tau) d\tau ds \end{aligned}$$

and $x(t) = \phi(t)$, $t \in [-r, 0]$.

Define $C_b^0 = \{y \in C_b : y^0 \cong 0\}$ and we now show that when using the control, the operator $F : C_b^0 \rightarrow C_b^0$, defined by

$$(Fy)(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ \int_0^t E^{-1}T(t-\eta)B\widetilde{W}^{-1}[x_1 - E^{-1}T(b)E\phi(0) \\ \quad - \int_0^b E^{-1}T(b-s) \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau) d\tau ds](\eta) d\eta \\ \quad + \int_0^t E^{-1}T(t-s) \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau) d\tau ds, & 0 \leq t \leq b \end{cases}$$

has a fixed point. This fixed point is then a solution of equation (2).

Clearly $x(b) = x_1$ which means that the control u steers the system (1) from the initial function ϕ to x_1 in time b , provided we can obtain a fixed point of the nonlinear operator F .

In order to study the controllability problem of (1), we introduce a parameter $\lambda \in (0, 1)$ and consider the following system

$$\begin{aligned} (Ex(t))' + Ax(t) &= \lambda(Bu)(t) + \lambda \int_0^t f(s, x_s) ds, \quad t \in J = [0, b], \\ x(t) &= \lambda\phi(t), \quad t \in [-r, 0]. \end{aligned} \tag{3}$$

First we obtain *a priori* bounds for the mild solution of the equation (3). Then from

$$\begin{aligned} x(t) &= \lambda E^{-1}T(t)E\phi(0) + \lambda \int_0^t E^{-1}T(t-\eta)B\widetilde{W}^{-1} \left[x_1 - E^{-1}T(b)E\phi(0) \right. \\ &\quad \left. - \int_0^b E^{-1}T(b-s) \int_0^s f(\tau, x_\tau) d\tau ds \right](\eta) d\eta \\ &\quad + \lambda \int_0^t E^{-1}T(t-s) \int_0^s f(\tau, x_\tau) d\tau ds, \end{aligned}$$

we have,

$$\begin{aligned} \|x(t)\| &\leq \|E^{-1}\|M_1e^{\omega t}|E\phi(0)| + \int_0^t \|E^{-1}\| \|T(t-\eta)\| M_2M_3[\|x_1\| \\ &\quad + \|E^{-1}\|M_1e^{\omega b}|E\phi(0)| + \int_0^b \|E^{-1}\|M_1e^{\omega(b-s)} \int_0^s m(\tau)\Omega(\|x_\tau\|) d\tau ds] d\eta \\ &\quad + \|E^{-1}\|M_1e^{\omega t} \int_0^t e^{-\omega s} \int_0^s m(\tau)\Omega(\|x_\tau\|) d\tau ds, \quad t \in [0, b]. \end{aligned}$$

We consider the function μ given by

$$\mu(t) = \sup\{\|x(s)\| : -r \leq s \leq t\}, \quad 0 \leq t \leq b.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = \|x(t^*)\|$. If $t^* \in [0, b]$, by the previous inequality, we have

$$\begin{aligned} e^{-\omega t}\mu(t) &\leq \|E^{-1}\|M_1|E\phi(0)| + \|E^{-1}\|M_1N \int_0^{t^*} e^{-\omega s} ds \\ &\quad + \|E^{-1}\|M_1 \int_0^{t^*} e^{-\omega s} \int_0^s m(\tau)\Omega(\|x_\tau\|) d\tau ds \\ &\leq \|E^{-1}\|M_1|E\phi(0)| \\ &\quad + \|E^{-1}\|M_1N \int_0^t e^{-\omega s} ds + \|E^{-1}\|M_1 \int_0^t e^{-\omega s} \int_0^s m(\tau)\Omega(\mu(\tau)) d\tau ds. \end{aligned}$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|$ and the previous inequality holds since $M_1 \geq 1$.

Denoting by $v(t)$, the right-hand side of the above inequality, we have $c = v(0) = \|E^{-1}\|M_1|E\phi(0)|$, $\mu(t) \leq e^{\omega t}v(t)$, $0 \leq t \leq b$ and

$$\begin{aligned} v'(t) &= \|E^{-1}\|M_1Ne^{-\omega t} + \|E^{-1}\|M_1e^{-\omega t} \int_0^t m(s)\Omega(\mu(s)) ds \\ &\leq \|E^{-1}\|M_1Ne^{-\omega t} + \|E^{-1}\|M_1e^{-\omega t} \int_0^t m(s)\Omega(e^{\omega s}v(s)) ds. \end{aligned}$$

We remark that

$$\begin{aligned} (e^{\omega t}v(t))' &= \omega e^{\omega t}v(t) + e^{\omega t}v'(t) \\ &\leq \omega e^{\omega t}v(t) + \|E^{-1}\|M_1N + \|E^{-1}\|M_1 \int_0^t m(s)\Omega(e^{\omega s}v(s)) ds \\ &\leq \omega e^{\omega t}v(t) + \|E^{-1}\|M_1N + \|E^{-1}\|M_1\Omega(e^{\omega t}v(t)) \int_0^t m(s) ds \\ &\leq \hat{m}(t)[e^{\omega t}v(t) + 1 + \Omega(e^{\omega t}v(t))]. \end{aligned}$$

This implies

$$\int_{v(0)}^{e^{\omega t}v(t)} \frac{ds}{1+s+\Omega(s)} \leq \int_0^b \hat{m}(s) ds < \int_c^\infty \frac{ds}{1+s+\Omega(s)}, \quad 0 \leq t \leq b.$$

This inequality implies that there is a constant K such that $v(t) \leq K$ and hence $\mu(t) \leq K$, $t \in [0, b]$. Since $\|x_t\| \leq \mu(t)$, $t \in [0, b]$, we have

$$\|x\|_1 = \sup\{\|x(t)\| : -r \leq t \leq b\} \leq K,$$

where K depends only on b and on the functions m and Ω .

Next we must prove that the operator F is a completely continuous operator. Let $B_k = \{y \in C_b^0 : \|y\|_1 \leq k\}$ for some $k \geq 1$.

We first show that the set $\{Fy : y \in B_k\}$ is equicontinuous. Let $y \in B_k$ and $t_1, t_2 \in [0, b]$. Then if $0 < t_1 < t_2 \leq b$,

$$\begin{aligned} &\|(Fy)(t_1) - (Fy)(t_2)\| \\ &\leq \left\| \int_0^{t_1} E^{-1}[T(t_1 - \eta) - T(t_2 - \eta)]B\widetilde{W}^{-1}[x_1 - E^{-1}T(b)E\phi(0)] \right. \\ &\quad \left. - \int_0^b E^{-1}T(b - s) \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau) d\tau ds(\eta) d\eta \right\| \\ &\quad + \left\| \int_{t_1}^{t_2} E^{-1}T(t_2 - \eta)B\widetilde{W}^{-1}[x_1 - E^{-1}T(b)E\phi(0)] \right. \end{aligned}$$

$$\begin{aligned}
 & - \int_0^b E^{-1}T(b-s) \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau) d\tau ds](\eta) d\eta \Big\| \\
 & + \Big\| \int_0^{t_1} E^{-1}[T(t_1-s) - T(t_2-s)] \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau) d\tau ds \Big\| \\
 & + \Big\| \int_{t_1}^{t_2} E^{-1}T(t_2-s) \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau) d\tau ds \Big\| \\
 \leq & \int_0^{t_1} \|E^{-1}\| |T(t_1-\eta) - T(t_2-\eta)| M_2 M_3 [\|x_1\| + \|E^{-1}\| M_1 e^{\omega b} |E\phi(0)| \\
 & + \|E^{-1}\| M_1 \int_0^b e^{\omega(b-s)} \int_0^s m(\tau) \Omega(k') d\tau ds] d\eta \\
 & + \int_{t_1}^{t_2} \|E^{-1}\| |T(t_2-\eta)| M_2 M_3 [\|x_1\| + \|E^{-1}\| M_1 e^{\omega b} |E\phi(0)| \\
 & + \|E^{-1}\| M_1 \int_0^b e^{\omega(b-s)} \int_0^s m(\tau) \Omega(k') d\tau ds] d\eta \\
 & + \int_0^{t_1} \|E^{-1}\| |T(t_1-s) - T(t_2-s)| \int_0^s m(\tau) \Omega(k') d\tau ds \\
 & + \int_{t_1}^{t_2} \|E^{-1}\| |T(t_2-s)| \int_0^s m(\tau) \Omega(k') d\tau ds,
 \end{aligned}$$

where $k' = k + \|\hat{\phi}\|$. The right hand side is independent of $y \in B_k$ and tends to zero as $t_2 - t_1 \rightarrow 0$, since the compactness of $T(t)$, for $t > 0$, implies the continuity in the uniform operator topology.

Thus the set $\{Fy; y \in B_k\}$ is equicontinuous.

Notice that we considered here only the case $0 < t_1 < t_2$, since the other cases $t_1 < t_2 < 0$ or $t_1 < 0 < t_2$ are very simple.

It is easy to see that the family FB_k is uniformly bounded. Next we show that $\overline{FB_k}$ is compact. Since we have shown that FB_k is an equicontinuous collection, it suffices, by the Arzela-Ascoli theorem, to show that F maps B_k into a precompact set in X .

Let $0 < t \leq b$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $y \in B_k$, we define

$$\begin{aligned}
 (F_\epsilon y)(t) & = \int_0^{t-\epsilon} E^{-1}T(t-\eta) B\widetilde{W}^{-1} [x_1 - E^{-1}T(b)E\phi(0) \\
 & - \int_0^b E^{-1}T(b-s) \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau) d\tau ds](\eta) d\eta \\
 & + \int_0^{t-\epsilon} E^{-1}T(t-s) \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau) d\tau ds \\
 & = T(\epsilon) \int_0^{t-\epsilon} E^{-1}T(t-\eta-\epsilon) B\widetilde{W}^{-1} [x_1 - E^{-1}T(b)E\phi(0) \\
 & - \int_0^b E^{-1}T(b-s) \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau) d\tau ds](\eta) d\eta
 \end{aligned}$$

$$+ T(\epsilon) \int_0^{t-\epsilon} E^{-1}T(t-s-\epsilon) \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau) d\tau ds.$$

Since $T(t)$ is a compact operator, the set $Y_\epsilon(t) = \{(F_\epsilon y)(t) : y \in B_k\}$ is precompact in X for every $\epsilon, 0 < \epsilon < t$. Moreover for every $y \in B_k$ we have

$$\begin{aligned} & \| (Fy)(t) - (F_\epsilon y)(t) \| \\ & \leq \int_{t-\epsilon}^t \| E^{-1}T(t-\eta)B\widetilde{W}^{-1}[x_1 - E^{-1}T(b)E\phi(0) \\ & \quad - \int_0^b E^{-1}T(b-s) \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau) d\tau ds](\eta) \| d\eta \\ & \quad + \int_{t-\epsilon}^t \| E^{-1}T(t-s) \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau) \| d\tau ds \\ & \leq \int_{t-\epsilon}^t \| E^{-1} \| \| T(t-\eta) \| M_2 M_3 [\| x_1 \| + | E^{-1} | M_1 e^{\omega b} | E\phi(0) | \\ & \quad + \| E^{-1} \| M_1 \int_0^b e^{\omega(b-s)} \int_0^s m(\tau)\Omega(k') d\tau ds] d\eta \\ & \quad + \int_{t-\epsilon}^t \| E^{-1} \| \| T(t-s) \| \int_0^s m(\tau)\Omega(k') d\tau ds. \end{aligned}$$

Therefore there are precompact sets arbitrarily close to the set $\{(Fy)(t) : y \in B_k\}$. Hence the set $\{(Fy)(t) : y \in B_k\}$ is precompact in X .

It remains to be shown that $F : C_b^0 \rightarrow C_b^0$ is continuous. Let $\{y_n\}_0^\infty \subseteq C_b^0$ with $y_n \rightarrow y$ in C_b^0 . Then there is an integer r such that $\|y_n(t)\| \leq r$ for all n and $t \in J$, so $y_n \in B_r$ and $y \in B_r$. By [C7], $f(t, y_n(t) + \hat{\phi}_t) \rightarrow f(t, y(t) + \hat{\phi}_t)$ for each $t \in J$ and since $|f(t, y_n(t) + \hat{\phi}_t) - f(t, y(t) + \hat{\phi}_t)| \leq 2g_{r'}(t), r' = r + \|\hat{\phi}\|$, we have, by dominated convergence theorem,

$$\begin{aligned} & \| Fy_n - Fy \| \\ & = \sup_{t \in J} \left\| \int_0^t E^{-1}T(t-\eta)B\widetilde{W}^{-1} \left[\int_0^b T(b-s) \right. \right. \\ & \quad \left. \left. \int_0^s [f(\tau, y_n(\tau) + \hat{\phi}_\tau) - f(\tau, y(\tau) + \hat{\phi}_\tau)] d\tau ds \right] (\eta) d\eta \right. \\ & \quad \left. + \int_0^t E^{-1}T(t-s) \int_0^s [f(\tau, y_n(\tau) + \hat{\phi}_\tau) - f(\tau, y(\tau) + \hat{\phi}_\tau)] d\tau ds \right\| \\ & \leq \int_0^b \| E^{-1} \| \| T(t-\eta) \| M_2 M_3 [M_1 \int_0^b e^{\omega(b-s)} \\ & \quad \int_0^s |f(\tau, y_n(\tau) + \hat{\phi}_\tau) - f(\tau, y(\tau) + \hat{\phi}_\tau)| d\tau ds] d\eta \\ & \quad + \int_0^b \| E^{-1} \| \| T(t-s) \| \int_0^s |f(\tau, y_n(\tau) + \hat{\phi}_\tau) - f(\tau, y(\tau) + \hat{\phi}_\tau)| d\tau ds \rightarrow 0 \\ & \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus F is continuous. This completes the proof that F is completely continuous.

Finally the set $\zeta(F) = \{y \in C_b^0 : y = \lambda Fy, \lambda \in (0, 1)\}$ is bounded, since for every solution y in $\zeta(F)$, the function $x = y + \hat{\phi}$ is a mild solution of (3) for which we have proved that $\|x\|_1 \leq K$ and hence

$$\|y\|_1 \leq K + \|\hat{\phi}\|.$$

Consequently, by Schaefer's theorem, the operator F has a fixed point in C_b^0 . This means that any fixed point of F is a mild solution of (1) on J satisfying $(Fx)(t) = x(t)$. Thus the system (1) is controllable on J . \square

Example. Consider the following partial integrodifferential equation of the form

$$\begin{aligned} & \frac{\partial}{\partial t}(z(t, y) - z_{yy}(t, y)) - z_{yy}(t, y) \\ &= Bu(t) + \int_0^t p(s, z(y, s - r)) ds, \quad 0 < y < \pi, t \in J = [0, b] \end{aligned} \tag{4}$$

with

$$z(0, t) = z(\pi, t) = 0, \quad t > 0, \quad z(t, y) = \phi(t, y), \quad -r \leq t \leq 0$$

where ϕ is continuous and $u \in L^2(J, U)$.

Assume that the following conditions hold with $X = Y = U = L^2(0, \pi)$.

[A₁] The operator $B : U \rightarrow Y$, is a bounded linear operator.

[A₂] The linear operator $W : L^2(J, U) \rightarrow X$, defined by

$$Wu = \int_0^b E^{-1}T(b - s)Bu(s) ds$$

has bounded inverse operator \widetilde{W}^{-1} which takes values in $L^2(J, U)/\ker W$.

[A₃] Further the function $p : J \times C \rightarrow Y$ is continuous in z and strongly measurable in t .

[A₄] Let $f(t, w_t)(y) = p(t, w(t - y))$, $0 < y < \pi$.

Define the operators $A : D(A) \subset X \rightarrow Y$, $E : D(E) \subset X \rightarrow Y$ by $Aw = -w_{yy}$, $Ew = w - w_{yy}$ respectively, where each domain $D(A), D(E)$ is given by $\{w \in X, w, w_y$ are absolutely continuous, $w_{yy} \in X, w(0) = w(\pi) = 0\}$.

With this choice of E, A, B and f , (1) is an abstract formulation of (4) (see [8]).

Then A and E can be written respectively as

$$Aw = \sum_{n=1}^{\infty} n^2(w, w_n)w_n, \quad w \in D(A),$$

$$Ew = \sum_{n=1}^{\infty} (1 + n^2)(w, w_n)w_n, \quad w \in D(E),$$

where $w_n(y) = \sqrt{2} \sin ny$, $n = 1, 2, 3, \dots$, is the orthogonal set of eigenvectors of A . Furthermore for $w \in X$ we have

$$E^{-1}w = \sum_{n=1}^{\infty} \frac{1}{1+n^2}(w, w_n)w_n,$$

$$-AE^{-1}w = \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2}(w, w_n)w_n,$$

$$T(t)w = \sum_{n=1}^{\infty} e^{\frac{-n^2}{1+n^2}t}(w, w_n)w_n,$$

It is easy to see that $-AE^{-1}$ generates a strongly continuous semigroup $T(t)$ on Y and $T(t)$ is compact such that $|T(t)| \leq e^{-t}$ for each $t > 0$.

[A₅] The function p satisfies the following conditions: There exists an integrable function $q : J \rightarrow [0, \infty)$ such that

$$|p(t, w(t - y))| \leq q(t)\Omega_1(\|w\|),$$

where $\Omega_1 : [0, \infty) \rightarrow (0, \infty)$ is continuous and nondecreasing.

Also we have

$$\int_0^b \hat{n}(s) ds < \int_c^\infty \frac{ds}{1+s+\Omega_1(s)},$$

where $c = |E^{-1}|e^{-t}|E\phi(0)|$, and $\hat{n}(t) = \max\{-1, |E^{-1}|e^{-t}N, |E^{-1}|e^{-t} \int_0^t q(s) ds\}$.

Here N depends on E, A, B , and p . Further all the conditions stated in the above theorem are satisfied. Hence the system (4) is controllable on J .

Remark. (See also [16].) Construction of \widetilde{W}^{-1} .

Let $Y = L^2[J, U]/\ker W$.

Since $\ker W$ is closed, Y is a Banach space under the norm

$$\|[u]\|_Y = \inf_{u \in [u]} \|u\|_{L^2[J, U]} = \inf_{W\hat{u}=0} \|u + \hat{u}\|_{L^2[J, U]}$$

where $[u]$ are the equivalence classes of u .

Define $\widetilde{W} : Y \rightarrow X$ by

$$\widetilde{W}[u] = Wu, \quad u \in [u].$$

Now \widetilde{W} is one-to-one and

$$\|\widetilde{W}[u]\|_X \leq \|W\| \|[u]\|_Y.$$

We claim that $V = \text{Range } W$ is a Banach space with the norm

$$\|v\|_V = \|\widetilde{W}^{-1}v\|_Y.$$

This norm is equivalent to the graph norm on $D(\widetilde{W}^{-1}) = \text{Range } W$, \widetilde{W} is bounded and since $D(\widetilde{W}) = Y$ is closed, \widetilde{W}^{-1} is closed and so the above norm makes $\text{Range } W = V$ a Banach space.

Moreover,

$$\begin{aligned} \|Wu\|_V &= \|\widetilde{W}^{-1}Wu\|_Y = \|\widetilde{W}^{-1}\widetilde{W}[u]\| \\ &= \|[u]\| = \inf_{u \in [u]} \|u\| \leq \|u\|, \end{aligned}$$

so

$$W \in \mathcal{L}(L^2[J, U], V).$$

Since $L^2[J, U]$ is reflexive and $\ker W$ is weakly closed, so that the infimum is actually attained. For any $v \in V$, we can therefore choose a control $u \in L^2[J, U]$ such that $u = \widetilde{W}^{-1}v$.

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