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LOG–PERIODOGRAM REGRESSION IN ASYMMETRIC LONG MEMORY[1]

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The long memory property of a time series has long been studied and several estimates of the memory or persistence parameter at zero frequency, where the spectral density function is symmetric, are now available. Perhaps the most popular is the log periodogram regression introduced by Geweke and Porter–Hudak [4]. In this paper we analyse the asymptotic properties of this estimate in the seasonal or cyclical long memory case allowing for asymmetric spectral poles or zeros. Consistency and asymptotic normality are obtained. Finite sample behaviour is evaluated via a Monte Carlo analysis.

1. INTRODUCTION

In the time series literature the behaviour of the spectral density around zero frequency has been of great interest and there exist several studies which measure the impact of a spectral divergence or zero at the origin. For instance several estimators of the slope of the logged spectral density of a long-range dependent process are now available. For a scalar real valued covariance stationary process $\{x_t, t = 0, \pm 1, \pm 2, \dots\}$ assume absolute continuity of the spectral distribution function so that there exist a spectral density $f(\lambda)$ such that the autocovariance of order j is

$$\gamma_j = E\{(x_1 - Ex_1)(x_{1+j} - Ex_1)\} = \int_{-\pi}^{\pi} f(\lambda)\cos(j\lambda)\,\mathrm{d}\lambda.$$

We say that a process has standard long memory when the spectral pole or zero occurs at zero frequency such that

$$f(\lambda) \sim C\lambda^{-2d} \text{ as } \lambda \to 0 \tag{1.1}$$

for C a positive constant and $|d| < 1/2$. The condition $d < 1/2$ is a stationarity condition and $d > -1/2$ is usually required for invertibility.

The long memory can appear at other frequencies different from zero (see for example Gray et al [6]) which is in accord with the concept of seasonal or cyclical

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long memory. Some parametric seasonal and cyclical long memory models have been used by Jonas [13], Carlin and Dempster [3], Hassler [9], Robinson [16] and Ooms [15]. For a review see Arteche and Robinson [2]. All these processes share a semiparametric condition on the spectral density function around the frequency ω where the spectral pole or zero occurs, namely

$$f(\omega+\lambda) \sim C|\lambda|^{-2d} \text{ as } \lambda \to 0, \tag{1.2}$$

for C a positive constant and $|d| < 1/2$. The spectral density function need not be symmetric at a frequency between zero and π as (1.2) and the existing parametric models assume. We can generalize (1.2) allowing for different spectral behaviour at each side of ω. We say that x_t has *asymmetric long memory* if its spectral density satisfies

$$\begin{aligned} f(\omega+\lambda) &\sim C\lambda^{-2d_1} \quad \text{as } \lambda \to 0^+ \\ f(\omega-\lambda) &\sim D\lambda^{-2d_2} \quad \text{as } \lambda \to 0^+ \end{aligned} \tag{1.3}$$

for $C, D \in (0,\infty)$ and $d_1, d_2 \in (-1/2, 1/2)$ and we allow

$$d_1 \neq d_2 \qquad \text{and/or} \qquad C \neq D.$$

For example $d_1 > d_2$ implies that cycles of period just lower than $2\pi/\omega$ are more persistent than cycles of period just larger. Clearly (1.3) nests (1.1) and (1.2) as special cases. In this paper we study the asymptotic properties of the perhaps most popular estimate of the memory parameter, the log periodogram regression of Geweke and Porter-Hudak [4], under seasonal or cyclical long memory. Our analysis allows for asymmetric long-memory as in (1.3). Noting (1.3) the log periodogram estimates of C and d_1 are obtained by applying least squares to

$$\log I(\omega+\lambda_j) = c + d_1(-2\log\lambda_j) + u_j \quad j = 1, \dots, m, \tag{1.4}$$

where at least $\frac{1}{m} + \frac{m}{n} \to 0$ as $n \to \infty$, $\lambda_j = \frac{2\pi j}{n}$ are Fourier frequencies, $I(\lambda) = |W(\lambda)|^2$ is the periodogram and $W(\lambda) = \frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n} x_t e^{it\lambda}$ is the discrete Fourier transform of x_t at frequency λ. The estimates of D and d_2 are similarly obtained using frequencies just before ω. The good properties of these estimates will hold if the u_j are uncorrelated and homoskedastic. However if we take $c = \log C - \eta$ where $\eta = 0.5772\ldots$ is Euler's constant, u_j can be considered as

$$u_j = \log\left(\frac{I(\omega+\lambda_j)}{C\lambda_j^{-2d_1}}\right) + \eta$$

and for $d_1 \neq 0$ they are not asymptotically uncorrelated nor identically distributed if $n \to \infty$ and j is held fixed (see Theorem 1 in Robinson [18] or Hurvich and Beltrao [10]). This invalidates the proofs of the asymptotic properties of this estimate claimed by Geweke and Porter-Hudak [4] and Hassler [7, 8] for the standard long memory case (1.1). In order to obtain the asymptotics Robinson [18] introduces a trimming number l such that the number of frequencies used in the regression (1.4) is from $j = l+1$ to $j = m$. Clearly l has to go to infinity more slowly than m such that $\frac{l}{m} \to 0$. Under Gaussianity and some other mild conditions Robinson [18] shows that when $\omega = 0$ (and because of the symmetry of the spectral density around 0, $d_1 = d_2$) $\sqrt{m}(\hat{d}_1 - d_1) \xrightarrow{d} N\left(0, \frac{\pi^2}{24}\right)$ where $\hat{d}_1$ is the log periodogram estimate of d_1. A gain in asymptotic efficiency is obtained by pooling J adjacent frequencies and regressing

$$Y_k^{(J)} = c^{(J)} + d_1(-2\log\lambda_k) + u_k^{(J)} \qquad k = l+J, l+2J, \ldots, m, \tag{1.5}$$

where $Y_k^{(J)} = \log(\sum_{j=1}^{J} I(\omega + \lambda_{k+j-J}))$ and J is fixed and assumed that $m-l$ is a multiple of J (if this condition does not hold the effects on the asymptotic properties is negligible because J is fixed and $\frac{m}{l} \to \infty$). Note that even if we use the pooling of J adjacent frequencies all the frequencies from $\omega + \lambda_{l+1}$ up to $\omega + \lambda_m$ are used in the estimation so that there is not loss of efficiency. The asymptotic distribution of the least squares estimate of d_1 in (1.5) is $\sqrt{m}(\hat{d}_1^{(J)} - d_1) \xrightarrow{d} N(0, \frac{J\psi'(J)}{4})$ where $\psi'(z) = \frac{d}{dz}\psi(z)$ and ψ is the digamma function, $\psi(z) = \frac{d}{dz}\log\Gamma(z)$ and Γ is the gamma function. The gain in efficiency comes up because $\psi'(1) = \pi^2/6$ and $J\psi'(J)$ decreases in J and goes to 1 as $J \to \infty$.

$u_k^{(J)}$ in (1.5) can be considered,

$$u_k^{(J)} = \log\left\{\sum_{j=1}^{J} \frac{I(\omega + \lambda_{k+j-J})}{C\lambda_{k+j-J}^{-2d_1}}\right\} - \psi(J) \qquad k = l+J, l+2J, \ldots, m. \tag{1.6}$$

If the $u_k^{(J)}$ are uncorrelated and homoskedastic with zero mean, least squares in (1.5) provides the best linear unbiased estimates of $c^{(J)}$ and d_1. The disturbances in (1.5) do not have those properties but in this paper we complement the work of Robinson [18] and show in Section 2 that least squares estimates have the same limiting distributional behaviour as if such properties held. Section 3 shows the finite sample performance of the log-periodogram estimate in asymmetric long memory time series and compares it with the Gaussian semiparametric estimate of Robinson [19] and Arteche [1] by means of a Monte Carlo analysis. Finally Section 4 concludes.

2. ASYMPTOTIC DISTRIBUTION

Let $\{x_{gt}, t = 0, \pm1, \pm2, \ldots\}$ and $\{x_{ht}, t = 0, \pm1, \pm2, \ldots\}$ be two real valued scalar processes with spectral density functions $f_g(\lambda)$ and $f_h(\lambda)$ respectively, integrables over $[-\pi, \pi]$, and cross-spectral density $f_{gh}(\lambda)$. Let us state the following assumptions:

(A.1) For a frequency $\omega \in (0, \pi)$ there exists $\alpha \in (0, 2]$ such that as $\lambda \to 0^+$

$$\begin{aligned} f_s(\omega + \lambda) &= C_s\lambda^{-2d_{1s}}(1 + O(\lambda^\alpha)) \\ f_s(\omega - \lambda) &= D_s\lambda^{-2d_{2s}}(1 + O(\lambda^\alpha)) \end{aligned}$$

for $s = g, h$ and $C_s, D_s \in (0, \infty)$, $d_{1s}, d_{2s} \in (-1/2, 1/2)$.

(A.2) In a neighbourhood $(-\epsilon,0)\cup(0,\epsilon)$ of ω f_{gh} is differentiable and as $\lambda\to 0^+$:

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}f_{gh}(\omega+\lambda)\right| &= O(\lambda^{-1-2d_1})\\ \left|\frac{\mathrm{d}}{\mathrm{d}\lambda}f_{gh}(\omega-\lambda)\right| &= O(\lambda^{-1-2d_2})\end{aligned}$$

where $2d_i=d_{ig}+d_{ih}$ for $i=1,2$.

(A.3) For some $\beta\in(0,2]$:

$$|R_{gh}(\omega+\lambda)-R_{gh}(\omega)|=O(\lambda^{\beta})\qquad \text{as } \lambda\to 0^+$$

where $R_{gh}(\lambda)=\frac{f_{gh}(\lambda)}{\sqrt{f_g(\lambda)f_h(\lambda)}}$ is the coherency between x_{gt} and x_{ht}.

The two main assumptions on the spectral density used in our univariate analysis are (A.1) and (A.2) because in that case $R_{gh}(\lambda)=1$ for all λ, but we introduce (A.3) to allow an easy multivariate extension of the results obtained in the univariate case. These two assumptions hold with $\alpha=\beta=2$ in the cases in Arteche and Robinson [2] (note that $\sin(\omega-\lambda)^{-2d_1}=(\omega-\lambda)^{-2d_1}(1+O((\omega-\lambda)^2))$ as $\lambda\uparrow\omega$ and $\sin(\lambda-\omega)^{-2d_2}=(\lambda-\omega)^{-2d_2}(1+O((\lambda-\omega)^2))$ as $\lambda\downarrow\omega$). Assumption (A.1) could be generalized allowing for different α's at frequencies just after and before ω but this would complicate the notation and the results we obtain hereafter would not be altered.

Let $W_s(\lambda)=\frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n}x_{st}e^{it\lambda}$ be the discrete Fourier transform of x_{st} $(s=g,h)$ where correction for an unknown mean of x_{st} is not necessary because $W_s(\lambda)$ is computed only at frequencies $\lambda_j=\frac{2\pi j}{n}$ for $j=1,\ldots,m$, where m is an integer less than $n/2$. Introduce the scaled discrete Fourier transform $v_s(\omega+\lambda)=\frac{W_s(\omega+\lambda)}{C_s^{\frac{1}{2}}\lambda^{-d_{1s}}}$ and denote $\bar{v}_s(\lambda)$ the complex conjugate of $v_s(\lambda)$.

Theorem 1. Let assumptions (A.1) – (A.3) hold and let $k=k(n)$ and $j=j(n)$ be two sequences of positive integers such that $j>k$ and $\frac{j}{n}\to 0$ as $n\to\infty$. Then as $n\to\infty$:

a) $E[v_g(\omega+\lambda_j)\bar{v}_h(\omega+\lambda_j)]=R_{gh}(\omega)+O\left(\frac{\log j}{j}\lambda_j^{-2(d_i-d_1)}+\left(\frac{j}{n}\right)^{\min(\alpha,\beta)}\right)$

b) $E[v_g(\omega+\lambda_j)v_h(\omega+\lambda_j)]=O\left(\frac{\log j}{j}\lambda_j^{-2(d_i-d_1)}\right)$

c) $E[v_g(\omega+\lambda_j)\bar{v}_h(\omega+\lambda_k)]=O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-(d_{ig}-d_{1g})}\lambda_k^{-(d_{ih}-d_{1h})}\right)$

d) $E[v_g(\omega+\lambda_j)v_h(\omega+\lambda_k)]=O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-(d_{ig}-d_{1g})}\lambda_k^{-(d_{ih}-d_{1h})}\right)$

where $i=1$ if $2d_1\geq 2d_2$ in a) and b) and if $d_{1s}\geq d_{2s}$, $s=g,h$ in c) and d) and $i=2$ otherwise.

Proof. See the Appendix. □

Remark. Even in the case $d_2 > d_1$, b) and d) are $O\left(\frac{\log j}{j}\right)$ and $O\left(\frac{\log j}{k}\right)$ respectively if $1/2 - d_{2s} + d_{1s} \geq 0$ for $s = g, h$ and $1/2 - d_{2s} + 2d_1 \geq 0$ for $s = g$ or h (see the Appendix). These conditions hold if $d_{1s} \geq 0$ for $s = g, h$, irrespective of the values of d_{2s}.

If $d_1 \geq d_2$ these results correspond to those obtained by Robinson [18, Theorem 2] for $\omega = 0$. We focus on frequencies just after ω because this theorem will be useful in the proof of the asymptotic normality of the estimates of C and d_1 in (1.3) which describe spectral behaviour at those frequencies. If we aim to estimate D and d_2 in (1.3) a similar result would be obtained for the scaled discrete Fourier transforms, $v(\omega - \lambda_j)$ and $v(\omega - \lambda_k)$.

Hereafter we will focus on the estimation of C and d_1 in the univariate case $g = h$ and $d_{ig} = d_{ih} = d_i$, $i = 1, 2$, in (A.1) and (A.2). In order to obtain the asymptotic distribution of the least squares estimates in (1.5) two further assumptions are needed:

(A.4) $\{x_t, t = 0, \pm 1, \pm 2, \dots\}$ is a Gaussian process.

(A.5) If $d_1 \geq d_2$,

$$\frac{\sqrt{m}\log m}{l} + \frac{l(\log m)^2}{m} + \frac{m^{1+\frac{1}{2\alpha}}}{n} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

and if $d_1 < d_2$,

$$\frac{\sqrt{m}n^{2(d_2-d_1)}\log m}{l^{1+2(d_2-d_1)}} + \frac{l(\log m)^2}{m} + \frac{m^{1+\frac{1}{2\alpha}}}{n} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

where m and l are the bandwidth and trimming numbers respectively so that we only use those frequencies $\omega + \frac{2\pi j}{n}$ such that $l < j \leq m$.

If $d_1 \geq d_2$ (A.5) is Assumption 6 in Robinson [18] and the proof of the asymptotic normality is basically the same noting Theorem 1. However when $d_1 < d_2$ a stronger condition must be imposed on the bandwidth and trimming numbers. To see the implications of (A.5) take $m \sim n^{\theta}$ and $l \sim n^{\phi}$. In this situation (A.5) entails

$$2(d_2 - d_1) + \frac{1}{2}\theta - \phi(1 + 2(d_2 - d_1)) < 0 \ , \ \phi < \theta \ , \ \theta\left(1 + \frac{1}{2\alpha}\right) < 1. \tag{2.1}$$

The first two conditions imply $\theta > \phi > 4(d_2 - d_1)/(1 + 4(d_2 - d_1))$, and incorporating the last condition in (2.1) indicates that we must have $\alpha > 2(d_2 - d_1)$. Because $|d_1 - d_2| < 1$, (A.5) can be satisfied for any d_1, d_2 if $\alpha = 2$. The larger d_2 with respect to d_1 the larger m and l-we need to get rid of the distorting influence of the periodogram at frequencies just before ω on the estimation of d_1 which describes spectral behaviour after ω (see Theorem 1). This trimming might be avoidable, or at least reducible, if we use a tapered discrete Fourier transform instead of $W(\lambda)$ (see Hurvich and Ray [11] and Velasco [20]).

Define $v(\lambda) = \frac{W(\omega+\lambda)}{C^{\frac{1}{2}}\lambda^{-d_1}} = v_R(\lambda) + iv_I(\lambda)$ where $v_R(\lambda)$ and $v_I(\lambda)$ are the real and imaginary parts of $v(\lambda)$. The $u_k^{(J)}$ in (1.6) can be written

$$u_k^{(J)} = \log[\sum_{j=1}^{J}\{v_R^2(\lambda_{k+j-J}) + v_I^2(\lambda_{k+j-J})\}e^{-\psi(J)}].$$

Introduce the 2×1 vector $\nu(\lambda) = (v_R(\lambda), v_I(\lambda))$. The second order moments of the elements of $\nu(\lambda_j)$ and $\nu(\lambda_k)$ can be deduced from those of $v(\lambda_j)$ and $v(\lambda_k)$ and their complex conjugate in Theorem 1. It indicates that the $\nu(\lambda_j)$ for j increasing adequately slowly with n can be regarded as approximately uncorrelated with zero mean (because $W(\lambda) = \frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n}(x_t - Ex_1)e^{it\lambda})$ and covariance matrix $\frac{1}{2}I_2$ where I_2 is the 2×2 identity matrix. Assumption (A.4) implies that the $\nu(\lambda_j)$ are Gaussian and thus the approximate uncorrelation can be interpreted as approximate independence. Introduce the two dimensional vector

$$V_j \sim NID\left(0, \frac{1}{2}I_2\right) \quad j = l+1, \dots, m \tag{2.2}$$

where $V_j = (V_{1,j}, V_{2,j})$ and the variates

$$w_k^{(J)} = \log[\sum_{j=1}^{J}\{V_{1,k+j-J}^2 + V_{2,k+j-J}^2\}e^{-\psi(J)}] \ , \ k = l+J, l+2J, \dots, m. \tag{2.3}$$

It follows that $\sum_{j=1}^{J}(V_{1,k+j-J}^2 + V_{2,k+j-J}^2) \sim \frac{1}{2}\chi_{2J}^2$ for each k. Thus (see Johnson and Kotz [12, p. 167 and 181]) $E[w_k^{(J)}] = 0$ and $w_k^{(J)}$ has finite moments of all orders and variance $\psi'(J)$ where $\psi'(z) = \frac{\mathrm{d}}{\mathrm{d}z}\psi(z)$. Further, independence of the V_j implies independence of $w_{l+J}^{(J)}, w_{l+2J}^{(J)}, \dots, w_m^{(J)}$. Consequently if the $u_k^{(J)}$ in (1.5) can be replaced by $w_k^{(J)}$ without affecting the limit distribution of the centered and adequately scaled least squares estimates in (1.5) we can apply the Lindeberg–Feller CLT and we will obtain the result stated in the following theorem:

Theorem 2. Let (A.1), (A.2) (with $g = h$), (A.4) and (A.5) hold. Then as $n \to \infty$,

$$\begin{bmatrix} \frac{\sqrt{m}}{\log n}(\hat{c}^{(J)} - c^{(J)}) \\ 2\sqrt{m}(\hat{d}_1^{(J)} - d_1) \end{bmatrix} \xrightarrow{d} N\left[0, J\psi'(J)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right]. \tag{2.4}$$

Proof. For $d_1 \geq d_2$ the assumptions and proof are equal to that in Robinson [18] for $\omega = 0$ noting Theorem 1. When $d_1 < d_2$ (A.5) differs from Assumption 6 in Robinson. Anyhow the steps followed in the proof are quite similar and therefore they will be presented briefly paying attention to the steps where (A.5) takes part. The proof is based on showing that each moment of the variates on the left-hand side of (2.4) converges to the corresponding moments of the normal distribution implied by the right-hand side and then appeal to the Frechet–Shohat "moment convergence theorem" (Loève [14, p. 187]) and the unique determination of the normal distribution by its moments. We use Theorem 1 to show that the moments differ negligibly from those which would arise if instead of $u_k^{(J)}$ we have $w_k^{(J)}$ and then apply the Lindeberg–Feller CLT.

Let $\hat{c}^{(J)}$ and $\hat{d}_1^{(J)}$ be the least squares estimates in (1.5). Then

$$\begin{bmatrix} \hat{c}^{(J)} - c^{(J)} \\ \hat{d}_1^{(J)} - d_1 \end{bmatrix} = (Z'Z)^{-1}Z'U$$

where U is a $\frac{m-l}{J} \times 1$ vector such that $U_j = u_k^{(J)}$ and Z is a $\frac{m-l}{J} \times 2$ matrix with the first column a vector of ones and the components of the second column are $z_j = -2\log\lambda_k$, for $k = l+jJ$, and $j = 1, 2, \dots, (m-l)/J$. From formula (5.2), (5.3) and (5.4) in Robinson [18] and under (A.5):

$$\frac{\sqrt{m}}{\log n}\bar{z} = 2\sqrt{m}(1 + O(\log n)^{-1}) \tag{2.5}$$

where $\bar{z} = \frac{J}{m-l}\sum_k z_k = -2\frac{J}{m-l}\sum_k \log\lambda_k$ and

$$\frac{J}{m-l}|Z'Z| = 4\frac{m}{J} + O(l(\log n)^2). \tag{2.6}$$

Now,

$$(Z'Z)^{-1} = \frac{1}{|Z'Z|}\frac{m-l}{J}\begin{bmatrix} \bar{z} \\ -1 \end{bmatrix}[\bar{z} \ \ -1] + \frac{J}{m-l}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$[\bar{z} \ \ -1]Z'U = [\bar{z} \ \ -1]\begin{bmatrix} \sum_k u_k^{(J)} \\ \sum_k z_k u_k^{(J)} \end{bmatrix} = 2\sum_k \left(\log\lambda_k - \frac{J}{m-l}\sum_k \log\lambda_k\right)u_k^{(J)}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}Z'U = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sum_k u_k^{(J)}$$

Define now the matrix:

$$\Delta = \begin{bmatrix} \frac{\sqrt{m}}{\log n} & 0 \\ 0 & 2\sqrt{m} \end{bmatrix}$$

It follows that

$$\begin{aligned} \Delta(Z'Z)^{-1}Z'U &= J^{\frac{1}{2}}\left(\frac{J}{m+o(m)}\right)^{\frac{1}{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\sum_k(\log\lambda_k - \frac{J}{m-l}\sum_k \log\lambda_k)u_k^{(J)} \\ &+ \frac{J}{m-l}\begin{bmatrix} \frac{\sqrt{m}}{\log n} \\ 0 \end{bmatrix}\sum_k u_k^{(J)}. \end{aligned} \tag{2.7}$$

The proof of the theorem is completed if as $n \to \infty$:

a) $\left(\frac{J}{m}\right)^{\frac{1}{2}} \sum_k \left(\log \lambda_k - \frac{J}{m-l} \sum_k \log \lambda_k) u_k^{(J)} \xrightarrow{d} N(0, \psi'(J)\right)$

b) $\frac{1}{\sqrt{m} \log n} \sum_k u_k^{(J)} \xrightarrow{p} 0$.

In order to prove a) and b) we claim that

$$\left(\frac{J}{m}\right)^{\frac{1}{2}} \sum_k a_k u_k^{(J)} \xrightarrow{d} N(0, \psi'(J)) \tag{2.8}$$

for any triangular array $a_{kn} = a_k$ satisfying as $n \to \infty$,

$$\max_k |a_k| = o(m)\ , \ \sum_k a_k^2 \sim \frac{m}{J} \ \text{ and } \ \sum_k |a_k|^p = O(m) \text{ for all } p \geq 1. \tag{2.9}$$

For b) $a_k = 1$ and for a) $a_k = \log k - \frac{J}{m-l} \sum_k \log k$ and (2.9) holds for both of them (see Robinson [18, p. 1067]). Thus if we can verify our claim (2.8) the proof is completed. If instead of $u_k^{(J)}$ in (2.8) we have $w_k^{(J)}$ a direct application of the Lindeberg–Feller CLT shows that (2.8) holds under (2.9). We show that the moments of $\left(\frac{J}{m}\right)^{\frac{1}{2}} \sum_k a_k u_k^{(J)}$ differ negligibly from those of $\left(\frac{J}{m}\right)^{\frac{1}{2}} \sum_k a_k w_k^{(J)}$ and then we use the Frechet–Shohat "moment convergence theorem" (Loève [14]).

Write $\chi_k = \left(\frac{J}{m}\right)^{\frac{1}{2}} a_k u_k^{(J)}$. Fix an integer N, $E[\sum_k \chi_k]^N$ is a sum of finitely many terms of the form:

$$\sum_{k_1} \cdots \sum_{k_M} E\left(\prod_{i=1}^{M} \chi_{k_i}^{N_{k_i}}\right) \tag{2.10}$$

where $N_{k_1}, N_{k_2}, \ldots, N_{k_M}$ are all positive and sum to N and $1 \leq M \leq N$. Fix such M and $N_{k_1}, \ldots, N_{k_M}$ and introduce the $2J \times 1$ vector $\nu_k^* = (\nu(\lambda_{k+1-J})', \ldots, \nu(\lambda_k)')'$ and the $2JM \times 1$ vector $\nu^* = (\nu_{k_1}^{*\prime}, \ldots, \nu_{k_M}^{*\prime})'$. Under (A.4) ν^* is normally distributed with zero mean and Theorem 1 implies that:

$$\begin{aligned} E[\nu_j^* \nu_k^{*\prime}] &= \frac{1}{2} I_{2J} + O\left(\left(\frac{j}{n}\right)^{\alpha} + \frac{\log j}{j} \lambda_j^{-2(d_2-d_1)}\right) \text{ if } j = k \\ &= O\left(\frac{\log j}{\sqrt{jk}} \lambda_j^{-(d_2-d_1)} \lambda_k^{-(d_2-d_1)}\right) \text{ if } j > k. \end{aligned}$$

as $n \to \infty$. It follows from (A.5) that:

$$\Sigma = E[\nu^* \nu^{*\prime}] = \frac{1}{2} I_{2JM} + O\left(\left(\frac{m}{n}\right)^{\alpha} + \frac{\log m}{l} \lambda_l^{-2(d_2-d_1)}\right) \tag{2.11}$$

$$= \frac{1}{2} I_{2JM} + o\left(m^{-\frac{1}{2}}\right) \tag{2.12}$$

as $n \to \infty$. Thus $\Sigma^{-1} = \Psi$ exists for n large enough. If φ_p is the density function of a p-dimensional standard normal variate (2.10) is:

$$\sum_{k_1} \cdots \sum_{k_M} |\Psi|^{\frac{1}{2}} \int \left(\prod_{i=1}^{M} \chi_{k_i}^{N_{k_i}}\right) \varphi_{2JM}\left(\Psi^{\frac{1}{2}} \nu^*\right) \mathrm{d}\nu^* \tag{2.13}$$

for n sufficiently large. Robinson [18] proved that the difference between (2.13) and

$$\sum_{k_1} \cdots \sum_{k_M} E\left[\prod_{i=1}^{M}\left(\left(\frac{J}{m}\right)^{\frac{1}{2}} a_{k_i} w_{k_i}\right)^{N_{k_i}}\right]$$

is negligible (tends to zero as $n \to \infty$) what completes the proof. □

Remark 1. $\hat{c}^{(J)}$ converges more slowly than $\hat{d}^{(J)}$ and there exists perfect negative correlation in the limiting joint distribution of $\hat{c}^{(J)}$ and $\hat{d}^{(J)}$. This distribution, as we could expect, is equal to that obtained by Robinson [18] for the case $\omega = 0$ and it only differs in a stronger condition on the bandwidth, m, and trimming, l, imposed to get asymptotic uncorrelation of the scaled discrete Fourier transforms of x_t.

Remark 2. C can be estimated from $\hat{c}^{(J)}$, $\hat{C}^{(J)} = \exp(\hat{c}^{(J)} - \psi(J))$ and a simple application of the "delta method" provides the asymptotic distribution of $\hat{C}^{(J)}$:

$$\frac{\sqrt{m}}{\log n}(\hat{C}^{(J)} - C) \xrightarrow{d} N(0, C^2 J\psi'(J)).$$

Remark 3. The multivariate extension is easily done following the steps in Robinson [18] substituting his Assumption 6 for our assumption (A.5) and noting Theorem 1. All his remarks follow in our case of spectral asymmetry around a positive frequency.

Remark 4. Using a similar result to Theorem 1 it can be readily shown the asymptotic independence of the memory estimates on both sides of ω in (1.3) which facilitates the construction of Wald tests of the hypothesis of spectral symmetry.

Remark 5. $\hat{d}_1^{(J)}$ is asymptotically less efficient than the Gaussian semiparametric estimate of Robinson [19] (see also Arteche [1] for the asymmetric long memory case) since $J\psi'(J) \downarrow 1$ as $J \to \infty$ and the asymptotic variance $1/4$ of the Gaussian semiparametric estimate is never achieved by $\hat{d}_1^{(J)}$. The main advantage of $\hat{d}_1^{(J)}$ is its simplicity because the Gaussian semiparametric estimate, unlike $\hat{d}_1^{(J)}$, requires iterative techniques and can not be written in closed form.

3. FINITE SAMPLE BEHAVIOUR

In this section we study via Monte Carlo analysis the effects of asymmetric long memory on the finite sample performance of the log periodogram regression estimates and compare it with the Gaussian semiparametric estimate of Robinson [19] that we denote $\tilde{d}_1$. From Arteche [1] $\tilde{d}_1$ is the argument that minimizes

$$R(d) = \log\left\{\frac{1}{m-l}\sum_{l+1}^{m}\lambda_j^{2d} I(\omega + \lambda_j)\right\} - \frac{2d}{m-l}\sum_{l+1}^{m}\log\lambda_j$$

over the closed set $[-0.499, 0.499]$. In order to generate an asymmetric long memory series we first generated two independent Gaussian processes $\{\epsilon_{1,t}\}$ and $\{\epsilon_{2,t}\}$ with zero means and lag-j autocovariances

$$\begin{aligned} \gamma_{1j} &= \sigma_1^2 \left(\delta_{j0} - \frac{\sin(j\omega)}{\pi j} \right), \\ \gamma_{2j} &= \sigma_2^2 \frac{\sin(j\omega)}{\pi j}, \end{aligned}$$

respectively, where $\delta_{j0} = 1$ if $j = 0$ and 0 otherwise. It follows that $\epsilon_{1,t}$ and $\epsilon_{2,t}$ have spectra

$$f_{\epsilon_1}(\lambda) = \begin{cases} 0, & 0 \le \lambda < \omega, \\ \frac{\sigma_1^2}{2\pi}, & \omega \le \lambda \le \pi, \end{cases} \tag{3.1}$$

and

$$f_{\epsilon_2}(\lambda) = \begin{cases} \frac{\sigma_2^2}{2\pi}, & 0 \le \lambda < \omega, \\ 0, & \omega \le \lambda \le \pi. \end{cases} \tag{3.2}$$

Now define the processes $\{x_{j,t}\}$, $j = 1, 2$, by

$$(1 - 2L\cos\omega + L^2)^{d_j} x_{j,t} = \epsilon_{j,t} \;, \; j = 1, 2, \quad t = 0 \pm 1 \ldots . \tag{3.3}$$

Thus the $\{x_{j,t}\}$ have spectra

$$f_{x_j}(\lambda) = \frac{f_{\epsilon_j}(\lambda)}{|1 - e^{i\lambda}\cos\omega + e^{2i\lambda}|^{2d_j}} \;, \; 0 \le \lambda < \pi \;, \; j = 1, 2,$$

and in view of (3.1) and (3.2) and independence of the $\{\epsilon_{j,t}\}$, $j = 1, 2$, $x_t = x_{1,t} + x_{2,t}$ has spectrum

$$f(\lambda) = f_{x_1}(\lambda) + f_{x_2}(\lambda) = \begin{cases} \frac{\sigma_1^2}{2\pi}|1 - 2e^{i\lambda}\cos\omega + e^{2i\lambda}|^{-2d_1} & \text{if} \quad \omega < \lambda \le \pi, \\ \frac{\sigma_2^2}{2\pi}|1 - 2e^{i\lambda}\cos\omega + e^{2i\lambda}|^{-2d_2} & \text{if} \quad 0 \le \lambda \le \omega, \end{cases} \tag{3.4}$$

which clearly satisfies (1.3). In order to generate realizations of x_t we rewrite (3.3) as

$$\sum_{s=0}^{\infty} C_s^{(d_j)}(\cos\omega) x_{j,t-s} = \epsilon_{j,t} \;, \; j = 1, 2, \;\; t = 0, \pm 1, \ldots , \tag{3.5}$$

(see Gray et al [6]) where the Gegenbauer polynomials $C_s^{(d)}(\eta)$ are of the form

$$C_s^{(d)}(\eta) = \sum_{k=0}^{[s/2]} \frac{(-1)^k \Gamma(s - k - d)(2\eta)^{s-2k}}{\Gamma(k+1)\Gamma(s-2k+1)\Gamma(-d)}.$$

We truncate the sum in (3.5) so that actually our generated $x_{j,t}$ are

$$x_{j,t} = -\sum_{s=1}^{1500} C_s^{(d_j)}(\cos\omega)x_{j,t-s} + \epsilon_{j,t}, \tag{3.6}$$

where $x_{j,t} = 0$ for $t \leq 0$. We prefer an autoregressive truncation over the moving average one of Gray et al [6] because autoregressive coefficients decay faster. The Gegenbauer functions are obtained via the recursion

$$C_s^{(d)}(\eta) = 2\eta\left(\frac{-d+s-1}{s}\right)C_{s-1}^{(d)}(\eta) - \left(\frac{-2d+s-2}{s}\right)C_{s-2}^{(d)}(\eta).$$

(see formula 8.933.1 in Gradshteyn and Ryzhik [5]). We carried out simulations for $\omega = \pi/4, \pi/2, 3\pi/4$ but report results only for $\omega = \pi/2$ because these are fairly typical. We took $d_1, d_2 = \{-0.4, -0.2, 0, 0.2, 0.4\}$ and $\sigma_1^2 = \sigma_2^2 = 1$. Assumptions (A.1) with $\alpha = 2$, (A.2) and (A.4) are then satisfied. We show the results for the log periodogram ($J = 1$), $\hat{d}_1^{(1)}$, and the Gaussian semiparametric, $\tilde{d}_1$, estimates of d_1 with sample size $n = 256$ and three different bandwidths, $m = 16$, 32 and 64. The effect of the trimming is only shown for $m = 64$ and three trimmings are analysed, $l = 2, 4, 8$. The number of replications was 1000 and all the calculations were done using GAUSS-386i version 3.2.8.

The bias and mean square error (MSE) of the untrimmed estimates are shown in Table 1. The bias tends to be positive for negative values of d_1 and when $d_1 < d_2$, and negative for positive d_1 and when $d_1 > d_2$ although a positive bias is more pervasive in $\hat{d}_1^{(1)}$. The bias and MSE are extremely high in the extreme cases $d_2 > d_1$ and both decrease with m because the more frequencies we use the less important the influence of periodogram ordinates close to ω "contaminated" by d_2. When we introduce the trimming we observe in Table 2 that the bias tends to reduce especially when the difference $d_2 - d_1$ is large. The MSE clearly increases with the trimming except in those cases where $d_2 - d_1$ is large and we omit a small number of frequencies ($l = 2$). It also becomes more invariant to the difference $d_2 - d_1$. Comparing $\hat{d}_1^{(1)}$ and $\tilde{d}_1$ it is noticeable the lower MSE of $\tilde{d}_1$ corroborating the higher efficiency in form of a lower asymptotic variance.

4. CONCLUSIONS

All the research done to date on processes with seasonal or cyclical long range dependence implicitly assume symmetry of the spectral poles or zeros and apply methods developed for the standard long memory at zero frequency. In these pages we have provided theoretical support to this practice for the log-periodogram estimate. However if the actual data generating process presents an asymmetric spectral pole or zero the estimation of the memory parameter using frequencies on both sides of the frequency of interest can lead to distorted results.

Table 1. Bias (MSE) of the untrimmed estimates of d_1, $n = 256$.

		$m = 16$				
$d_1 \backslash d_2$		-0.4	-0.2	0	0.2	0.4
-0.4	$\tilde{d}_1^{(1)}$	0.0338 (0.0446)	0.0420 (0.0466)	0.0725 (0.0494)	0.1523 (0.0718)	0.3293 (0.1673)
	$\tilde{d}_1$	0.0408 (0.0192)	0.0477 (0.0206)	0.0705 (0.0257)	0.1395 (0.0468)	0.2999 (0.1298)
-0.2	$\tilde{d}_1^{(1)}$	0.0059 (0.0454)	0.0097 (0.0442)	0.0206 (0.0444)	0.0568 (0.0500)	0.1742 (0.0817)
	$\tilde{d}_1$	-0.0085 (0.0275)	-0.0051 (0.0276)	0.0065 (0.0281)	0.0431 (0.0322)	0.1490 (0.0572)
0	$\tilde{d}_1^{(1)}$	-0.0032 (0.0444)	-0.0029 (0.0448)	0.0004 (0.0446)	0.0129 (0.0463)	0.0736 (0.0509)
	$\tilde{d}_1$	-0.0231 (0.0312)	-0.0217 (0.0312)	-0.0174 (0.0312)	-0.0024 (0.0319)	0.0521 (0.0348)
0.2	$\tilde{d}_1^{(1)}$	-0.0053 (0.0452)	-0.0059 (0.0460)	-0.0040 (0.0451)	0.0012 (0.0452)	0.0238 (0.0446)
	$\tilde{d}_1$	-0.0271 (0.0306)	-0.0266 (0.0306)	-0.0248 (0.0305)	-0.0192 (0.0303)	0.0021 (0.0297)
0.4	$\tilde{d}_1^{(1)}$	0.0013 (0.0450)	0.0012 (0.0448)	0.0016 (0.0447)	0.0032 (0.0451)	0.0105 (0.0452)
	$\tilde{d}_1$	-0.0446 (0.0229)	-0.0444 (0.0228)	-0.0439 (0.0227)	-0.0423 (0.0224)	-0.0368 (0.0214)
		$m = 32$				
$d_1 \backslash d_2$		-0.4	-0.2	0	0.2	0.4
-0.4	$\tilde{d}_1^{(1)}$	0.0244 (0.0187)	0.0297 (0.0196)	0.0484 (0.0209)	0.0956 (0.0291)	0.2050 (0.0663)
	$\tilde{d}_1$	0.0224 (0.0098)	0.0275 (0.0104)	0.0429 (0.0124)	0.0896 (0.0210)	0.2014 (0.0596)
-0.2	$\tilde{d}_1^{(1)}$	0.0023 (0.0189)	0.0049 (0.0185)	0.0121 (0.0187)	0.0341 (0.0205)	0.1057 (0.0318)
	$\tilde{d}_1$	-0.0080 (0.0127)	-0.0055 (0.0127)	0.0021 (0.0127)	0.0259 (0.0139)	0.0961 (0.0244)
0	$\tilde{d}_1^{(1)}$	-0.0083 (0.0187)	-0.0078 (0.0189)	-0.0052 (0.0189)	0.0028 (0.0195)	0.0404 (0.0203)
	$\tilde{d}_1$	-0.0195 (0.0131)	-0.0185 (0.0131)	-0.0156 (0.0130)	-0.0057 (0.0131)	0.0293 (0.0143)
0.2	$\tilde{d}_1^{(1)}$	-0.0141 (0.0192)	-0.0144 (0.0196)	-0.0128 (0.0193)	-0.0095 (0.0192)	0.0055 (0.0186)
	$\tilde{d}_1$	-0.0253 (0.0134)	-0.0248 (0.0134)	-0.0237 (0.0133)	-0.0198 (0.0132)	-0.0053 (0.0130)
0.4	$\tilde{d}_1^{(1)}$	-0.0139 (0.0189)	-0.0138 (0.0188)	-0.0135 (0.0189)	-0.0124 (0.0191)	-0.0071 (0.0190)
	$\tilde{d}_1$	-0.0335 (0.0113)	-0.0332 (0.0113)	-0.0328 (0.0113)	-0.0314 (0.0112)	-0.0268 (0.0109)
		$m = 64$				
$d_1 \backslash d_2$		-0.4	-0.2	0	0.2	0.4
-0.4	$\tilde{d}_1^{(1)}$	0.0634 (0.0131)	0.0664 (0.0136)	0.0772 (0.0151)	0.1046 (0.0207)	0.1699 (0.0405)
	$\tilde{d}_1$	0.0519 (0.0083)	0.0550 (0.0088)	0.0649 (0.0101)	0.0949 (0.0157)	0.1699 (0.0384)
-0.2	$\tilde{d}_1^{(1)}$	0.0301 (0.0099)	0.0316 (0.0099)	0.0358 (0.0102)	0.0488 (0.0117)	0.0913 (0.0185)
	$\tilde{d}_1$	0.0187 (0.0062)	0.0201 (0.0063)	0.0244 (0.0065)	0.0387 (0.0076)	0.0827 (0.0140)
0	$\tilde{d}_1^{(1)}$	0.0038 (0.0088)	0.0040 (0.0089)	0.0055 (0.0089)	0.0105 (0.0093)	0.0334 (0.0103)
	$\tilde{d}_1$	-0.0084 (0.0059)	-0.0078 (0.0059)	-0.0060 (0.0059)	0.0001 (0.0059)	0.0218 (0.0066)
0.2	$\tilde{d}_1^{(1)}$	-0.0193 (0.0092)	-0.0196 (0.0094)	-0.0187 (0.0093)	-0.0164 (0.0091)	-0.0068 (0.0089)
	$\tilde{d}_1$	-0.0317 (0.0068)	-0.0314 (0.0068)	-0-0307 (0.0068)	-0.0281 (0.0066)	-0.0188 (0.0062)
0.4	$\tilde{d}_1^{(1)}$	-0.0385 (0.0101)	-0.0383 (0.0101)	-0.0384 (0.0101)	-0.0376 (0.0101)	-0.0335 (0.0100)
	$\tilde{d}_1$	-0.0522 (0.0083)	-0.0520 (0.0083)	-0.0517 (0.0083)	-0.0507 (0.0082)	-0.0471 (0.0078)

Table 2. Bias (MSE) of the trimmed estimates of d_1, $n = 256$, $m = 64$.

		$l=2$				
$d_1 \backslash d_2$		-0.4	-0.2	0	0.2	0.4
-0.4	$\tilde{d}_1^{(1)}$	0.0799 (0.0168)	0.0805 (0.0169)	0.0826 (0.0175)	0.0914 (0.0189)	0.1196 (0.0241)
	$\tilde{d}_1$	0.0651 (0.0104)	0.0658 (0.0106)	0.0679 (0.0109)	0.0748 (0.0121)	0.0981 (0.0164)
-0.2	$\tilde{d}_1^{(1)}$	0.0461 (0.0133)	0.0461 (0.0135)	0.0475 (0.0137)	0.0528 (0.0142)	0.0696 (0.0159)
	$\tilde{d}_1$	0.0294 (0.0084)	0.0299 (0.0084)	0.0312 (0.0086)	0.0355 (0.0089)	0.0501 (0.0101)
0	$\tilde{d}_1^{(1)}$	0.0147 (0.0120)	0.0145 (0.0121)	0.0152 (0.0123)	0.0180 (0.0124)	0.0289 (0.0127)
	$\tilde{d}_1$	-0.0024 (0.0081)	-0.0021 (0.0081)	-0.0013 (0.0082)	0.0012 (0.0082)	0.0099 (0.0083)
0.2	$\tilde{d}_1^{(1)}$	-0.0148 (0.0127)	-0.0148 (0.0126)	-0.0145 (0.0126)	-0.0128 (0.0126)	-0.0070 (0.0126)
	$\tilde{d}_1$	-0.0324 (0.0097)	-0.0322 (0.0097)	-0.0317 (0.0097)	-0.0303 (0.0096)	-0.0253 (0.0093)
0.4	$\tilde{d}_1^{(1)}$	-0.0415 (0.0147)	-0.0416 (0.0146)	-0.0417 (0.0147)	-0.0414 (0.0147)	-0.0377 (0.0146)
	$\tilde{d}_1$	-0.0608 (0.0124)	-0.0606 (0.0124)	-0.0604 (0.0123)	-0.0597 (0.0123)	-0.0573 (0.0119)
		$l=4$				
$d_1 \backslash d_2$		-0.4	-0.2	0	0.2	0.4
-0.4	$\tilde{d}_1^{(1)}$	0.0951 (0.0227)	0.0952 (0.0229)	0.0960 (0.0233)	0.0991 (0.0241)	0.1111 (0.0260)
	$\tilde{d}_1$	0.0767 (0.0137)	0.0770 (0.0138)	0.0780 (0.0140)	0.0810 (0.0146)	0.0912 (0.0166)
-0.2	$\tilde{d}_1^{(1)}$	0.0576 (0.0178)	0.0572 (0.0181)	0.0577 (0.0182)	0.0606 (0.0186)	0.0680 (0.0195)
	$\tilde{d}_1$	0.0364 (0.0108)	0.0367 (0.0108)	0.0373 (0.0109)	0.0393 (0.0112)	0.0465 (0.0119)
0	$\tilde{d}_1^{(1)}$	0.0214 (0.0157)	0.0210 (0.0157)	0.0214 (0.0159)	0.0232 (0.0163)	0.0287 (0.0167)
	$\tilde{d}_1$	-0.0003 (0.0100)	-0.0002 (0.0101)	0.0002 (0.0101)	0.0015 (0.0102)	0.0062 (0.0103)
0.2	$\tilde{d}_1^{(1)}$	-0.0141 (0.0164)	-0.0141 (0.0164)	-0.0137 (0.0164)	-0.0124 (0.0163)	-0.0088 (0.0163)
	$\tilde{d}_1$	-0.0324 (0.0117)	-0.0322 (0.0118)	-0.0317 (0.0118)	-0.0303 (0.0118)	-0.0253 (0.0116)
0.4	$\tilde{d}_1^{(1)}$	-0.0471 (0.0191)	-0.0472 (0.0190)	-0.0472 (0.0191)	-0.0466 (0.0190)	-0.0439 (0.0187)
	$\tilde{d}_1$	-0.0705 (0.0152)	-0.0704 (0.0152)	-0.0703 (0.0152)	-0.0699 (0.0152)	-0.0682 (0.0149)
		$l=8$				
$d_1 \backslash d_2$		-0.4	-0.2	0	0.2	0.4
-0.4	$\tilde{d}_1^{(1)}$	0.1152 (0.0371)	0.1153 (0.0371)	0.1158 (0.0374)	0.1171 (0.0373)	0.1227 (0.0381)
	$\tilde{d}_1$	0.0949 (0.0216)	0.0951 (0.0216)	0.0956 (0.0217)	0.0969 (0.0220)	0.1011 (0.0231)
-0.2	$\tilde{d}_1^{(1)}$	0.0687 (0.0300)	0.0686 (0.0301)	0.0689 (0.0299)	0.0705 (0.0301)	0.0745 (0.0307)
	$\tilde{d}_1$	0.0443 (0.0174)	0.0445 (0.0175)	0.0449 (0.0175)	0.0460 (0.0177)	0.0496 (0.0182)
0	$\tilde{d}_1^{(1)}$	0.0235 (0.0266)	0.0230 (0.0266)	0.0236 (0.0270)	0.0249 (0.0277)	0.0281 (0.0278)
	$\tilde{d}_1$	-0.0017 (0.0166)	-0.0016 (0.0166)	-0.0012 (0.0167)	-0.0004 (0.0167)	0.0022 (0.0168)
0.2	$\tilde{d}_1^{(1)}$	-0.0216 (0.0279)	-0.0214 (0.0279)	-0.0210 (0.0281)	-0.0198 (0.0283)	-0.0179 (0.0282)
	$\tilde{d}_1$	-0.0467 (0.0195)	-0.0466 (0.0196)	-0.0463 (0.0196)	-0.0458 (0.0196)	-0.0440 (0.0195)
0.4	$\tilde{d}_1^{(1)}$	-0.0642 (0.0332)	-0.0643 (0.0329)	-0.0645 (0.0332)	-0.0641 (0.0332)	-0.0620 (0.0331)
	$\tilde{d}_1$	-0.0941 (0.0252)	-0.0941 (0.0252)	-0.0939 (0.0252)	-0.0937 (0.0252)	-0.0927 (0.0250)

APPENDIX: PROOF OF THEOREM 1

a) The proof of a) is in two parts. First we show that

$$E[W_g(\omega+\lambda_j)\bar{W}_h(\omega+\lambda_j)] = f_{gh}(\omega+\lambda_j) + O\left(\frac{\log j}{j}\lambda_j^{-2d_i}\right) \tag{4.1}$$

and then that

$$f_{gh}(\omega+\lambda_j) - C_g^{\frac{1}{2}}C_h^{\frac{1}{2}}\lambda_j^{-2d_1}R_{gh}(\omega) = O\left(\lambda_j^{\min(\alpha,\beta)-2d_1}\right). \tag{4.2}$$

In order to prove (4.1) write the left hand side of the equality as

$$\frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{s=1}^{n}\gamma_{gh}(t-s)e^{is(\omega+\lambda_j)}e^{-it(\omega+\lambda_j)} = \int_{-\pi}^{\pi} f_{gh}(\lambda)\,K(\omega+\lambda_j-\lambda)\,\mathrm{d}\lambda$$

where $\gamma_{gh}(t-s)$ is the covariance between x_{gt} and x_{hs} and $K(\lambda)=\frac{1}{2\pi n}\sum_t\sum_s e^{i(t-s)\lambda}$ is Fejer's kernel. Since $\int_{-\pi}^{\pi}K(\omega+\lambda_j-\lambda)\,\mathrm{d}\lambda=1$ we have to study the order of magnitude of

$$\int_{-\pi}^{\pi}\{f_{gh}(\lambda)-f_{gh}(\omega+\lambda_j)\}K(\omega+\lambda_j-\lambda)\,\mathrm{d}\lambda. \tag{4.3}$$

Due to assumptions (A.1) and (A.2) we can pick ϵ so small that for some $C_\epsilon<\infty$:

$$\begin{aligned}|f_{gh}(\omega+\lambda)| &\le f_g^{\frac{1}{2}}(\omega+\lambda)f_h^{\frac{1}{2}}(\omega+\lambda)\le C_\epsilon|\lambda|^{-2d_1}\\ \left|\tfrac{\mathrm{d}}{\mathrm{d}\lambda}f_{gh}(\omega+\lambda)\right| &\le C_\epsilon|\lambda|^{-1-2d_1}\end{aligned}$$

for $\lambda\in(-\epsilon,0)\cup(0,\epsilon)$ and $2d_1=d_{g1}+d_{h1}$. Because $\omega\in(0,\pi)$ and $\frac{j}{n}\to 0$ as $n\to\infty$ we can choose ϵ such that for n large enough:

$$\epsilon>2\lambda_j,\quad 2\omega+\lambda_j-\epsilon>0,\quad 2\omega+\lambda_j+\epsilon<2\pi, \tag{4.4}$$

what will be necessary for subsequent analysis. For such ϵ we have that the integral over $\Omega=[-\pi,\omega-\epsilon]\cup[\omega+\epsilon,\pi]$ is bounded in absolute value by

$$\begin{aligned}&\left\{\max_{\lambda\in\Omega}K(\omega+\lambda_j-\lambda)\right\}\int_{-\pi}^{\pi}\{|f_{gh}(\lambda)|+|f_{gh}(\omega+\lambda_j)|\}\,\mathrm{d}\lambda\\ =\;& O(n^{-1}(1+\lambda_j^{-2d_1}))=O\left(\frac{1}{j}\lambda_j^{-2d_1}\right).\end{aligned}$$

The first equality comes from the following facts that will be useful in subsequent analysis:

$$\begin{aligned}K(\lambda) &= \frac{|D(\lambda)|^2}{2\pi n} && (4.5)\\ |D(\lambda)| &= \left|\sum_t e^{it\lambda}\right|\le\frac{1}{|\sin\frac{\lambda}{2}|}\quad\text{if } 0<\lambda<2\pi && (4.6)\\ |K(\lambda)| &= O(n^{-1}\lambda^{-2})\quad\text{for } 0<|\lambda|<\pi && (4.7)\\ |f_{gh}(\lambda)| &\le f_g^{\frac{1}{2}}(\lambda)f_h^{\frac{1}{2}}(\lambda)\text{ and }\int_{-\pi}^{\pi}f_i(\lambda)\,\mathrm{d}\lambda=\mathrm{var}(x_{it})<\infty\,,\; i=g,h && (4.8)\end{aligned}$$

and the second one because:

$$n^{-1} = O\left(\left(\frac{j}{n}\right)^{1+2d_1}\frac{1}{j}\lambda_j^{-2d_1}\right) \text{ and } 1+2d_1 > 0. \tag{4.9}$$

Now decompose the remainder of the integral :

$$\int_{\omega-\epsilon}^{\omega+\epsilon} = \int_{\omega-\epsilon}^{\omega-\frac{\lambda_j}{2}} + \int_{\omega-\frac{\lambda_j}{2}}^{\omega+\frac{\lambda_j}{2}} + \int_{\omega+\frac{\lambda_j}{2}}^{\omega+2\lambda_j} + \int_{\omega+2\lambda_j}^{\omega+\epsilon}. \tag{4.10}$$

The first integral in (4.10) is bounded in modulus by

$$\begin{aligned}
& \left\{\max_{\omega-\epsilon\le\lambda\le\omega-\frac{\lambda_j}{2}} |f_{gh}(\lambda)|\right\}\int_{\omega-\epsilon}^{\omega-\frac{\lambda_j}{2}} K(\omega+\lambda_j-\lambda)\,\mathrm{d}\lambda \\
& \qquad + |f_{gh}(\omega+\lambda_j)|\int_{\omega-\epsilon}^{\omega-\frac{\lambda_j}{2}} K(\omega+\lambda_j-\lambda)\,\mathrm{d}\lambda \\
= & \left\{\max_{\frac{\lambda_j}{2}\le\lambda\le\epsilon} |f_{gh}(\omega-\lambda)|\right\}\int_{\frac{\lambda_j}{2}}^{\epsilon} K(\lambda_j+\lambda)\,\mathrm{d}\lambda \\
& \qquad + |f_{gh}(\omega+\lambda_j)|\int_{\frac{\lambda_j}{2}}^{\epsilon} K(\lambda_j+\lambda)\,\mathrm{d}\lambda \\
\le & \left\{\max_{\frac{\lambda_j}{2}\le\lambda\le\epsilon+\lambda_j} \frac{|f_{gh}(\omega-\lambda)|}{\lambda^{\frac{1}{2}-d_{2h}}}\right\}\int_{\frac{\lambda_j}{2}}^{\epsilon+\lambda_j} K(\lambda)\lambda^{\frac{1}{2}-d_{2h}}\,\mathrm{d}\lambda \\
& \qquad + |f_{gh}(\omega+\lambda_j)|\int_{\frac{\lambda_j}{2}}^{\epsilon+\lambda_j} K(\lambda)\,\mathrm{d}\lambda \\
= & O\left(n^{-1}\lambda_j^{-1-2d_2} + n^{-1}\lambda_j^{-1-2d_1}\right) = O\left(j^{-1}\lambda_j^{-2d_i}\right)
\end{aligned}$$

because of (4.7). Similarly the last integral in (4.10) is $O(j^{-1}\lambda_j^{-2d_1})$. Now using the mean value theorem,

$$\begin{aligned}
\left|\int_{\omega+\frac{\lambda_j}{2}}^{\omega+2\lambda_j}\right| &= \left|\int_{\frac{\lambda_j}{2}}^{2\lambda_j}\{f_{gh}(\omega+\lambda)-f_{gh}(\omega+\lambda_j)\}K(\lambda_j-\lambda)\,\mathrm{d}\lambda\right| \\
&\le \left\{\max_{\frac{\lambda_j}{2}\le\lambda\le2\lambda_j}|f'_{gh}(\omega+\lambda)|\right\}\int_{\frac{\lambda_j}{2}}^{2\lambda_j}|\lambda-\lambda_j|K(\lambda_j-\lambda)\,\mathrm{d}\lambda \\
&= O(n^{-1}\lambda_j^{-1-2d_1}\int_{\frac{\lambda_j}{2}}^{2\lambda_j}|D(\lambda_j-\lambda)|\,\mathrm{d}\lambda) = O\left(\frac{\log j}{j}\lambda_j^{-2d_1}\right)
\end{aligned}$$

because of (4.5) and

$$|D(\lambda)| \le 2|\lambda|^{-1}, \quad 0<|\lambda|<\pi \tag{4.11}$$

$$\int_{-C\lambda_j}^{C\lambda_j}|D(\lambda)|\,\mathrm{d}\lambda = O(\log j) \quad \text{for } C<\infty. \tag{4.12}$$

For the property (4.11) on Dirichlet's kernel $D(\lambda)$ see Zygmund [21, pp. 49–51] and (4.12) is Lemma 5 of Robinson [17]. To complete the proof of (4.1) $\int_{\omega-\frac{\lambda_j}{2}}^{\omega+\frac{\lambda_j}{2}}$ in (4.10) is bounded in absolute value by

$$\begin{aligned}
&\left\{\max_{-\frac{\lambda_j}{2}\le\lambda\le\frac{\lambda_j}{2}} K(\lambda_j-\lambda)\right\}\int_{-\frac{\lambda_j}{2}}^{\frac{\lambda_j}{2}}\{|f_{gh}(\omega+\lambda)|+|f_{gh}(\omega+\lambda_j)|\}\,\mathrm{d}\lambda\\
=\;& O\left(n^{-1}\lambda_j^{-2}\lambda_j^{1-2d_i}\right)=O\left(j^{-1}\lambda_j^{-2d_i}\right).
\end{aligned}$$

Proceeding as in the proof of Theorem 2 in Robinson [18] we have that (4.2) is $O(\lambda_j^{\alpha-2d_1})+O(\lambda_j^{\beta-2d_1})$ under assumption (A.1) and (A.3) what completes the proof of a).

b) To prove b) write $E[W_g(\omega+\lambda_j)W_h(\omega+\lambda_j)]$ as

$$\begin{aligned}
&\frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{s=1}^{n}\gamma_{gh}(t-s)e^{it(\omega+\lambda_j)}e^{is(\omega+\lambda_j)}\\
=\;&\int_{-\pi}^{\pi}\frac{1}{2\pi n}f_{gh}(\lambda)D(\omega+\lambda_j+\lambda)D(\omega+\lambda_j-\lambda)\,\mathrm{d}\lambda.
\end{aligned}$$

The integral over $\Omega=[-\pi,-\omega-\epsilon]\cup[-\omega+\epsilon,\omega-\epsilon]\cup[\omega+\epsilon,\pi]$ is bounded in absolute value by

$$\begin{aligned}
&\frac{1}{2\pi n}\left\{\max_{\lambda\in\Omega}|D(\omega+\lambda_j+\lambda)||D(\omega+\lambda_j-\lambda)|\right\}\int_{-\pi}^{\pi}|f_{gh}(\lambda)|\,\mathrm{d}\lambda\\
=\;& O(n^{-1})=O\left(j^{-1}\lambda_j^{-2d_1}\right)
\end{aligned}$$

using (4.4), (4.6) and (4.9). Now $\left|\int_{-\omega-\epsilon}^{-\omega-2\lambda_j}\right|$ is bounded by

$$\begin{aligned}
&\frac{1}{2\pi n}\left\{\max_{2\lambda_j\le\lambda\le\epsilon}|f_{gh}(-\lambda-\omega)|\right\}\int_{2\lambda_j}^{\epsilon}|D(\lambda_j-\lambda)||D(2\omega+\lambda_j+\lambda)|\,\mathrm{d}\lambda\\
\le\;&\frac{1}{2\pi n}\left\{\max_{\lambda_j\le\lambda\le\epsilon}\frac{|f_{gh}(-\omega-\lambda)|}{\lambda^{\frac12-d_{1h}}}\right\}\left\{\max_{\lambda_j\le\lambda\le\epsilon}\frac{1}{|\sin\frac{2\omega+\lambda_j+\lambda}{2}|}\right\}\int_{\lambda_j}^{\epsilon}2\lambda^{-\frac12-d_{1h}}\,\mathrm{d}\lambda\\
=\;& O\left(n^{-1}\lambda_j^{-\frac12-d_{1g}}\right)=O\left(\frac{1}{j}\lambda_j^{-2d_1}\left(\frac{j}{n}\right)^{\frac12+d_{1h}}\right)=O\left(j^{-1}\lambda_j^{-2d_1}\right)
\end{aligned}$$

the first inequality because of (4.4), (4.6) and (4.11) and the last equality because $1/2+d_{1g}>0$. Similarly

$$\left|\int_{\omega+2\lambda_j}^{\omega+\epsilon}\right|=O\left(j^{-1}\lambda_j^{-2d_1}\right).$$

Proceeding in the same manner the integral over $[-\omega + \frac{\lambda_j}{2}, -\omega + \epsilon]$ is bounded in modulus by

$$\frac{1}{2\pi n}\left\{\max_{\frac{\lambda_j}{2}\le\lambda\le\epsilon+\lambda_j}\frac{|f_{gh}(-\omega+\lambda)|}{\lambda^{\frac{1}{2}-d_{2g}}}\right\}\left\{\max_{\frac{\lambda_j}{2}\le\lambda\le\epsilon}\frac{1}{|\sin\frac{2\omega+\lambda_j-\lambda}{2}|}\right\}2\int_{\frac{\lambda_j}{2}}^{\epsilon+\lambda_j}\lambda^{-\frac{1}{2}-d_{2g}}\,\mathrm{d}\lambda$$
$$= O\left(n^{-1}\lambda_j^{-\frac{1}{2}-d_{2h}}\right) = O\left(j^{-1}\lambda_j^{-2d_2}\right)$$

and under the conditions in the remark to Theorem 1 this is $O(j^{-1}\lambda_j^{-2d_1}\lambda_j^{\frac{1}{2}-d_{2h}+2d_1})$ $= O(j^{-1}\lambda_j^{-2d_1})$. Similarly the integral over $[\omega-\epsilon, \omega-\lambda_j/2]$ is $O(j^{-1}\lambda_j^{-2d_2})$ and $O(j^{-1}\lambda_j^{-2d_1})$ under the conditions in the remark to Theorem 1. The rest of integrals are

$$\left|\int_{\omega+\frac{\lambda_j}{2}}^{\omega+2\lambda_j} + \int_{-\omega-2\lambda_j}^{-\omega-\frac{\lambda_j}{2}}\right|$$
$$= O\left(\frac{1}{2\pi n}\left\{\max_{\frac{\lambda_j}{2}\le\lambda\le 2\lambda_j}|f_{gh}(\omega+\lambda)|\right\}\left\{\max_{\frac{\lambda_j}{2}\le\lambda\le 2\lambda_j}\frac{1}{|\sin\frac{2\omega+\lambda_j+\lambda}{2}|}\right\}\int_{-\lambda_j}^{\lambda_j}|D(\lambda)|\,\mathrm{d}\lambda\right)$$
$$= O\left(n^{-1}\lambda_j^{-2d_1}\log j\right) = O\left(\frac{\log j}{j}\lambda_j^{-2d_1}\right)$$

the second inequality because of (4.4) and (4.6) and the first equality due to (4.12). To complete the proof of b) the integral over $[\omega - \frac{\lambda_j}{2}, \omega + \frac{\lambda_j}{2}]$ is bounded in absolute value by

$$\frac{1}{2\pi n}\left\{\max_{-\frac{\lambda_j}{2}\le\lambda\le\frac{\lambda_j}{2}}|D(2\omega+\lambda_j+\lambda)|\right\}\left\{\max_{-\frac{\lambda_j}{2}\le\lambda\le\frac{\lambda_j}{2}}|D(\lambda_j-\lambda)|\right\}\int_{-\frac{\lambda_j}{2}}^{\frac{\lambda_j}{2}}|f_{gh}(\omega+\lambda)|\,\mathrm{d}\lambda$$
$$= O\left(n^{-1}\lambda_j^{-1}(\lambda_j^{1-2d_1}+\lambda_j^{1-2d_2})\right) = O\left(j^{-1}\lambda_j^{-2d_i}\right)$$

the first equality because of (4.4), (4.8) and (4.11), and under the conditions in the remark to Theorem 1 this is $O(j^{-1}\lambda_j^{-2d_1}(\lambda_j + \lambda_j^{1-2d_2+2d_1})) = O(j^{-1}\lambda_j^{-2d_1})$. The analysis for the integral over $[-\omega \pm \frac{\lambda_j}{2}]$ is similar and this concludes the proof of b).

c) To prove c) write $E[W_g(\omega+\lambda_j)\bar{W}_h(\omega+\lambda_k)]$ as

$$\frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{s=1}^{n}\gamma_{gh}(s-t)e^{it(\omega+\lambda_j)}e^{-is(\omega+\lambda_k)} = \int_{-\pi}^{\pi} f_{gh}(\lambda)E_{jk}(\lambda)\,\mathrm{d}\lambda$$

where $E_{jk}(\lambda) = \frac{1}{2\pi n}D(\omega+\lambda_j-\lambda)D(\lambda-\omega-\lambda_k)$. Since $\int_{-\pi}^{\pi} e^{i(s-t)\lambda}\,\mathrm{d}\lambda = 0$ for $s \neq t$ and 2π for $s = t$, and $\sum_{t=1}^{n} e^{it(\lambda_j-\lambda_k)} = 0$ for $0 < j-k < n/2$ then

$$\int_{-\pi}^{\pi} E_{jk}(\lambda)\,\mathrm{d}\lambda = 0. \tag{4.13}$$

Thus we can expand the integral as

$$\left\{\int_{-\pi}^{\omega+\frac{\lambda_k}{2}} + \int_{\omega+2\lambda_j}^{\pi}\right\}\{f_{gh}(\lambda) - f_{gh}(\omega+\lambda_j)\}E_{jk}(\lambda)\,\mathrm{d}\lambda \tag{4.14}$$

$$+ \int_{\omega+\frac{\lambda_k+\lambda_j}{2}}^{\omega+2\lambda_j}\{f_{gh}(\lambda) - f_{gh}(\omega+\lambda_j)\}E_{jk}(\lambda)\,\mathrm{d}\lambda \tag{4.15}$$

$$+ \int_{\omega+\frac{\lambda_k}{2}}^{\omega+\frac{\lambda_k+\lambda_j}{2}}\{f_{gh}(\lambda) - f_{gh}(\omega+\lambda_k)\}E_{jk}(\lambda)\,\mathrm{d}\lambda \tag{4.16}$$

$$- \{f_{gh}(\omega+\lambda_j) - f_{gh}(\omega+\lambda_k)\}\int_{\omega+\frac{\lambda_k}{2}}^{\omega+\frac{\lambda_k+\lambda_j}{2}} E_{jk}(\lambda)\,\mathrm{d}\lambda. \tag{4.17}$$

Now (4.15) is bounded by

$$\frac{1}{\pi n}\left\{\max_{(\lambda_k+\lambda_j)/2\le\lambda\le 2\lambda_j}|f'_{gh}(\omega+\lambda)|\right\}\int_{\frac{\lambda_j+\lambda_k}{2}}^{2\lambda_j}|D(\lambda-\lambda_k)|\,\mathrm{d}\lambda$$
$$= O(n^{-1}\lambda_j^{-1-2d_1}\log j) = O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\left(\frac{k}{j}\right)^{\frac{1}{2}+d_{1h}}\right) = O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right)$$

for $j > k$. The absolute value of (4.16) is bounded by

$$\frac{1}{\pi n}\left\{\max_{\lambda_k/2\le\lambda\le(\lambda_j+\lambda_k)/2}|f'_{gh}(\omega+\lambda)|\right\}\int_{\frac{\lambda_k}{2}}^{\frac{\lambda_k+\lambda_j}{2}}|D(\lambda_j-\lambda)|\,\mathrm{d}\lambda$$
$$= O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right)$$

if $j/2 \le k < j$, and when $k < j/2$ (4.16) is bounded by

$$\frac{1}{\pi n}\left\{\max_{\lambda_k\le 2\lambda\le(\lambda_j+\lambda_k)}|f_{gh}(\omega+\lambda)| + |f_{gh}(\omega+\lambda_j)|\right\}(\lambda_j-\lambda_k)^{-1}\int_{\frac{\lambda_k}{2}}^{\frac{\lambda_k+\lambda_j}{2}}|D(\lambda_j-\lambda)|\,\mathrm{d}\lambda$$
$$= O\left((\lambda_j^{-2d_1}+\lambda_k^{-2d_1})(j-k)^{-1}\log j\right)$$
$$= O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right).$$

Now (4.17) is bounded in modulus by

$$\frac{1}{2\pi n}\left\{\max_{\lambda_k\le\lambda\le\lambda_j}|f'_{gh}(\omega+\lambda)|\right\}\int_{\frac{\lambda_k}{2}}^{\frac{\lambda_j+\lambda_k}{2}}|D(\lambda-\lambda_k)|\,\mathrm{d}\lambda$$
$$= O\left(n^{-1}\lambda_k^{-1-d_1}\log j\right) = O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right)$$

if $k \geq j/2$, and when $k < j/2$ (4.17) is bounded by

$$\begin{aligned} & \frac{1}{\pi n}\{|f_{gh}(\omega+\lambda_j)| + |f_{gh}(\omega+\lambda_k)|\}(\lambda_j-\lambda_k)^{-1}\int_{\frac{\lambda_k}{2}}^{\frac{\lambda_j+\lambda_k}{2}} |D(\lambda-\lambda_k)|\,\mathrm{d}\lambda \\ = \; & O((\lambda_j^{-2d_1}+\lambda_k^{-2d_1})(j-k)^{-1}\log j) = O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right) \end{aligned}$$

as in the evaluation of (4.16). Proceeding similarly and as in Robinson [18] it is straightforward to show that (4.14) is $O\left((jk)^{-0.5}\lambda_j^{-d_{ig}}\lambda_k^{-d_{ih}}\right)$ which completes the proof of c).

d) Write $E[W_g(\omega+\lambda_j)W_h(\omega+\lambda_k)]$ as

$$\begin{aligned} & \sum_{t=1}^{n}\sum_{s=1}^{n}\frac{1}{2\pi n}\gamma_{gh}(t-s)e^{it(\omega+\lambda_j)}e^{is(\omega+\lambda_k)} \\ = \; & \int_{-\pi}^{\pi}\frac{1}{2\pi n}f_{gh}(\lambda)D(\omega+\lambda_j+\lambda)D(\omega+\lambda_k-\lambda)\,\mathrm{d}\lambda. \end{aligned}$$

Proceeding as in b),

$$\left|\int_{-\pi}^{-\omega-\epsilon}+\int_{-\omega+\epsilon}^{\omega-\epsilon}+\int_{\omega+\epsilon}^{\pi}\right| = O(n^{-1}) = O\left(\frac{1}{\sqrt{kj}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right).$$

Now the integral over $[-\omega-\epsilon, -\omega-2\lambda_j]$ is bounded in absolute value by

$$\begin{aligned} & \frac{1}{2\pi n}\left\{\max_{2\lambda_j\leq\lambda\leq\epsilon}|f_{gh}(-\omega-\lambda)|\right\}\int_{2\lambda_j}^{\epsilon}|D(\lambda_j-\lambda)||D(2\omega+\lambda_k+\lambda)|\,\mathrm{d}\lambda \\ \leq \; & \frac{1}{\pi n}\left\{\max_{2\lambda_j\leq\lambda\leq\epsilon}\frac{1}{|\sin\frac{2\omega+\lambda_k+\lambda}{2}|}\right\}\left\{\max_{\lambda_j\leq\lambda\leq\epsilon}\frac{|f_{gh}(-\omega-\lambda)|}{\lambda^{\frac{1}{2}-d_{1h}}}\right\}\int_{\lambda_j}^{\epsilon}\lambda^{-\frac{1}{2}-d_{1h}}\,\mathrm{d}\lambda \\ = \; & O\left(n^{-1}\lambda_j^{-\frac{1}{2}-d_{1g}}\right) = O\left(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right) \end{aligned}$$

and similarly the integral over $[\omega+2\lambda_k, \omega+\epsilon]$ is $O((jk)^{-0.5}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}})$. The integral over $[-\omega-2\lambda_j, -\omega-\frac{\lambda_j}{2}]$ is bounded in absolute value by

$$\begin{aligned} & \frac{1}{2\pi n}\left\{\max_{\frac{\lambda_j}{2}\leq\lambda\leq2\lambda_j}|f_{gh}(-\omega-\lambda)|\right\}\int_{\frac{\lambda_j}{2}}^{2\lambda_j}|D(\lambda_j-\lambda)||D(2\omega+\lambda_k+\lambda)|\,\mathrm{d}\lambda \\ = \; & O\left(n^{-1}\lambda_j^{-2d_1}\log j\right) = O\left(\frac{\log j}{\sqrt{kj}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right) \end{aligned}$$

and similarly the integral over $[\omega+\lambda_k/2, \omega+2\lambda_k]$ is $O((kj)^{-0.5}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}})$. The integral over $[-\omega \pm \frac{\lambda_j}{2}]$ is bounded in absolute value by

$$\frac{1}{2\pi n}\left\{\max_{-\frac{\lambda_j}{2}\le\lambda\le\frac{\lambda_j}{2}} |D(\lambda_j+\lambda)|\,|D(2\omega+\lambda_k-\lambda)|\right\}\int_{-\frac{\lambda_j}{2}}^{\frac{\lambda_j}{2}} |f_{gh}(-\omega+\lambda)|\,\mathrm{d}\lambda$$
$$= O\left(n^{-1}\lambda_j^{-1}\lambda_j^{1-2d_i}\right) = O\left(\frac{1}{\sqrt{kj}}\lambda_j^{-d_{ig}}\lambda_k^{-d_{ih}}\right)$$

and under the conditions stated in the remark this is

$$O\left(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\left[\lambda_j^{\frac{1}{2}-d_{1h}}\lambda_k^{\frac{1}{2}+d_{1h}}+\lambda_j^{\frac{1}{2}-2d_2+d_{1g}}\lambda_k^{\frac{1}{2}+d_{1h}}\right]\right) = O\left(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right).$$

We obtain similarly the same result for the integral over $[\omega \pm \frac{\lambda_k}{2}]$. The integral over $[-\omega+\frac{\lambda_j}{2}, -\omega+\epsilon]$ is bounded in absolute value by

$$\frac{1}{\pi n}\left\{\max_{\frac{\lambda_j}{2}\le\lambda\le\epsilon} |D(2\omega+\lambda_k-\lambda)|\right\}\left\{\max_{\lambda_j\le\lambda\le\lambda_j+\epsilon}\frac{|f_{gh}(\lambda-\omega)|}{\lambda^{\frac{1}{2}-d_{2h}}}\right\}\int_{\lambda_j}^{\epsilon+\lambda_j}\lambda^{-\frac{1}{2}-d_{2h}}\,\mathrm{d}\lambda$$
$$= O\left(n^{-1}\lambda_j^{-\frac{1}{2}-d_{2g}}\right) = O\left(\frac{1}{\sqrt{kj}}\lambda_j^{-d_{2g}}\lambda_k^{-d_{2h}}\right)$$

and under the conditions in the remark

$$O\left(\frac{1}{\sqrt{kj}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\lambda_j^{-d_{2g}+d_{1g}}\lambda_k^{\frac{1}{2}+d_{1h}}\right) = O\left(\frac{1}{\sqrt{kj}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right).$$

We obtain similarly the same upper bound for the integral over $[\omega-\epsilon, \omega-\frac{\lambda_j}{2}]$. Finally the absolute value of the integral over $[\omega-\frac{\lambda_j}{2}, \omega-\frac{\lambda_k}{2}]$ is bounded by

$$\frac{1}{2\pi n}\left\{\max_{\frac{\lambda_k}{2}\le\lambda\le\frac{\lambda_j}{2}} |D(2\omega+\lambda_j-\lambda)|\right\}\left\{\max_{\lambda_k\le\lambda\le\lambda_j}\frac{|f_{gh}(\omega-\lambda)|}{\lambda^{\frac{1}{2}-d_{2g}}}\right\}\int_{\lambda_k}^{\lambda_j}\lambda^{-\frac{1}{2}-d_{2g}}\,\mathrm{d}\lambda$$
$$= O\left(n^{-1}\lambda_k^{-\frac{1}{2}-d_{2h}}\lambda_j^{\frac{1}{2}-d_{2g}}\right) = O\left(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{2g}}\lambda_k^{-d_{2h}}\right)$$

and under the conditions in the remark this is

$$O\left(k^{-1}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\lambda_j^{\frac{1}{2}-d_{2g}+d_{1g}}\lambda_k^{\frac{1}{2}-d_{2h}+d_{1h}}\right) = O\left(\frac{1}{k}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right)$$

and the proof is completed. □

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REFERENCES

[1] J. Arteche: Gaussian semiparametric estimation in seasonal/cyclical long memory time series. Kybernetika *36* (2000), 279–310.

[2] J. Arteche and P. M. Robinson: Seasonal and cyclical long memory. In: Asymptotics, Nonparametrics and Time Series (S. Ghosh ed.), Marcel Dekker, New York 1999, pp. 115–148.

[3] J. B. Carlin and A. P. Dempster: Sensitivity analysis of seasonal adjustments: empirical case studies. J. Amer. Statist. Assoc. *84* (1989), 6–20.

[4] J. Geweke and S. Porter–Hudak: The estimation and application of long-memory time series models. J. Time Ser. Anal. *4* (1983), 221–238.

[5] J. S. Gradshteyn and I. W. Ryzhik: Table of Integrals, Series and Products. Academic Press, Florida 1980.

[6] H. L. Gray, N. F. Zhang and W. A. Woodward: On generalized fractional processes. J. Time Ser. Anal. *10* (1989), 233–257.

[7] U. Hassler: Regression of spectral estimators with fractionally integrated time series. J. Time Ser. Anal. *14* (1993), 369–380.

[8] U. Hassler: Corrigendum: The periodogram regression. J. Time Ser. Anal. *14* (1993), 549.

[9] U. Hassler: (Mis)specification of long-memory in seasonal time series. J. Time Ser. Anal. *15* (1994), 19–30.

[10] C. M. Hurvich and K. I. Beltrao: Asymptotics for the low frequency ordinates of the periodogram of long memory time series. J. Time Ser. Anal. *14* (1993), 455–472.

[11] C. M. Hurvich and B. K. Ray: Estimation of the memory parameter for nonstationary or noninvertible fractionally integrated processes. J. Time Ser. Anal. *16* (1995), 17–42.

[12] N. L. Johnson and S. Kotz: Continuous Univariate Distributions – I. Wiley, New York 1970.

[13] A. J. Jonas: Persistent Memory Random Processes. Ph.D. Thesis. Department of Statistics, Harvard University, 1983.

[14] M. Loève: Probability Theory I. Springer, Berlin 1977.

[15] M. Ooms: Flexible seasonal long-memory and economic time series. Preprint, Erasmus University Rotterdam 1995.

[16] P. M. Robinson: Efficient tests of non-stationary hypothesis. J. Amer. Statist. Assoc. *89* (1994), 1420–1437.

[17] P. M. Robinson: Rates of convergence and optimal spectral bandwith for long-range dependence. Probab. Theory Related Fields *99* (1994), 443–473.

[18] P. M. Robinson: Log-periodogram regression of time series with long-range dependence. Ann. Statist. *23* (1995), 1048–1072.

[19] P. M. Robinson Gaussian semiparametric estimation of long-range dependence. Ann. Statist. *23* (1995), 1630–1661.

[20] C. Velasco: Non-stationary log-periodogram regression. J. Econometrics *91* (1999), 325–371.

[21] A. Zygmund: Trigonometric Series. Cambridge University Press, Cambridge U.K. 1977.

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