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CONTINUOUS-TIME PERIODIC SYSTEMS IN H_2 AND H_∞
Part II: State Feedback Problems

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This paper deals with some state-feedback H_2/H_∞ control problems for continuous time periodic systems. The derivation of the theoretical results underlying such problems has been presented in the first part of the paper. Here, the parametrization and optimization problems in H_2, H_∞ and mixed H_2/H_∞ are introduced and solved.

1. STATE-FEEDBACK PROBLEMS

In the paper, we consider the periodic system described by

$$\dot{x} = A(t)\,x + B_1(t)\,w + B_2(t)\,u \tag{1}$$

$$z_1 = C_1(t)\,x + D_1(t)\,u \tag{2}$$

$$z_2 = C_2(t)\,x + D_2(t)\,u \tag{3}$$

where

$$A(\cdot),\ B_1(\cdot),\ B_2(\cdot),\ C_1(\cdot),\ D_1(\cdot),\ C_2(\cdot),\ D_2(\cdot)$$

are T-periodic piecewise continuous function matrices. The signal $u(t)$ is the control input, $w(t)$ is an input disturbance and $z_1(t), z_2(t)$ are controlled output variables.

The paper benefits from the development of the theory of H_∞ control for shift-invariant systems. In this regard, specially important is the celebrated paper [1], the additional parametrization results given in [2], the parametrization of memoryless state-feedback controllers via LMI and the mixed H_2/H_∞ control results in [3].
The application of the above theory to periodic systems is far from being trivial, since it requires, besides non standard results on the differential periodic Riccati equations, an appropriate extension of the mathematical machinery concerning system theoretical aspects such as spectral properties, Youla–Kučera parametrization, small gain results, H_2 and H_∞ norm, BIBO stability of feedback systems and so on so forth. All these arguments are collected and studied in the first part of the paper [4].

The present paper deals with the following state-feedback problems

(1) Find a necessary and sufficient condition for the existence of a T-periodic causal controller fed by (x, w) and yielding u such that the H_∞ norm (defined in [4]) from w to z_1 is less than a prescribed positive attenuation level γ.

(2) Parametrize all stabilizing T-periodic controllers fed by (x, w) and yielding u such that the H_∞ norm from w to z_1 is less than a prescribed positive attenuation level γ.

(3) Parametrize all memoryless T-periodic controllers ($u(t) = K(t)\,x(t)$) such that the H_∞ norm from w to z_1 is less than (or equal to) a prescribed positive attenuation level γ.

(4) Find a memoryless T-periodic controller ($u(t) = K(t)\,x(t)$) which minimizes the H_2 norm (defined in [4]) between w and z_2.

(5) Find a memoryless T-periodic controller of the kind $u(t) = K(t)\,x(t)$ which minimizes the H_2 norm between w and z_2 while keeping the H_∞ norm from w to z_1 less than or equal to a prescribed positive attenuation level γ.

Section 2 contains two theorems concerning the parametrization of stabilizing memoryless state-feedback controllers (Theorem 2.1) and the optimal H_2 control problem (Theorem 2.2). The H_∞ full-information control problem (Theorem 3.1), the parametrization of H_∞ performant controllers (Theorem 3.2) and the parametrization of memoryless state-feedback controllers via differential LMI (Theorem 4.1) are the object of Section 3. Finally, in Section 4, the so-called convex and post-optimization procedures for the mixed H_2/H_∞ control problem (Theorems 5.1, 5.2) are described.

2. H_2 CONTROL

Here we limit our attention to the control law

$$u(t) = K(t)\,x(t)$$

where $K(\cdot)$ is a T-periodic control gain to be designed.
The aim of this section is twofold. First we want to characterize the set of all stabilizing periodic gains, and, in addition, tackle the so-called H_2 control problem. Let us begin with the first problem. As is well known a T-periodic gain $K(\cdot)$ is stabilizing if and only if there exists a periodic positive definite solution $P(\cdot)$ of the differential Lyapunov inequality

$$\dot{P}(t) - (A(t) + B_2(t)\,K(t))\,P(t) - P(t)\,(A(t) + B_2(t)\,K(t))' > 0.$$

Now consider the new differential inequality in two unknows $P(\cdot)$ and $W(\cdot)$

$$-\dot{P}(t) + A(t)\,P(t) + B_2(t)\,W(t)' + P(t)\,A(t)' + B_2(t)\,P(t) < 0. \tag{4}$$

The following theorem, whose proof is trivial, characterizes the set of all stabilizing gains in terms of a suitable convex set.

Theorem 2.1. The set of all T-periodic pairs $(P(\cdot), W(\cdot))$, with $P(\cdot)$ positive definite, satisfying (4) is convex. Any stabilizing T-periodic gain $K(\cdot)$ can be written as $K(t) = W(t)'P(t)^{-1}$.

Let us now move the the H_2 problem for system (1), (3) and denote by $T(z_2, w, K)$ the input-output operator between the input w and the output z_2 once the control law is applied to the system. The H_2 optimal control problem consists in finding a periodic $K(\cdot)$ in such a way that

(i) the closed-loop system is stable

(ii) the H_2 norm of $T(z_2, w, K)$ is minimized.

The proof of the theorem below is based on the periodic Lyapunov equation (eq. (13) in [4]) by exploiting the monotonicity property of periodic Riccati equations and the theorem of the existence of the unique stabilizing periodic solution, see [5].

Theorem 2.2. Assume that $(A(\cdot), B_2(\cdot))$ is stabilizable, $D_2(t)$ is full column rank for each t and that the periodic system (A, B_2, C_2, D_2) does not have invariant zeros in the unit circle. Then the optimal solution of the H_2 problem is

$$K^o(t) = -(D_2'(t)\, D_2(t))^{-1}(B_2'\bar{Q}_2(t) + D_2'(t)\, C_2(t))$$

where $\bar{Q}_2(t)$ is the unique stabilizing periodic positive semidefinite solution of the periodic Riccati equation

$$\begin{aligned} -\dot{Q}(t) = {} & A'(t)Q(t) + Q(t)\, A(t) + C_2'(t)\, C_2(t) \\ & -(B_2'Q(t) + D_2'(t)\, C_2(t))'(D_2'(t)\, D_2(t))^{-1}\, (B_2'\bar{Q}(t) + D_2'(t)\, C_2(t)). \end{aligned}$$

Notice that, under the given assumption, the Riccati equation may well admit more than one positive semidefinite periodic solutions. The uniqueness of such a solution is ensured if the stronger assumption is made that the system $(A(\cdot), B_2(\cdot), C_2(\cdot), D_2(\cdot))$ does not have invariant zeros outside the open unit disk.

3. H_∞ CONTROL

Let us now be given a positive scalar γ. The so-called full-information control problem for system (1), (2) consists in finding

(i) a necessary and sufficient condition for the existence of a T-periodic causal controller $K : (x, w) \longrightarrow u$, such that the closed-loop system is stable and the H_∞ norm of $T(z_1, w, K)$ is less than γ;

(ii) the family to which all such controllers belong.

The derivation of the main result will be made under the following assumptions, which are standard in the literature of H_∞ control.

A1) $C_1(t)'D_1(t) = 0$, $\forall t$.

A2) $D_1(t)'D_1(t) = I$, $\forall t$.

A3) The pair $(A(\cdot), B_2(\cdot))$ is stabilizable.

A4) The pair $(A(\cdot), C_2(\cdot))$ is detectable.

Consider first the H_2 periodic Riccati equation:

$$-\dot{\Pi}(t) = A(t)'\Pi(t) + \Pi(t)A(t) - \Pi(t)B_2(t)\,B_2(t)'\Pi(t) + C_1(t)'C_1(t) \qquad (5)$$

and let $\Pi(\cdot)$ be the unique stabilizing semidefinite T-periodic solution (whose existence is ensured by assumptions A3, A4). Now, let a new variable v be defined as follows:

$$v(t) = u(t) + B_2(t)'\Pi(t)\,x(t).$$

The output z_1 can be rewritten as

$$z_1(t) = \mathcal{R}_{\rm op}(\tau)\,B_1(t)\,w(t) + \mathcal{U}_{\rm op}(\tau)v(t) \qquad (6)$$

where $\mathcal{R}$ and $\mathcal{U}$ are the following T-periodic stable systems:

$$\mathcal{R} = (A - B_2B_2'\Pi, I, C_1 - D_1B_2'\Pi, 0) \qquad (7)$$

$$\mathcal{U} = (A - B_2B_2'\Pi, B_2, C_1 - D_1B_2'\Pi, D_1). \qquad (8)$$

Define also a T-periodic matrix $D_+(\cdot)$ such that $D_+(t)'D_1(t) = 0$, $\forall t$ and $D_+(t)\,D_+(t)' = I$, $\forall t$, and consider the T-periodic stable system

$$\mathcal{U}_+ = (A - B_2B_2'\Pi, -\Pi^{-1}C_1'D_+, C_1 - D_1B_2'\Pi, D_+).$$

Hereafter, the symbol $\mathcal{A}_{\rm op}(\tau)$ denotes the input-output operator associated with a periodic system $\mathcal{A}$, with zero initial condition at $t = \tau$. The following lemma is now in order.

Lemma 3.1. Systems $\mathcal{U}$ and $\mathcal{U}_+$ are inner at $t = \tau$ and $\mathcal{U}_{\rm op}(\tau)^{\sim}\mathcal{U}_{+{\rm op}}(\tau) = 0$.

Proof. Let x_1, x_2, x_3 and x_4 be the state variables of $\mathcal{U}$, $\mathcal{U}^{\sim}$, $\mathcal{U}_+$ and $\mathcal{U}_+^{\sim}$, respectively. Simple computations show that

(i) $\mathcal{U}_{1{\rm op}}(\tau) = \mathcal{U}_{\rm op}(\tau)^{\sim}\mathcal{U}_{\rm op}(\tau)$ is such that $\mathcal{U}_1 = (-A' - \Pi B_2B_2', 0, B_2', I)$ (with state variable $x_5 = x_2 - \Pi x_1$). Hence $\mathcal{U}_{1{\rm op}}(\tau) = I$.

(ii) $\mathcal{U}_{2{\rm op}}(\tau) = \mathcal{U}_{+{\rm op}}(\tau)^{\sim}\mathcal{U}_{\rm op}(\tau)$ is such that $\mathcal{U}_2 = (A - B_2B_2'\Pi, 0, -D_+'C_1, 0)$ (with state variable $x_6 = \Pi^{-1}x_4 - x_1$). Hence $\mathcal{U}_{2{\rm op}}(\tau) = 0$.

(iii) $\mathcal{U}_{3{\rm op}}(\tau) = \mathcal{U}_{+{\rm op}}(\tau)^{\sim}\mathcal{U}_{+{\rm op}}(\tau)$ is such that $\mathcal{U}_3 = (A + \Pi^{-1}C_1'C_1, 0, D_+'C_1, I)$ (with state variable $x_7 = x_3 - \Pi^{-1}x_4$). Hence $\mathcal{U}_{3{\rm op}}(\tau) = I$. □

To state the main theorem, we need to introduce the class Ξ of all T-periodic causal controllers $\mathcal{K}$ fed by x and w and yielding u such that the resulting closed-loop system is stable and satisfies $||T(z_1, w, \mathcal{K})||_\infty < \gamma$.

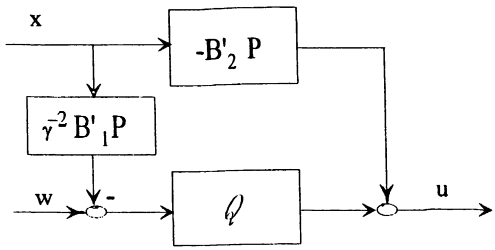

Fig. 2. Parametrization of the set $\Xi_R \subseteq \Xi$.

Theorem 3.1. Consider system $\mathcal{P}$ given by equations (1), (2), let $\gamma > 0$ be a given scalar, and let the assumptions A1 – A4 be fulfilled. Then,

(a) there exists a (full-information) T-periodic controller $\mathcal{K}$ such that the closed-loop system is stable and $||T(z_1, w, \mathcal{K})||_\infty < \gamma$ if and only if there exists a T-periodic positive semidefinite solution of

$$\begin{aligned} -\dot{P}(t) &= A(t)'P(t) + P(t)\,A(t) \\ &+ P(t)\,(B_1(t)\,B_1(t)'\gamma^{-2} - B_2(t)\,B_2(t)')\,P(t) + C_1(t)'C_1(t) \quad (9) \end{aligned}$$

i. e. such that $A(\cdot) + (B_1(\cdot)\,B_1(\cdot)'\gamma^{-2} - B_2(\cdot)\,B_2(\cdot)')\,P(\cdot)$ is stable.

(b) Suppose that there exists a periodic stabilizing solution of (9) and denote by Ξ_R the set of all controllers such that (see Figure 2)

$$u(t) = -B_2(t)'P(t)\,x(t) + \mathcal{Q}_{\rm op}(\tau)\,(w(t) - \gamma^{-2}B_1(t)'P(t)\,x(t)) \qquad (10)$$

where $\mathcal{Q}$ is a stable T-periodic system with $||\mathcal{Q}||_\infty < \gamma$. Then,

(b_1) The set Ξ_R is included in the set Ξ, i. e. $\Xi_R \subseteq \Xi$.

(b_2) If $\mathcal{K} \in \Xi$ then there exists a controller $\hat{\mathcal{K}} \in \Xi_R$ yielding the same input-output operator, i. e. $T(z_1, w, \mathcal{K}) = T(z_1, w, \hat{\mathcal{K}})$.

Proof. We suppose without loss of generality that $\gamma = 1$. This can be done by suitably scaling matrices B_2 and C_1 as follows: $B_2 \to B_2\gamma$, $C_1 \to C_1\gamma^{-1}$. Consequently, $\mathcal{K} \to \mathcal{K}\gamma^{-1}$.

Point (b) and sufficiency of (a). In this part we first suppose that there exists the stabilizing T-periodic positive semidefinite solution $P(\cdot)$ of the Riccati equation (9). We shall prove (i) that the controllers of the proposed family Ξ_R (see equation (10)) do the job and (ii) that any controller in Ξ has a counterpart in the family Ξ_R yielding the same closed-loop operator.

i) Let $r(t) = w(t) - B_1(t)'P(t)\,x(t)$ and $\mathcal{Q}$ be as follows

$$\dot{\sigma} = \bar{F}(t)\sigma + \bar{G}(t)\,r \tag{11}$$
$$q = \bar{H}(t)\sigma + \bar{E}(t)\,r. \tag{12}$$

Then the overall T-periodic system can be written as:

$$\dot{p} = \tilde{A}p(t) + \tilde{B}(t)\,w \tag{13}$$
$$z = \tilde{C}(t)\,p + \tilde{D}(t)\,w \tag{14}$$

where

$$\tilde{A}(t) = \begin{bmatrix} A(t) - B_2(t)\,B_2(t)'P(t) - B_2(t)\bar{E}(t)\,B_1(t)' & B_2(t)\bar{H}(t) \\ -\bar{G}(t)\,B_1(t)' & \bar{F}(t) \end{bmatrix}$$

$$\tilde{B}(t) = \begin{bmatrix} B_1(t) + B_2(t)\bar{E}(t) \\ \bar{G}(t) \end{bmatrix}$$

$$\tilde{C}(t) = \begin{bmatrix} C_1(t) - D_1(t)\,B_2(t)'P(t) - D_1(t)\bar{E}(t)\,B_1(t)'P(t) & D_1(t)\bar{H}(t)) \end{bmatrix}$$

$$\tilde{D}(t) = D_1(t)\bar{E}(t).$$

We know from the assumption that system $\mathcal{Q} = (\bar{F}, \bar{G}, \bar{H}, \bar{E})$ is well-posed and stable with norm less that 1. In particular, $\bar{F}(\cdot)$ is stable and $I - \bar{E}(\cdot)'\bar{E}(\cdot) > 0$. In view of Lemma 2.6 of [4] there exists a T-periodic positive semidefinite stabilizing solution $\Gamma(\cdot)$ of:

$$-\dot{\Gamma}(t) = \bar{F}(t)'\Gamma(t) + \Gamma(t)\bar{F}(t) + \bar{H}(t)'\bar{H}(t)$$
$$+ (\Gamma(t)\bar{G}(t) + \bar{H}(t)'\bar{E}(t))\,(I - \bar{E}(t)'\bar{E}(t))^{-1}(\bar{G}(t)'\Gamma(t) + \bar{E}(t)'\bar{H}(t)). \tag{15}$$

It is just a matter of cumbersome matrix manipulation to check that

$$S(t) = \begin{bmatrix} P(t) & 0 \\ 0 & \Gamma(t) \end{bmatrix} \tag{16}$$

is a T-periodic positive semidefinite stabilizing solution of the periodic Riccati equation:

$$-\dot{S}(t) = \tilde{A}(t)'S(t) + S(t)\tilde{A}(t) + \tilde{C}(t)'\tilde{C}(t)$$
$$+ (S(t)\tilde{B}(t) + \tilde{C}(t)'\tilde{D}(t))(I - \tilde{D}(t)'\tilde{D}(t))^{-1}(\tilde{D}(t)'\tilde{C}(t) + \tilde{B}(t)'S(t)). \tag{17}$$

Since $I - \tilde{D}(t)'\tilde{D}(t) = I - \bar{E}(t)'\bar{E}(t) > 0$, $\forall\, t$, from Lemma 2.6 of [4] it follows that $\tilde{A}(\cdot)$ is stable and the overall system has H_∞ norm less than one.

(ii) Suppose that there exists a controller $\mathcal{K} \in \Xi$, i.e. a controller such than system (13), (14) is stable with H_∞ norm less that one and let

$$\hat{\mathcal{K}} = (A_K, [B_K\ \bar{B}_K], C_K, [D_K\ \bar{D}_K]).$$

By letting $r(t) = w(t) - B_1(t)'P(t)\,x(t)$ and $q(t) = u(t) + B_2(t)'P(t)\,x(t)$ one can form in correspondence system $\mathcal{Q} = (\bar{F}, \bar{G}, \bar{H}, \bar{E})$ in (11), (12) with

$$\bar{F}(t) = \begin{bmatrix} A_K(t) & B_K(t)\,B_1(t)'P(t) + \bar{B}_K(t) \\ B_2(t)\,C_K(t) & A(t) + B_1(t)\,B_1(t)'P(t) + B_2(t)\,(D_K(t)\,B_1(t)'P(t) + \bar{D}_K(t)) \end{bmatrix}, \tag{18}$$

$$\bar{G}(t) = \begin{bmatrix} B_K(t) \\ B_2(t)\,D_K(t) + B_1(t) \end{bmatrix} \tag{19}$$

$$\bar{H}(t) = \begin{bmatrix} C_K(t) & D_K(t)\,B_1(t)'P(t) + \bar{D}_K(t) + B_2(t)'P(t) \end{bmatrix} \tag{20}$$

$$\bar{E}(t) = D_K(t). \tag{21}$$

This leads to a controller $\hat{\mathcal{K}}$ which belongs to the family Ξ_R given by equation (10). We have to prove that system $\mathcal{Q} = (\bar{F}, \bar{G}, \bar{H}, \bar{E})$ in equation (11), (12), (18), (19) is stable with H_∞ norm less than one. From Lemma 2.6 of [4] we know that there exists a periodic positive semidefinite stabilizing solution of equation (17). Consider now equation (9) and the quadratic form $v(x,t) = x'P(t)\,x$. It is easy to see that $\dot{v}(x(t),t) = -r(t)'r(t) + w(t)'w(t) + q(t)'q(t) - z(t)'z(t)$. Recalling that $x(\tau) = x(\infty) = 0$ and $||T(z_1, w, \mathcal{K})\,w||_2 < ||w||_2$, $\forall\, w \in L_2[\tau\, \infty)$, $w \neq 0$, it follows that

$$||q||_2 < ||r||_2,\ \forall\, r \in L_2[0\, \infty),\ r \neq 0. \tag{22}$$

Since there exists the stabilizing solution $S(\cdot)$ of (17), the Hamiltonian matrix

$$H_c(t) = \begin{bmatrix} A_c(t) & B_c(t)\,B_c(t)' \\ -C_c(t)'C_c(t) & -A_c(t)' \end{bmatrix}$$

where

$$\begin{aligned} A_c(t) &= \tilde{A}(t) + \tilde{B}(t)\,(I - \tilde{D}(t)'\tilde{D}(t))^{-1}\tilde{D}(t)'\tilde{C}(t) \\ B_c(t)\,B_c(t)' &= \tilde{B}(t)I - \tilde{D}(t)'\tilde{D}(t))^{-1}\tilde{B}(t) \\ C_c(t)'C_c(t) &= \tilde{C}(t)'I - \tilde{D}(t)\tilde{D}(t)')^{-1}\tilde{C}(t) \end{aligned}$$

does not have unit-modulus characteristic multipliers. Easy but cumbersome matrix manipulations show that

$$Z(t)H_c(t)Z(t)^{-1} + \dot{Z}(t)Z(t)^{-1} = \begin{bmatrix} F_c(t) & G_c(t)\,G_c(t)' & ? & 0 \\ -H_c(t)'H_c(t) & -F_c(t)' & ? & 0 \\ 0 & 0 & ? & 0 \\ ? & ? & ? & ? \end{bmatrix}$$

where

$$
\begin{aligned}
F_c(t) &= \bar{F}(t) + \bar{G}(t)\,(I - \bar{E}(t)'\bar{E}(t))^{-1}\bar{E}(t)'\bar{H}(t) \\
G_c(t)\,G_c(t)' &= \bar{G}(t)\,(I - \bar{E}(t)'\bar{E}(t))^{-1}\bar{G}(t) \\
H_c(t)'H_c(t) &= \bar{H}(t)'(I - \bar{E}(t)\bar{E}(t)')^{-1}\bar{H}(t) \\
Z(t) &= \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ -P(t) & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}
\end{aligned}
$$

and ? denotes the blocks which are irrelevant. It follows that

$$
\tilde{H}_c(t) = \begin{bmatrix} F_c(t) & G_c(t)\,G_c(t)' \\ -H_c(t)'H_c(t) & -F_c(t)' \end{bmatrix}
$$

does not have unit-modulud characterstic multiplies. This fact, together with (22) implies by Lemma 2.5 of [4] that there exists a stabilizing T-periodic solution of eq. (15). Since the stabilizing solution of (17) is unique and the matrix (16) is a stabilizing solution, we conclude that $S(\cdot)$ as in (16) is that solution. Since $S(t) \geq 0$, $\forall t$, it follows that $\Gamma(t) \geq 0$, $\forall t$, as well. Lemma 2.6 of [4] concludes the proof of (ii).

Necessity of (a)

Suppose that there exists a stabilizing controller $\mathcal{K}$ yielding $\|T(z_1, w, \mathcal{K})\|_\infty < 1$. We first show that there is no loss of generality in assuming the observability of the periodic pair $(A(\cdot), C_1(\cdot))$. This is done by resorting to the Kalman canonical decomposition of a periodic system into its observable and unobservable parts, see e. g. [6]. Now, let the system state $x = \begin{bmatrix} x_1' & x_2' \end{bmatrix}'$ be decomposed accordingly to this partition, so that

$$
A(t) = \begin{bmatrix} A_{11}(t) & 0 \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \quad B(t) = \begin{bmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{bmatrix}, \quad C_1(t) = \begin{bmatrix} C_{11}(t) & 0 \end{bmatrix}.
$$

The existing T-periodic controller

$$
\mathcal{K} = (\bar{A}, [\bar{B}_1\ \bar{B}_2], \bar{C}, [\bar{D}_1\ \bar{D}_2])
$$

fed by (x, w) and yielding u can be partitioned accordingly, i. e. $\bar{B}_1(t) = [\bar{B}_{11}(t)\bar{B}_{12}(t)]$ and $\bar{D}_1(t) = [\bar{D}_{11}(t)\ \bar{D}_{12}(t)]$. Hence, by grouping state x_2 with the controller state, a controller $\hat{\mathcal{K}}$ for the observable part of the system is obtained as $\hat{\mathcal{K}} = (\hat{A}, [\hat{B}_1\ \hat{B}_2], \hat{C}, [\hat{D}_1\ \hat{D}_2])$ with

$$
\hat{A}(t) = \begin{bmatrix} A_{22}(t) + B_{22}(t)\bar{D}_{12}(t) & B_{22}(t) \\ \bar{B}_{12}(t) & \bar{A}(t) \end{bmatrix}
$$

$$\hat{B}(t) = \left[\begin{array}{cc} \hat{B}_1(t) & \hat{B}_2(t) \end{array} \right] = \left[\begin{array}{cc} A_{21}(t) + B_{22}(t)\bar{D}_{11}(t) & B_{21}(t) + B_{22}(t)\bar{D}_2(t) \\ \bar{B}_{11}(t) & \bar{B}_2(t) \end{array} \right]$$

$$\hat{C}(t) = \left[\begin{array}{cc} \bar{D}_{12}(t) & \bar{C}(t) \end{array} \right], \; \hat{D}(t) = \left[\begin{array}{cc} \bar{D}_{11}(t) & \bar{D}_2(t) \end{array} \right].$$

Such a controller is stabilizing and such that $\|T(z_1, w, \hat{\mathcal{K}})\|_\infty < 1$. If we now show that there exists the T-periodic stabilizing positive semidefinite solution $P_1(\cdot)$ of the periodic Riccati equation associated with the observable part $(A_{11}, [B_{11}\, B_{12}], C_{11})$ of system (1), (2), then the T-periodic stabilizing and positive semidefinite solution of (9) takes on the form

$$P(t) = \left[\begin{array}{cc} P_1(t) & 0 \\ 0 & 0 \end{array} \right].$$

Hence, we can assume that $(A(\cdot), C(\cdot))$ is observable from the very beginning, and proceed to prove the existence of the periodic stabilizing positive semidefinite solutions of the periodic Riccati equation (9).

Lemma 3.1 together with expression (6) entail that

$$\left[\begin{array}{c} \mathcal{U}_{\rm op}(\tau)^\sim \\ \mathcal{U}_{+\rm op}(\tau)^\sim \end{array} \right] z = \left[\begin{array}{c} \mathcal{U}_{\rm op}^\sim(\tau)\, \mathcal{R}_{\rm op}(\tau)\, B_1(\cdot)\, w + v \\ \mathcal{U}_{+\rm op}(\tau)^\sim \mathcal{R}_{\rm op}(\tau)\, B_1(\cdot)\, w \end{array} \right]$$

with

$$\|z\|_2 = \left\| \left[\begin{array}{c} \mathcal{U}_{\rm op}(\tau)^\sim \\ \mathcal{U}_{+\rm op}(\tau)^\sim \end{array} \right] z \right\|_2 . \tag{23}$$

On the other hand, the assumption that the H_∞ norm is less than one, definition $v(t) = u(t) + B_2(t)'P(t)\, x(t)$, the fact that $x \in L_2[\tau\ \infty)$ and $x(\tau) = 0$ imply

$$1 > \sup_{w \in L_2[\tau + \infty),\, \|w\|_2 = 1} \; \inf_{v \in L_2[\tau + \infty} \|z\|_2 .$$

Recall now the operator $\mathcal{M}_{\rm op}$ previously introduced and associate it with system $\mathcal{G} = [\ \mathcal{G}_1 \quad \mathcal{G}_2\ ]$, where

$$\left[\begin{array}{cc} \mathcal{G}_{1\rm op}(\tau) & \mathcal{G}_{2\rm op}(\tau) \end{array} \right] = B_1(\cdot)'\mathcal{R}_{\rm op}(\tau) \left[\begin{array}{cc} \mathcal{U}_{\rm op}(\tau) & \mathcal{U}_{+\rm op}(\tau) \end{array} \right]$$

By taking into account equations (23) and (27) in [4] , and defining

$$\hat{v}(t) = v(t) + \Omega_+ \mathcal{U}_{\rm op}(\tau)^\sim \mathcal{R}_{\rm op}(\tau)\, B_1(t)\, w(t)$$

we obtain

$$1 > \sup_{w \in L_2[\tau\ \infty),\, \|w\|_2 = 1} \; \inf_{\hat{v} \in L_2[\tau + \infty)} \left\| \mathcal{M}_{\rm op}(\tau)^\sim w + \left[\begin{array}{c} \hat{v} \\ 0 \end{array} \right] \right\|_2 .$$

But $\mathcal{M}_{\rm op}^\sim w$ and $[\hat{v}' \quad 0]'$ are orthogonal so that

$$1 > \sup_{w \in L_2[\tau\ +\infty),\, \|w\|_2 = 1} \|\mathcal{M}_{\rm op}^\sim w\|_2 = \|\mathcal{M}_{\rm op}^\sim\| = \|\mathcal{M}_{\rm op}\|.$$

Hence

$$\sup_{\bar{q}\neq 0,\, \bar{q}\in\Psi} \|\mathcal{M}_{\rm op}\bar{q}\|_2 < 1 \tag{24}$$

where

$$\mathcal{M}_{\rm op}\bar{q} = \Omega_+ B_1(\cdot)'\mathcal{R}_{\rm op}(\tau)^{\sim} \left[\ \mathcal{U}_{\rm op}(\tau) \quad \mathcal{U}_{+{\rm op}}(\tau)\ \right] \begin{bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{bmatrix}.$$

It is easy to see from equations (7), (8) that two possible realizations of

$$B_1(\cdot)'\mathcal{R}_{\rm op}(\tau)^{\sim}\mathcal{U}_{\rm op}(\tau)$$

and

$$B_1(\cdot)'\tilde{\mathcal{R}}_{\rm op}(\tau)\mathcal{U}_{+{\rm op}}(\tau)$$

respectively, are as follows:

$$B_1(\cdot)'\mathcal{R}_{\rm op}(\tau)^{\sim}\mathcal{U}_{\rm op}(\tau) = (A - B_2B_2'\Pi, -B_2, B_1'\Pi, 0)$$

$$B_1(\cdot)'\mathcal{R}_{\rm op}(\tau)^{\sim}\mathcal{U}_{+{\rm op}}(\tau) = (A - B_2B_2'\Pi, \Pi^{-1}C_1'D_+, B_1'\Pi, 0).$$

Equation (24) in particular implies that

$$\|B_1(\cdot)'\mathcal{R}_{\rm op}(\tau)^{\sim}\mathcal{U}_{+{\rm op}}(\tau)\bar{q}_2\|_2 < \|\bar{q}_2\|_2,\ \forall\ \bar{q}_2 \in L_2[\tau + \infty)$$

Moreover, $A(\cdot) - B_2(\cdot)\,B_2(\cdot)'\Pi(\cdot)$ is stable. Hence Lemma 2.6 of [4] ensures the existence of the stabilizing positive semidefinite solution $W(\cdot)$ of

$$\begin{aligned}-\dot{W}(t) &= (A(t) - B_2(t)\,B_2(t)'\Pi(t))'W(t) + W(t)\,(A(t) - B_2(t)\,B_2(t)'\Pi(t)) \\ &\quad + W(t)\Pi(t)^{-1}C_1(t)'C_1(t)\Pi(t)^{-1}W(t) + \Pi(t)B(t)^{-1}B_1(t)'\Pi(t).\end{aligned} \tag{25}$$

We have here used the fact $C_1(t)'D_+(t)\,D_+(t)'C_1(t) = C_1(t)'C_1(t)$, $\forall t$. Notice also that $\Pi(t)^{-1}$ is the controllability Grammian of system $B_1(\cdot)'\mathcal{R}_{\rm op}(\tau)^{\sim}\ [\mathcal{U}_{\rm op}(\tau)\ \mathcal{U}_{+{\rm op}}(\tau)]$. The Hamiltonian matrices $T_1(\cdot)$ and $T_2(\cdot)$ associated with equation (9) and (25), respectively, are related by the matrix

$$T_3(t) = \begin{bmatrix} -I & \Pi(t)^{-1} \\ -\Pi(t) & 0 \end{bmatrix}.$$

Actually, $T_1(t) = (T_3(t)\,T_2(t) + \dot{T}_3(t))\,T_3(t)^{-1}$. Hence equation (9) admits the positive semidefinite T-periodic stabilizing solution $P(t) = (I - \Pi(t)^{-1}W(t))^{-1}\Pi(t) = \Pi(t)\,(\Pi(t) - W(t))^{-1}\Pi(t)$ provided that we show that $\Pi(t) - W(t)$ is positive definite. This last condition follows from Lemmas 2.8, 2.9 and 2.1 of [4] applied to system $\mathcal{G}$ given by equations (4),(5) of [4] with $\mathcal{G}_{\rm op}(\tau) = B_1(\cdot)'\mathcal{R}_{\rm op}(\tau)^{\sim}\ [\ \mathcal{U}_{\rm op}(\tau) \quad \mathcal{U}_{+{\rm op}}(\tau)\ ]$ and

$$\begin{aligned}F(t) &= A(t) - B_2(t)\,B_2(t)'\Pi(t) \\ G(t) &= [-B_2(t) \quad \Pi(t)^{-1}C_1(t)'D_+(t)] \\ H(t) &= B_1(t)'\Pi(t) \\ E(t) &= [0 \quad 0])\end{aligned}$$

Notice however that Lemmas 2.1 and 2.9 of [4] require the reachability of this last system. As a matter of fact we now finally show that this condition is ensured by the assumption of observability of the T-periodic pair $(A(\cdot), C_1(\cdot))$. Suppose by contradiction that $\mathcal{G}$ is not reachable. In view of the PBH test (equation (7) of [4]) there exists a nonzero periodic vector δ such that

$$\begin{bmatrix} \lambda I - A(t)' + B_2(t)\, B_2(t)'\Pi(t) \\ B_2(t)'\delta \\ D_+(t)'C_1(t)\Pi(t)^{-1} \end{bmatrix} \delta = \begin{bmatrix} \dot{\delta} \\ 0 \\ 0 \end{bmatrix}, \quad t \geq \tau. \tag{26}$$

It is easy to recognize from (5) and (26) that $\mu(t) = \Pi(t)^{-1}\delta(t)$ is a nonzero periodic solution of

$$\begin{bmatrix} \lambda I + A(t) \\ C_1(t) \end{bmatrix} \mu = \begin{bmatrix} \dot{\mu} \\ 0 \\ 0 \end{bmatrix}, \quad t \geq \tau.$$

Hence, the assumed observability of $(A(\cdot), C_1(\cdot))$ is violated (recall the PBH test, equation (7) of [4].

Remark 3.1. Notice that the family Ξ of H_∞ performant periodic controllers can obviously include non-dynamic periodic controllers such that

$$u(t) = \hat{E}_1(t)\, w(t) + \hat{E}_2(t)\, x(t)$$

where $\hat{E}_1(\cdot)$ and $\hat{E}_2(\cdot)$ are T-periodic matrices. Based on the line of reasoning used in deriving equations (18),(19) it turns out that a controller in Ξ_R yielding the same closed-loop operator can be constructed by defining in (10) system $\mathcal{Q}$ of (11),(12) with

$$\begin{aligned}
\bar{F}(t) &= (A(t) + B_1(t)\, B_1(t)'P(t) + B_2(t)\hat{E}_1(t)'B_1(t)'P(t) + B_2(t)\hat{E}_2(t) \\
\bar{G}(t) &= B_1(t) + B_2(t)\hat{E}_1(t) \\
\bar{H}(t) &= \hat{E}_2(t) + B_2(t)'P(t) + \hat{E}_1(t)\, B_1(t)'P \\
\bar{E}(t) &= \hat{E}_1(t)
\end{aligned}$$

As a matter of fact, if ξ is the state of this system and x the state of system (1) it follows that $\xi - x$ is the state of an unreachable T-periodic system with $\bar{F}(\cdot)$ as dynamic matrix.

Theorem 3.2. Consider system $\mathcal{P}$ given by equations (1), (2), let γ be a given positive scalar, and let Assumptions A1 − A4 be fulfilled. Moreover, assume that there exists the stabilizing periodic positive semidefinite solution $P(\cdot)$ of the periodic Riccati equation (9). Then, the class Ξ of all periodic stabilizing controllers $\mathcal{K}$ such that $||T(z_1, w, \mathcal{K})||_\infty < \gamma$ is shown in Figure 3, where $\mathcal{Z}_1$ and $\mathcal{Q}$ are any stable periodic systems and $||\mathcal{Q}||_\infty < \gamma$.

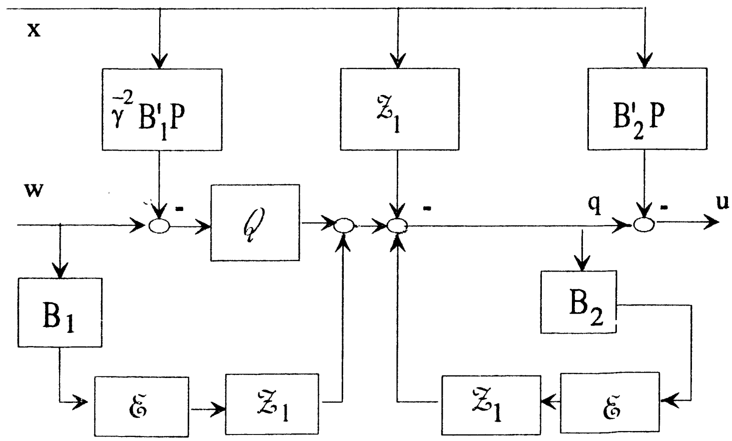

Fig. 3. Parametrization of the set Ξ.

Proof. Let

$$u(t) = -B_2(t)'P(t)\,x(t) + q(t) \tag{27}$$

and form the closed-loop system $\mathcal{P}_q$ as

$$\dot{x} = (A(t) - B_2(t)\,B_2(t)'P(t))\,x + B_1(t)\,w + B_2(t)q$$

$$z = (C_1(t) - D_1(t)\,B_2(t)'P(t))\,x + D_1(t)q$$

$$y = \begin{bmatrix} I \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} w.$$

Denote by $\mathcal{P}_{qij}, i, j = 1, 2$ the corresponding subsystems of $\mathcal{P}_q$, i. e.

$$\mathcal{P}_{q11} = (A - B_2B_2'P, B_1, C_1 - D_1B_2'P, 0)$$

$$\mathcal{P}_{q12} = (A - B_2B_2'P, B_2, C_1 - D_1B_2'P, D)$$

$$\mathcal{P}_{q21} = \left(A - B_2B_2'P, B_1, \begin{bmatrix} I \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ I \end{bmatrix}\right)$$

$$\mathcal{P}_{q22} = \left(A - B_2B_2'P, B_2, \begin{bmatrix} I \\ 0 \end{bmatrix}, 0\right).$$

As shown in Lemma 2.11 of [4], the generic periodic controller ($\mathcal{K}_\mathcal{F}$: $y \to q$) which stabilizes $\mathcal{P}_{q22}$ is given by

$$\mathcal{K}_{F\text{op}}(\tau) = [\mathcal{Y}_{\text{op}}(\tau) - \mathcal{S}_{\text{op}}(\tau)\mathcal{Z}_{\text{op}}(\tau)][\mathcal{X}_{\text{op}}(\tau) - \mathcal{N}_{\text{op}}(\tau)\mathcal{Z}_{\text{op}}(\tau)]^{-1}$$

where $\mathcal{Z}$ is any stable periodic controller. Now, since $\mathcal{P}_{q22}$ is stable per se, it is possible to choose $\mathcal{N}_{\text{op}}(\tau) = \mathcal{P}_{q22\text{op}}(\tau)$, $\mathcal{Y}_{\text{op}}(\tau) = 0$, $\mathcal{Z}_{\text{op}}(\tau) = I$, and $\mathcal{X}_{\text{op}}(\tau) = I$, so that

$$\mathcal{K}_{F\text{op}}(\tau) = -\mathcal{Z}_{\text{op}}(\tau)\,(I - \mathcal{P}_{q22\text{op}}(\tau)\,\mathcal{Z}_{\text{op}}(\tau))^{-1}. \tag{28}$$

Plugging this controller in the system, it results:

$$z(t) = [(\mathcal{P}_{q11\text{op}}(\tau) - \mathcal{P}_{q12\text{op}}(\tau)\,\mathcal{Z}_{\text{op}}(\tau)\mathcal{P}_{q21\text{op}}(\tau))\,w]\,(t). \tag{29}$$

On the other hand, all the possible periodic input-output stable operators $\mathcal{P}(\mathcal{K})_{\text{op}}(\tau)$ with $||\mathcal{P}(\mathcal{K})||_\infty < \gamma$ are such that

$$z(t) = [\mathcal{P}_{q11\text{op}}(\tau)\,w]\,(t) + [\mathcal{P}_{q12\text{op}}(\tau)\,\mathcal{V}_{\text{op}}(\tau)\mathcal{Q}_{\text{op}}(\tau)\,(I - \gamma^{-2}B_1'P\mathcal{E}_{\text{op}}(\tau)\,B_1)\,w]\,(t) \tag{30}$$

where

$$\mathcal{E} = (A - B_2B_2'P, I, I, 0)$$

and

$$\mathcal{V}_{\text{op}}(\tau) = (I + \gamma^{-2}\mathcal{Q}_{\text{op}}(\tau)\,B_1(\cdot)'P(\cdot)\,\mathcal{E}_{\text{op}}(\tau)\,B_2(\cdot))^{-1}.$$

By comparing (29) with (30) it follows

$$\mathcal{P}_{q12\text{op}}(\tau)\left[\mathcal{Z}_{\text{op}}(\tau)\,\mathcal{P}_{q21\text{op}}(\tau) + \mathcal{V}_{\text{op}}(\tau)\mathcal{Q}_{\text{op}}(\tau)\,(I - \gamma^{-2}B_1(\cdot)'P\mathcal{E}_{\text{op}}(\tau)\,B_1(\cdot))\right] = 0.$$

Since $D_1(t)'D_1(t) = I$, $\forall\, t$, $\mathcal{P}_{q12\text{op}}(\tau)$ is left-invertible, so that

$$-\mathcal{Z}_{\text{op}}(\tau)\,\mathcal{P}_{q21\text{op}}(\tau) = \mathcal{V}_{\text{op}}(\tau)\mathcal{Q}_{\text{op}}(\tau)\,(I - \gamma^{-2}B_1(\cdot)'P(\cdot)\mathcal{E}_{\text{op}}(\tau)\,B_1(\cdot)).$$

Moreover, since

$$\mathcal{P}_{q21\text{op}}(\tau) = \begin{bmatrix} \mathcal{E}_{\text{op}}(\tau)B_1(\cdot) \\ I \end{bmatrix} \tag{31}$$

it is readily seen that

$$I - \gamma^{-2}B_1(\cdot)'P(\cdot)\,\mathcal{E}_{\text{op}}(\tau)\,B_1(\cdot) = \begin{bmatrix} -\gamma^{-2}B_1(\cdot)'P(\cdot) & I \end{bmatrix}\mathcal{P}_{q21\text{op}}(\tau)$$

so that

$$\left(\mathcal{Z}_{\text{op}}(\tau) + \mathcal{V}_{\text{op}}(\tau)\,\mathcal{Q}_{\text{op}}(\tau)\begin{bmatrix} -\gamma^{-2}B_1(\cdot)'P(\cdot) & I \end{bmatrix}\right)\mathcal{P}_{q21\text{op}}(\tau) = 0. \tag{32}$$

Assume for the moment that $\mathcal{V}$ is stable (this condition will be proved later on). Hence, a particular solution of equation (32) is

$$\bar{\mathcal{Z}}_{\text{op}}(\tau) = \mathcal{V}_{\text{op}}(\tau)\,\mathcal{Q}_{\text{op}}(\tau)\begin{bmatrix} \gamma^{-2}B_1(\cdot)'P(\cdot) & -I \end{bmatrix}$$

whereas the general solution is given by

$$\mathcal{Z}_{\text{op}}(\tau) = \bar{\mathcal{Z}}_{\text{op}}(\tau) + \hat{\mathcal{Z}}_{\text{op}}(\tau)$$

where $\hat{\mathcal{Z}}$ is any stable T-periodic system solution of the homogeneous equation

$$\hat{\mathcal{Z}}_{\text{op}}(\tau)\,\mathcal{P}_{q21\text{op}}(\tau) = 0.$$

Since $\mathcal{V}$ is stable with stable inverse (recall that $\mathcal{Q}$ is stable and P is stabilizing so that $\mathcal{V}^{-1}$ is stable as well), without any loss of generality we can write

$$\hat{\mathcal{Z}}_{\rm op}(\tau) = [\ \mathcal{V}_{\rm op}(\tau)\,\mathcal{Z}_{1\rm op}(\tau) \quad \mathcal{Z}_{2\rm op}(\tau)\ ]$$

with $\mathcal{Z}_1$ and $\mathcal{Z}_2$ stable. Recalling now equation (31) it follows that

$$\mathcal{Z}_{2\rm op}(\tau) = -\mathcal{V}_{\rm op}(\tau)\mathcal{Z}_{1\rm op}(\tau)\,\mathcal{E}_{\rm op}(\tau)B_1$$

so that

$$\hat{\mathcal{Z}}_{\rm op}(\tau) = \mathcal{V}_{\rm op}(\tau)\,\mathcal{Z}_{1\rm op}(\tau)\,[\ I \quad -\mathcal{E}_{\rm op}(\tau)B_1(\cdot)\ ]\,.$$

Hence,

$$\mathcal{Z}_{\rm op}(\tau)=\mathcal{V}_{\rm op}(\tau)\,[\,\mathcal{Z}_{1\rm op}(\tau)+\gamma^{-2}\mathcal{Q}_{\rm op}(\tau)\,B_1(\cdot)'P(\cdot) \quad -\mathcal{Q}_{\rm op}(\tau)-\mathcal{Z}_{1\rm op}(\tau)\,\mathcal{E}_{\rm op}(\tau)B_1(\cdot)\,]$$

with $\mathcal{Z}_1$ and $\mathcal{Q}$ stable and $\|\mathcal{Q}\|_\infty < \gamma$. Putting this last expression of $\mathcal{Z}$ into (28) and recalling (27) it follows that the family Ξ of all periodic stabilizing performant controllers is represented by the generic controller ($\mathcal{K}_F$: $y \to u$)

$$\mathcal{K}_{F\rm op}(\tau)=[\ -B_2(\cdot)'P(\cdot) \quad 0\ ]\,(I-\mathcal{Z}_{1\rm op}(\tau)\mathcal{E}_{\rm op}(\tau)\,B_2(\cdot))^{-1}\times$$
$$\times\,[\,-\mathcal{Z}_{1\rm op}(\tau)-\gamma^{-2}\mathcal{Q}_{\rm op}(\tau)\,B_1(\cdot)'P(\cdot) \quad \mathcal{Q}_{\rm op}(\tau)+\mathcal{Z}_{1\rm op}(\tau)\,\mathcal{E}_{\rm op}(\tau)B_1(\cdot)\,] \tag{33}$$

with $\mathcal{Z}_1$ and $\mathcal{Q}$ stable and $\|\mathcal{Q}\|_\infty < \gamma$. By inspection, this controller corresponds to the block scheme of Figure 3.

Finally, we only miss to show that $\mathcal{V}$ is a stable periodic system, where

$$\mathcal{V}_{\rm op}(\tau) = (I+\gamma^{-2}\mathcal{Q}_{\rm op}(\tau)\,B_1(\cdot)'P(\cdot)\mathcal{E}_{\rm op}(\tau)\,B_2(\cdot))^{-1}.$$

Actually, consider the periodic system $\begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \end{bmatrix}$, given by

$$\begin{aligned}
\dot{\xi} &= (A(t)-B_2(t)\,B_2(t)'P(t))\xi + B_2(t)q \\
z_1 &= (C_1(t)-D_1(t)\,B_2(t)'P(t))\xi \\
z_2 &= \gamma^{-1}B_1(t)'P(t)\xi
\end{aligned}$$

The input-output operator of $\mathcal{L}_2$ is exactly $\gamma^{-1}B_1(\cdot)'P(\cdot)\,\mathcal{E}_{\rm op}(\tau)B_2(\cdot)$. Moreover, consider the periodic Riccati equation (9), equivalently rewritten as follows

$$\begin{aligned}
-\dot{P}(t) &= P(t)\,(A(t)-B_2(t)\,B_2(t)'P(t))+(A(t)-B_2(t)\,B_2(t)'P(t))'P(t) \\
&\quad + (C_1(t)-D_1(t)\,B_2(t)'P(t))'(C_1(t)-D_1(t)\,B_2(t)'P(t)) \\
&\quad + P(t)\,B_1(t)\,B_1(t)'P(t)\gamma^{-2}.
\end{aligned}$$

Thanks to Lemma 2.6 of [4] it follows $\|\mathcal{L}\|_\infty<1$. From $\mathcal{L}_{2\rm op}(\tau)=\gamma^{-1}B_1(\cdot)'P(\cdot)\,\mathcal{E}_{\rm op}(\tau)$ $B_2(\cdot)$ and $\|\mathcal{L}_2\|_\infty \le \|\mathcal{L}\|_\infty$; it then results $\|\gamma^{-1}B_1(\cdot)'P(\cdot)\,\mathcal{E}_{\rm op}(\tau)B_2(\cdot)\|_\infty < 1$. Moreover, it is $\|-\gamma^{-1}\mathcal{Q}_{\rm op}(\tau)\|_\infty < 1$. Since $\mathcal{E}$ and $\mathcal{Q}$ are both stable, Lemma 2.12 of [4] can be applied to yield the conclusion that the system with input output-operator

$$(I+\gamma^{-2}\mathcal{Q}_{\rm op}(\tau)\,B_1(\cdot)'P(\cdot)\,\mathcal{E}_{\rm op}(\tau)\,B_2(\cdot))^{-1}$$

is stable as well. □

Remark 3.2. As obvious, the set Ξ_R can be recovered from Ξ by selecting $\mathcal{Z}_1 = 0$. It is now interesting to point out further the relationship in the time-domain setting between the sets Ξ and Ξ_R. As a matter of fact, if the controller initial state at $t = \tau$ is zero, then, from equation (33) it follows

$$q(t) = [\mathcal{Z}_{1\text{op}}(\tau)\,(\mathcal{E}_{\text{op}}(\tau)\,B_2(\cdot)q + \mathcal{E}_{\text{op}}(\tau)B_1(\cdot)\,w - x)]\,(t) + \\ +[\mathcal{Q}_{\text{op}}(\tau)\,(w - \gamma^{-2}B_1(\cdot)'P(\cdot)x)]\,(t).$$

But, from the system equation $\dot{x} = (A(t) - B_2(t)\,B_2(t)'P(t))\,x + B_1(t)\,w + B_2(t)q$ it follows that letting $\hat{A}(t) = A(t) - B_2(t)\,B_2(t)'P(t)$,

$$x(t) = [\mathcal{E}_{\text{op}}(\tau)\,w]\,(t) + [\mathcal{E}_{\text{op}}(\tau)q]\,(t) + \Phi_{\hat{A}}(t,\tau)\,x(\tau),\ t \geq \tau$$

so that

$$u(t) = -B_2(t)'P(t)\,x(t) + \mathcal{Z}_{1\text{op}}(\tau)\Phi_{\hat{A}}(t,\tau)\,x(\tau) + [\mathcal{Q}_{\text{op}}(\tau)\,(w - B_1(\cdot)'P(\cdot)\,x)]\,(t). \quad (34)$$

By comparing the control pattern in equation (34) (corresponding to the set Ξ), with the one given by equation (10) (corresponding to the set Ξ_R) the conclusion is drawn that the only difference is played by the possible nonzero system initial state $x(\tau)$.

Remark 3.3. The solution of the full information control problem provided in Theorem 2.2 is the starting point in order to properly extend to periodic systems the so called output estimation problem in H_∞. As in the time-invariant case, it can be shown that the solution of this latter problem derives directly from the solution of the disturbance feedforward problem, which, in turn, is intimately related to the solution of the former.

4. PARAMETRIZATION OF MEMORYLESS H_∞ STATE-FEEDBACK CONTROLLERS

In this section we want to characterize the set of stabilizing control laws

$$u(t) = K(t)\,x(t)$$

with $K(\cdot)$ T-periodic, which render the H_∞ norm of $T(z_1, w, K)$ bounded from above by a given positive attenuation value γ. Let us denote by $\mathcal{K}_\gamma$ the set of periodic stabilizing control gains such that $T_\tau(z_1, w, K) \leq \gamma$ and define the square matrix

$$W(t) = \begin{bmatrix} W_1(t) & W_2(t) \\ W_2(t)' & W_3(t) \end{bmatrix} \quad (35)$$

such that, $\forall\, t$

$$W(t) \geq 0 \quad (36)$$

$$\begin{aligned} \dot{W}_1(t) \ \geq\ & A(t)\,W_1(t) + W_1(t)\,A'(t) \\ +\ & \gamma^{-2}(W_1(t)\,C_1'(t)\,C_1(t)\,W_1(t) + W_2(t)\,W_2'(t)) \\ +\ & B_2(t)\,W_2'(t) + W_2(t)\,B_2'(t) + B_1(t)\,B_1'(t). \end{aligned} \quad (37)$$

Now, denote by $\mathcal{W}_\gamma$ the set of all periodic pairs $(W(\cdot), \gamma^2)$ satisfying eqs. (35) – (37).

Theorem 4.1. Let assumptions A1 – A4 hold and suppose that $B_1(t)\,B_1'(t) > 0, \forall t$. Then,

(a) The set $\mathcal{W}_\gamma$ is convex.

(b) Each $(W(\cdot),\gamma^2) \in \mathcal{W}_\gamma$ is such that $W_1(t) > 0,\ \forall t$.

(c) $\mathcal{K}_\gamma = \{(W_2'(\cdot)\,W_1(t)^{-1}) : (W(\cdot),\gamma^2) \in \mathcal{W}_\gamma\}$.

Proof. (a) To prove convexity it is sufficient to rewrite inequality (37) in affine form. Indeed, by using Schur complements and letting Letting

$$\begin{aligned}\theta(W(t)) &= -\dot{W}_1(t) + A(t)\,W_1(t) + W_1(t)\,A(t)' \\ &\quad + B_2(t)\,W_2(t)' + W_2(t)\,B_2(t)' + B_1(t)\,B_1(t)' \\ R(t) &= \begin{bmatrix} C_1(t)'C_1(t) & 0 \\ 0 & I \end{bmatrix} \\ U &= \begin{bmatrix} I & 0 \end{bmatrix}\end{aligned}$$

it is easy to see that inequelity (37) can be rewritten as

$$S(W(t)) = \begin{bmatrix} \theta(W) & UW(t)R(t)^{1/2} \\ R(t)^{1/2}W(t)U' & -\gamma^2 I \end{bmatrix} \leq 0.$$

Hence, since $\theta(W(\cdot))$ is affine, $A(W(\cdot)$ is affine as well so showing convexity of $\mathcal{W}_\gamma$.
(b) This condition follows from the assumption that $B_1(t)\,B_1(t))'$ is positive definite, for each t. Indeed, assume, by contradiction, that $W_1(t)$ is singular for some time instant $\xi \in [0,T)$, i.e. $W_1(\xi)\,x = 0,\ x \neq 0$, and let $y(t) = x'W_1(t)\,x$. Since $W(\cdot)$ is positive semidefinite it must be also $W_2(\xi)'x = 0$. By premultipling inequality (37) by x' and postmultipling by x it then follows that $\dot{y}(\xi) \geq x'B_1(\xi)\,B_1(\xi)'x > 0$. This, together with $y(\xi) = 0$ and $y(t) \geq 0, \forall t$, leads to contradiction.
(c) This point simply derives from Lemma 2.7 of [4] by comparing the two inequalities (37) and (26) of [4] and letting $W_2(t)' = K(t)\,W_1(t)$. Actually, such a comparizon yields $W_1(t) = P(t)$. Indeed, if $K(\cdot) \in \mathcal{K}_\gamma$ then let

$$F(t) = A(t) + B_2(t)\,K(t) \tag{38}$$

$$G(t) = B_1(t) \tag{39}$$

$$H(t) = C_1(t) + D_1(t)\,K(t) \tag{40}$$

$$E(t) = 0. \tag{41}$$

Notice that the pair $(F(\cdot), G(\cdot))$ is reachable since $G(t)\,G(t)' > 0,\ \forall t$. In view of Lemma 2.7 in [4] there exists a T-periodic positive semidefinite solution $Q(\cdot)$ of inequality (26) of [4]. In addition $Q(\cdot)$ is positive definite thanks to the positive definiteness of $B_1(t)\,B_1(t)' = G(t)\,G(t)'$. Now, let

$$W(t) = \begin{bmatrix} Q(t) & Q(t)\,K(t)' \\ K(t)Q(t) & K(t)Q(t)\,K(t)' \end{bmatrix}.$$

It is simple to verify that the periodic matrix $W(\cdot)$ satisfies all required conditions since it is positive semidefinite with $W_1(t) = Q(t)$ positive definite and such that inequality (37) is satisfied. Conversely assume that $W(\cdot) \in \mathcal{W}_\gamma$. Then, by letting $K(t) = W_2(t)'W_1(t)^{-1}$, and $F(\cdot)$, $G(\cdot)$, $H(\cdot)$, $E(\cdot)$ as in (38)-(41) it follows that $Q(t) = W_1(t)$ is a positive definite solution of inequality (26) of [4] so that, in view of Lemma 2.7 of [4] the conclusion that $K(\cdot) \in \mathcal{K}_\gamma$ follows. □

The interest of the theorem above mainly relies on the convexity property of the set $\mathcal{W}_\gamma$.

The results relative to the parametrization of periodic stabilizing controllers guaranteeing the strict H_∞ inequality, namely for $\|T(z_1, w, K)\|_\infty$, are still obtained from Theorem 4.1 by replacing the inequality sign ≥ 0 in equation (37) with the strict inequality sign $>$.

5. THE MIXED H_2/H_∞ CONTROL PROBLEM

In this section we want to address the problem of finding a periodic stabilizing gain $K(\cdot)$ which minimizes $\|T(z_0, w, K)\|_2$ while keeping $\|T(z_1, w, K)\|_\infty \leq \gamma$, i.e.

$$J_m = \min_K \{\|T(z_0, w, K)\|_2 : \|T(z_1, w, K)\|_\infty \leq \gamma\}. \tag{42}$$

The exact solution to this problem is not yet available in the literature, even in the time-invariant case. This fact spurred the research activity in the direction of finding *suboptimal* solutions to problem (42). One way consists in exploiting the convex structure of the set $\mathcal{W}_\gamma$ introduced in the previous section and trying to replace the nonconvex objective function with a convex (linear) function of $W(\cdot)$. By doing so, a suboptimal solution is recovered by solving the convex optimization problem stated in the result below.

Theorem 5.1. Let assumptions A1 – A4 hold and also the assumptions of Theorem 2.2 be fulfilled. Moreover, let $L(t) := [\,C_2(t) \quad D_2(t)\,]$ and assume that $\bar{W}(\cdot)$ is the optimal solution of the convex problem

$$J_{\text{sub}} = \min W(\cdot) \in \mathcal{W}_\gamma \text{trace} \int_0^T L(t)\, W(t) L(t)'\, \mathrm{d}t.$$

Then, $K^o(t) = \bar{W}_2(t)'\bar{W}_1(t)^{-1}$ minimizes an upper bound of the objective function of problem (42).

Proof. Let $K(t) = W_2(t)'W_1(t)^{-1}$ and consider eqs. (38) – (39). Direct comparizon of (12) in [4] and (37), this last inequality with $W_2(t)' = K(t)\, W_1(t)$, shows that $W_1(t) \geq P_2(t)$ so that

$$\text{trace} \int_0^T L(t)\, W(t) L(t)'\, \mathrm{d}t$$

$$+ \text{trace} \int_0^T (C_2(t) + D_2(t)\,K(t))\,W_1(t)\,(C_2(t) + D_2(t)\,K(t))'\,\mathrm{d}t$$
$$+ \text{trace} \int_0^T D_2(t)\,(W_3(t) - K(t)\,W_1(t)\,K(t)')\,D_2(t)'\,\mathrm{d}t$$
$$\geq \text{trace} \int_0^T (C_2(t)+D_2(t)\,K(t))\,P_2(t)\,(C_2(t) + D_2(t)\,K(t))'\,\mathrm{d}t = \|T(z_2, w, K)\|_2^2. \;\square$$

The rest of this section is devoted to present an algorithm (α iteration procedure) providing a suboptimal solution of the mixed problem which performs better than the convex optimization procedure described in Theorem 5.1. Let $K_1(\cdot)$ be a stabilizing periodic gain, and $\alpha \in [0,\ 2]$ a scalar parameter. Moreover let

$$F(t) = A(t) + B_2(t)\,K_1(t)\,(1-\alpha)^2 + (\alpha^2 - 2\alpha)\,B_2(t)(D_2(t)'D_2(t))^{-1}D_2(t)'C_2(t)$$

$$G(t) = B_2(t)\,(D_2(t)'D_2(t))^{-1/2}(2\alpha - \alpha^2)^{1/2}$$

$$H(t) = C_2(t) + D_2(t)\,K_1(t)\,(1-\alpha) - \alpha D_2(t)(D_2(t)'D_2(t))^{-1}D_2(t)'C_2(t).$$

Consider now the standard periodic differential Riccati equation

$$-\dot{\Pi}_2(t) = F(t)'\Pi_2(t) + \Pi_2(t)F(t) + H(t)'H(t) - \Pi_2(t)\,G(t)\,G(t)'\Pi_2(t). \tag{43}$$

We are in a position to prove the following result.

Theorem 5.2. Consider system (1) – (3) and let Assumptions A1 – A4 hold. Moreover, let $K_1(\cdot)$ be any stabilizing T-periodic feedback control gain. Then for any real $\alpha \in [0,2]$

(i) The periodic differential Riccati equation (43) admits the stabilizing T-periodic positive semidefinite solution $\Pi_2(\cdot)$, $\forall\,\alpha \in [0,2]$.

(ii) The periodic feedback gain

$$K_2(t) = (1-\alpha)\,K_1(t) - \alpha(D_2(t)'D_2(t))^{-1}(B_2(t)'\Pi_2(t) + D_2(t)'C_2(t)) \tag{44}$$

is stabilizing, i.e. $A(\cdot) + B_2(\cdot)\,K_2(\cdot)$ is stable.

(iii) The H_2 norm due to $K_2(\cdot)$ is less than (or equal to) the H_2 norm due to $K_2(\cdot)$, i.e.

$$\|T(z_2, w, K_2)\|_2 \leq \|T(z_2, w, K_1)\|_2, \;\forall\,\alpha \in [0,2].$$

(iv) The H_2 norm due to $K_2(\cdot)$ is monotonically nonincresing for $\alpha \in [0,1]$, and monotonically nondecreasing for $\alpha \in [1,2]$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\|T(z_2, w, K_2)\|_2 \leq 0, \quad \forall\,\alpha \in [0,1]$$
$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\|T(z_2, w, K_2)\|_2 \geq 0, \quad \forall\,\alpha \in [1,2].$$

Proof. *Point (i).* Equation (43) is a standard periodic differential Riccati equation (in the unknown $\Pi_2(\cdot)$) of the kind encountered in the H_2 design context. The first point to be proven is that the pairs $(F(\cdot), G(\cdot))$ and $(F(\cdot), H(\cdot))$ are stabilizable and detectable, respectively. Indeed, let

$$\hat{A}_2(t) = A(t) + B_2(t)\,(D_2(t)'D_2(t))^{-1}D_2(t)'C_2(t)$$
$$\hat{C}_2(t) = (I - D_2(t)\,(D_2(t)'D_2(t))^{-1}D_2(t)')\,C_2(t).$$

In view of the assumption on the zeros of system (A, B_2, C_2, D_2) and the fact that $D_2(\cdot)$ is full column rank, it follows that the pair $(\hat{A}_2(\cdot), \hat{C}_2(\cdot))$ is detectable. Now, notice that

$$F(t) = \hat{A}_2(t) + B_2(t)\,(1-\alpha)\,S_2(t)$$
$$H(t) = \hat{C}_2(t) + D_2(t)\,S_2(t)$$

where $S_2(t) = (1-\alpha)\,(K_1(t) + (D_2(t)'D_2(t))^{-1}D_2(t)'C_2(t))$. Now assume by contradiction that the pair the pair $(F(\cdot), H(\cdot))$ is not detectable. This means that there exists a nonzero periodic solution of

$$\begin{bmatrix} -\lambda I + F(t) \\ H(t) \end{bmatrix} \theta = \begin{bmatrix} \dot{\theta} \\ 0 \end{bmatrix}, \quad t \geq \tau$$

with $Re(\lambda) \geq 0$. The second equation with the definition of $\hat{C}_2(t)$ and the facts that $\hat{C}_2(t)'D_2(t)$ and $D_2(t)$ is full column rank, implies that

$$S_2(t)\,x(t) = 0, \quad \hat{C}_2(t)\,x(t) = 0, \quad \forall t.$$

Hence

$$(-\lambda I + F(t))\,x(t) = (\lambda I + \hat{A}_2(t))\,x(t) = \dot{x}(t)$$

which, together with $Re(\lambda) \geq 0$, $\hat{C}_2(t)\,x(t) = 0$, $\forall t$ and $x(\cdot) \neq 0$, contradicts the detectability of $(\hat{A}_2(\cdot), \hat{C}_2(\cdot))$. As for the stabilizability of $(F(\cdot), G(\cdot))$, it easily follows from that of $(\hat{A}_2(\cdot), B_2(\cdot))$, which in turn is equivalent to the stated condition of stabilizability of $(A(\cdot), B_2(\cdot))$. Therefore we have shown that the pair $F(\cdot), H(\cdot))$ is detectable and the pair $F(\cdot), G(\cdot))$ is stabilizable. Hence, the equation (see [7]) admits a stabilizing solution whenever $\alpha \in [0, 2]$. Actually, under the stated assumptions, this solution is also the unique positive semidefinite one.

Point (ii) In order to check the stability of $A(\cdot) + B_2(\cdot)\,K_2(\cdot)$, with $K_2(\cdot)$ given by eq. (44), consider again eq. (43). It can be rewritten in the following way

$$\begin{aligned} -\dot{\Pi}_2(t) &= (A(t) + B_2(t)\,K_2(t))'\Pi_2(t) + \Pi_2(t)\,(A(t) + B_2(t)\,K_2(t)) \\ &\quad + (C_2(t) + D_2(t)\,K_2(t))'(C_2(t) + D_2(t)\,K_2(t)). \end{aligned} \tag{45}$$

Now notice that

$$A(t) + B_2(t)\,K_2(t) = \hat{A}_2(t) + B_2(t)\,S(t)$$
$$C_2(t) + D_2(t)\,K_2(t) = \hat{C}_2(t) + D_2(t)\,S(t)$$
$$0 = D_2'(t)\hat{C}_2(t)$$

where the periodic matrix $S(\cdot)$ is defined as follows

$$S(t) = [(1-\alpha)(K_1(t) + (D_2(t)'D_2(t))^{-1}D_2(t)'C_2(t)) \\ - \alpha(D_2(t)'D_2(t))^{-1}B_2(t)'\Pi_2(t)].$$

Thus, the stated detectability assumption together with the above equations imply that the pair $(A(\cdot)+B_2(\cdot)\,K_2(\cdot), C_2(\cdot)+D_2(\cdot)\,K_2(\cdot))$ is detectable as well. This fact and the existence of a solution $\Pi_2(t) \geq 0$ of (45), entails, by an inertia argument (see [7]) that $A(\cdot)+B_2(\cdot)\,K_2(\cdot)$ is stable.

Point (iii). For $i = 1,2$ let $Q_i(t)$ be the periodic solutions of the differential Lyapunov equations

$$-\dot{\Pi}_i(t) = (A(t) + B_2(t)\,K_i(t))'\Pi_i(t) + \Pi_i(A(t) + B_2(t)\,K_i(t)) \\ + (C_2(t) + D_2(t)\,K_i(t))'(C_2(t) + D_2(t)\,K_i(t)) \qquad (46)$$

respectively. Of course,

$$\text{trace} \int_0^T [B_1'(t)\Pi_i(t)\,B_1(t)] = \|T(z_2, w, K_i)\|_2^2.$$

By means of standard algebraic manipultions it is possible to conclude that the difference between the two periodic solutions $\Pi_1(t)$ and $\Pi_2(t)$ of (46) satisfies the equation

$$-(\dot{\Pi}_1(t) - \dot{\Pi}_2(t)) = (A(t) + B_2(t)\,K_1(t))'(\Pi_1(t) - \Pi_2(t)) \\ + (\Pi_1(t) - \Pi_2(t))\,(A(t) + B_2(t)\,K_1(t)) + \hat{K}_1(t)'\hat{K}_1(t) \\ - \hat{K}_2(t)'\hat{K}_2(t)$$

where, for $i = 1,2$,

$$\hat{K}_i(t) := K_i(t) + (D_2(t)'D_2(t))^{-1}[B_2(t)'\Pi_2(t) + D_2(t)'C_2(t)]\,(D_2(t)'D_2(t))^{1/2}.$$

Indeed, with this choice the above equation becomes

$$-(\dot{\Pi}_1(t) - \dot{\Pi}_2(t)) = (A(t) + B_2(t)\,K_1(t))'(\Pi_1(t) - \Pi_2(t)) \\ + (\Pi_1(t) - \Pi_2(t))\,(A(t) + B_2(t)\,K_1(t)) \\ + \hat{K}_1(t)'\hat{K}_1(t)(2\alpha - \alpha^2)$$

which shows that $\Pi_1(\cdot) \geq \Pi_2(\cdot)$ whenever $\alpha \in [0,2]$.

Point (iv). Notice first that $\Pi_2(t)$ enjoys a symmetric property with respect to α, i.e. $\Pi_2(t)$ at α is equal to $\Pi_2(t)$ at $2-\alpha$. Denoting with $\Gamma_2(\cdot)$ the periodic matrix which is the derivative of $\Pi_2(\cdot)$ with respect to α it is possible to verify that such a matrix satisfies the following Lyapunov equation

$$-\dot{\Gamma}_2(t) = \hat{F}(t)'\Gamma_2(t) + \Gamma_2(t)\hat{F}(t) - 2(1-\alpha)\hat{K}_1(t)'\hat{K}_1(t) \qquad (47)$$

where $\hat{F}(t) = F(t) - G(t)\,G(t)'\Pi_2(t)$ is a stable matrix since $\Pi_2(t)$ is the stabilizing solution of eq. (43). Hence from eq. (47) it follows that $\Gamma_2(t) \leq 0$ for $\alpha \in [0,1]$ and $\Gamma_2(t) \geq 0$ for $\alpha \in [1,2]$. □

Remark 5.1. The above results can be exploited in the following way. Consider system (1)-(3) and suppose that $K_1(\cdot)$ is a given periodic matrix such that $A(\cdot) + B_2(\cdot)K_1(\cdot)$ is stable and $||T(z_1, w; K_1)||_\infty < \gamma$ (for instance the matrix resulting from the convex programming problem (42)). Then one one dimensional search in the interval $[0, 2]$ for α with eq. (43) taken into account, allows to determine the value of α^o corresponding to which the control law $u(t) = K_2(t)\,x(t)$ minimizes the H_2 norm while keeping the H_∞ norm not greater than γ. Incidentally, notice that the choice $\alpha = 1$ corresponds to the optimal unconstrained H_2 control law, see Theorem 2.2. Obviously α^o and the relevant value for the H_2 norm depend on the chosen $K_1(\cdot)$. Hence the α procedure consists in finding α^o as follows:

$$\begin{aligned}
\alpha^o &= \operatorname{argmin}\{1-\alpha_1, \alpha_2 - 1\} \\
\alpha_1 &= \max\{\alpha|\ \alpha \in [0,1], ||T(z_1, w; s)||_\infty \le \gamma\} \\
\alpha_2 &= \min\{\alpha|\ \alpha \in [1,2], ||T(z_1, w; s)||_\infty \le \gamma\}.
\end{aligned}$$

6. CONCLUDING REMARKS

In this paper a number of important state-feedback control problems for continuous-time periodic systems are tackled. Particular attention is devoted to the parametrization of memoryless H_∞ controllers and to the discussion of a new procedure for the suboptimal solution of the mixed H_2/H_∞ control problem. The exposition greatly exploits the theory of periodic Riccati and Lyapunov equations and does not take any advantage of the recent developments of the "transfer function" approach for continuous-time periodic systems.

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