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GAUSSIAN SEMIPARAMETRIC ESTIMATION IN SEASONAL/CYCLICAL LONG MEMORY TIME SERIES[1]

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Gaussian semiparametric or *local Whittle* estimation of the memory parameter in standard long memory processes was proposed by Robinson [18]. This technique shows several advantages over the popular log-periodogram regression introduced by Geweke and Porter–Hudak [7]. In particular under milder assumptions than those needed in the log periodogram regression it is asymptotically more efficient. We analyse the asymptotic behaviour of the Gaussian semiparametric estimate of the memory parameter in *seasonal or cyclical long memory* processes allowing for asymmetric spectral divergences or zeros. Consistency and asymptotic normality are obtained.

1. INTRODUCTION

Let $\{x_t, t = 0, \pm 1, \pm 2, \ldots\}$ be a real-valued covariance stationary process with spectral density $f(\lambda)$. We say that x_t has *standard long memory* if

$$f(\lambda) \sim C\lambda^{-2d} \quad \text{as } \lambda \to 0, \tag{1.1}$$

where $0 < C < \infty$ and the memory parameter d satisfies $|d| < 1/2$. Whittle [19] proposed an estimation technique based on the minimization of the function

$$\Upsilon_n(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \log f(x) + \frac{I_n^c(x)}{f(x)} \right\} \mathrm{d}x \tag{1.2}$$

where θ is the vector of parameters to estimate, $f(z)$ is the spectral density (absolute knowledge of this function up to the vector of parameters θ is thus assumed) and $I_n^c(z)$ is the centered periodogram of $\{x_t, t = 1, 2, \ldots, n\}$ at frequency z,

$$I_n^c(z) = \frac{1}{2\pi n} \left| \sum_{t=1}^{n} e^{itz}(x_t - \bar{x}_n) \right|^2, \quad \bar{x}_n = \frac{1}{n} \sum_{t=1}^{n} x_t.$$

This technique was originally proposed for short memory processes with a smooth spectral density and in that case is asymptotically equivalent to maximum likelihood.

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The application of this methodology to standard long memory processes with a spectral density satisfying (1.1) has been analysed by Fox and Taqu [6], Dahlhaus [5], Giraitis and Surgailis [8] and Heyde and Gay [12] and $\sqrt{n}$-consistency and asymptotic normality have been proved, and when x_t is actually Gaussian asymptotic efficiency has been found. However these properties depend strongly on a correct specification of $f(\lambda)$ and if some kind of misspecification occurs the estimates will in general be inconsistent. Particularly the estimates of long memory parameters will be inconsistent if short memory components are misspecified. To overcome this inconvenience Kunsch [15] and Robinson [18] considered a semiparametric discrete version of (1.2) and assumed only partial knowledge of the spectral density so that the *Gaussian semiparametric* estimates of d and C are obtained by minimizing

$$Q(C,d) = \frac{1}{m}\sum_{j=1}^{m}\left\{\log C\lambda_j^{-2d} + \frac{\lambda_j^{2d}}{C} I_n(\lambda_j)\right\} \tag{1.3}$$

where $I_n(\lambda) = |W(\lambda)|^2$ and $W(\lambda) = (2\pi n)^{-1/2}\sum_{t=1}^{n}\exp(it\lambda)\,x_t$ are the (uncentered) periodogram and discrete Fourier transform respectively, $\lambda_j = 2\pi j/n$ are Fourier frequencies and m is the bandwidth number such that λ_m is the last frequency used in the estimation. It is required that at least $m^{-1}+mn^{-1} \to 0$ as $n \to \infty$ such that the proportion of the frequency band involved in the estimation degenerates relatively slowly to 0 as n increases. Since $j = 0$ is omitted in (1.3) we do not need specification or estimation of an unknown mean. Robinson [18] proved that if $\tilde{d} = \arg\min Q(C,d)$ over a closed interval of admissible estimates $\Theta = [\Delta_1, \Delta_2]$, such that $-1/2 < \Delta_1 < \Delta_2 < 1/2$, then $\sqrt{m}(\tilde{d}-d) \xrightarrow{d} N(0,1/4)$. This estimate overcomes the log-periodogram proposed by Geweke and Porter–Hudak [7] in the sense that its asymptotic variance is smaller. Moreover, unlike the log-periodogram regression, the proof of the asymptotic properties does not need to trim out low frequencies and much weaker assumptions than Gaussianity are imposed (see Robinson [17]). The main disadvantage is that $\tilde{d}$ can not be defined in a closed form.

The drawback of this estimate with respect to the parametric one is that only $\sqrt{m}$ consistency is achieved. Therefore the semiparametric estimate is much less efficient than the parametric ones when they are based on a complete and correct specification of $f(\lambda)$. This loss in efficiency is the price to pay for guaranteeing consistency under misspecification of the spectral density at frequencies far from the one we are interested in.

In this paper we study the properties of the Gaussian semiparametric estimate proposed by Robinson [18] in the case of *seasonal or cyclical long memory* such that the spectral density satisfies

$$\begin{aligned} f(\omega+\lambda) &\sim C\lambda^{-2d_1} \quad \text{as } \lambda \to 0^+ \\ f(\omega-\lambda) &\sim D\lambda^{-2d_2} \quad \text{as } \lambda \to 0^+ \end{aligned} \tag{1.4}$$

for C, $D \in (0,\infty)$ and d_1, $d_2 \in (-1/2, 1/2)$ and we allow

$$d_1 \neq d_2 \quad \text{and/or} \quad C \neq D.$$

Thus we allow for asymmetric spectral poles or zeros in the sense that the parameters d_1, C governing spectral behaviour just after ω may be different from d_2, D which govern spectral behaviour at frequencies just before ω. Some parametric seasonal or cyclical long memory models have been discussed by Jonas [14], Carlin and Dempster [4], Gray et al [9], Hassler [11] and Robinson [16]. For a review see Arteche and Robinson [2]. The case of an asymmetric spectral pole or zero at frequency $\omega \neq 0$ presents a peculiarity with respect to the analysis at the origin where the spectral density function is symmetric. If the parameter we want to estimate is d_1 such that $d_1 < d_2$ we need to trim out some frequencies close to ω in order to get rid of the influence of the periodogram at frequencies just before ω where the spectral density is governed by d_2. Thus the Gaussian semiparametric estimates of d_1 and C are obtained by minimizing

$$Q(C,d) = \frac{1}{m-l}\sum_{j=l+1}^{m}\left\{\log C\lambda_j^{-2d} + \frac{\lambda_j^{2d}}{C} I_j\right\} \tag{1.5}$$

where $\lambda_j = \frac{2\pi j}{n}$, $j = l+1,\ldots,m$, $I_j = I_n(\omega+\lambda_j)$ is the periodogram at frequency $\omega+\lambda_j$ and $l = 0$ if $d_1 \geq d_2$ and $l \to \infty$ more slowly than m as $n \to \infty$ if $d_1 < d_2$. Concentrating C out of the objective function we have that minimizing (1.5) is equivalent to minimize

$$R(d) = \log \tilde{C}(d) - 2d\frac{1}{m-l}\sum_{l+1}^{m}\log\lambda_j \tag{1.6}$$

where

$$\tilde{C}(d) = \frac{1}{m-l}\sum_{l+1}^{m}\lambda_j^{2d} I_j. \tag{1.7}$$

Then the procedure consists in obtaining an estimate of d_1, $\tilde{d}_1 = \arg\min_{d\in\Theta} R(d)$ where $\Theta = [\Delta_1, \Delta_2]$ is the set of admissible values for d_1 and then plug $\tilde{d}_1$ in (1.7) to obtain an estimate of C, $\tilde{C}(\tilde{d}_1)$. A similar procedure, using frequencies just before ω, can be used to estimate D and d_2. In Section 2 we proved the consistency of $\tilde{d}_1$ while the asymptotic distribution is obtained in Section 3. Finally technical lemmas are placed in the Appendix.

2. CONSISTENCY OF THE ESTIMATE

In order to prove the consistency of $\tilde{d}_1$ we need the following assumptions:

A.1. For $\alpha \in (0,2]$ and $\omega \in (0,\pi)$, as $\lambda \to 0^+$:

$$\begin{aligned} f(\omega+\lambda) &\sim C\lambda^{-2d_1}(1+O(\lambda^\alpha)) \\ f(\omega-\lambda) &\sim D\lambda^{-2d_2}(1+O(\lambda^\alpha)) \end{aligned}$$

where $C, D \in (0,\infty)$, $|d_2| < 1/2$ and $d_1 \in \Theta = [\Delta_1, \Delta_2]$ where $-1/2 < \Delta_1 < \Delta_2 < 1/2$. The choice of Δ_1 and Δ_2 reflects prior knowledge on d_1, for example if we know that $f(\omega+\lambda) \not\to 0$ as $\lambda \to 0^+$ a reasonable choice is $\Delta_1 = 0$.

A.2. In a neighbourhood $(-\delta, 0) \cup (0, \delta)$ of ω, $f(\lambda)$ is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \log f(\omega \pm \lambda) \;=\; O(\lambda^{-1}) \qquad \text{as } \lambda \to 0^+.$$

A.3. $x_t - Ex_1 = \sum_{j=0}^{\infty} \alpha_j \epsilon_{t-j}$ and $\sum_{j=0}^{\infty} \alpha_j^2 < \infty$ where $E[\epsilon_t|F_{t-1}] = 0$, $E[\epsilon^2|F_{t-1}] = 1$ for $t = 0, \pm 1, \pm 2, \ldots$, F_t is the σ-field generated by ϵ_s, $s \leq t$ and there exists a random variable ε such that $E\varepsilon^2 < \infty$ and for all $\eta > 0$ and some $\kappa < 1$, $P(|\epsilon_t| > \eta) \leq \kappa P(|\varepsilon| > \eta)$.

A.4. If $d_1 \geq d_2$:

$$\frac{1}{m} + \frac{m}{n} \to 0 \text{ as } n \to \infty$$

and if $d_1 < d_2$:

$$\frac{m}{n} + \frac{l}{m} \log m + \frac{n^{d_2 - d_1}}{l^{\frac{1}{2} + (d_2 - d_1)}} (\log m)^{\frac{3}{2}} \to 0 \text{ as } n \to \infty.$$

Assumption A.1 and A.2 are those imposed in Arteche [1] for the asymptotics of the log-periodogram regression and correspond to those in Robinson [17] for $\omega = 0$. Assumption A.3 says that the innovations in the Wold decomposition of x_t are a square integrable martingale difference sequence that satisfies a milder homogeneity restriction than strictly stationarity. Assumption A.4 distinguishes the cases $d_1 \geq d_2$ and $d_1 < d_2$. In the former m tends to ∞ (necessary for consistency) but more slowly than n (due to our semiparametric specification of $f(\lambda)$). In the latter we introduce the trimming number l which has to go to infinity with n at a slower rate than m but its rate of divergence as $n \to \infty$ is higher the larger the difference $d_2 - d_1$. This is because the higher d_2 with respect to d_1 the more influential the periodogram at frequencies just before ω (where the spectral density is governed by d_2) in the estimation of d_1 (see Lemma 1) such that a larger trimming is needed. The more restrictive case comes up when $d_2 - d_1$ approaches 1. In that case $\frac{n}{l^{\frac{3}{2}}} (\log m)^{\frac{3}{2}} \to 0$ as $n \to \infty$ so that if $m \sim n^{\theta}$ and $l \sim n^{\phi}$, then $1 > \theta > \phi > 2/3$ ensures A.4.

The following theorem establishes the consistency of the Gaussian semiparametric estimate of d_1 in (1.4). We only focus on the case $d_1 < d_2$. The proof when $d_1 \geq d_2$ is a straightforward extension of that in Robinson [18] when $\omega = 0$.

Theorem 1. Let assumptions A.1 – A.4 hold. Then as $n \to \infty$

$$\tilde{d}_1 \xrightarrow{p} d_1.$$

Proof. $\tilde{d}_1 = \arg\min_{\Theta} R(d)$ where $R(d)$ is defined in (1.6). Write $S(d) = R(d) - R(d_1)$ and $N_\delta = \{d : |d - d_1| < \delta\}$ for $0 < \delta < 1/4$ and $\bar{N}_\delta = (-\infty, \infty) - N_\delta$. Then

$$\begin{aligned} & P(|\tilde{d}_1 - d_1| \geq \delta) \\ = \;& P(\tilde{d}_1 \in \bar{N}_\delta \cap \Theta) = P\left(\inf_{\bar{N}_\delta \cap \Theta} R(d) \leq \inf_{N_\delta \cap \Theta} R(d)\right) \leq P\left(\inf_{\bar{N}_\delta \cap \Theta} S(d) \leq 0\right) \end{aligned}$$

because $d_1 \in N_\delta \cap \Theta$. Now define the following subsets of the set of admissible values Θ,

$$\Theta_1 = \{d : \Delta \le d \le \Delta_2\} \text{ such that } \begin{cases} \Delta = \Delta_1 & \text{if} \quad d_1 < \Delta_1 + \frac{1}{2} \\ d_1 \ge \Delta > d_1 - \frac{1}{2} & \text{if} \quad d_1 \ge \Delta_1 + \frac{1}{2} \end{cases}$$

$$\Theta_2 = \begin{cases} \{d : \Delta_1 \le d < \Delta\} & \text{if } d_1 \ge \Delta_1 + \frac{1}{2} \\ \emptyset & \text{otherwise} \end{cases}.$$

Thus

$$P(|\tilde{d}_1 - d_1| \ge \delta) \le P\left(\inf_{\bar{N}_\delta \cap \Theta_1} S(d) \le 0\right) + P\left(\inf_{\Theta_2} S(d) \le 0\right).$$

Write $S(d) = U(d) - T(d)$ where $U(d)$ is the deterministic part of $S(d)$ obtained by replacing I_j by $C\lambda_j^{-2d_1}$ and sums by integrals, and $T(d)$ is the remainder.

$$\begin{aligned} U(d) &= 2(d-d_1) - \log\{2(d-d_1)+1\} \\ T(d) &= \log\left\{\frac{\tilde{C}(d_1)}{C}\right\} - \log\left\{\frac{\tilde{C}(d)}{C(d)}\right\} - \log\left\{\frac{1}{m-l}\sum_{j=l+1}^{m}\left(\frac{j}{m-l}\right)^{2(d-d_1)}\{2(d-d_1)+1\}\right\} \\ &+ \ 2(d-d_1)\left\{\frac{1}{m-l}\sum_{j=l+1}^{m}\log j - \log(m-l) + 1\right\} \end{aligned}$$

where

$$C(d) = C\frac{1}{m-l}\sum_{l+1}^{m}\lambda_j^{2(d-d_1)}. \tag{2.1}$$

Note that $U(d)$ achieves a unique minimum in Θ_1 for $d = d_1$. Now,

$$P\left(\inf_{\bar{N}_\delta \cap \Theta_1} S(d) \le 0\right) = P\left(\inf_{\bar{N}_\delta \cap \Theta_1}(U(d) - T(d)) \le 0\right) \le P\left(\sup_{\Theta_1}|T(d)| \ge \inf_{\bar{N}_\delta \cap \Theta_1} U(d)\right).$$

Using the mean value theorem we have

$$\log(1+x) \le x - \frac{1}{8}x^2, \quad -\log(1-x) \ge x + \frac{1}{2}x^2$$

for $0 < x < 1$. It follows that

$$\inf_{\bar{N}_\delta \cap \Theta_1} U(d) \ge \min\left(2\delta - \log\{2\delta+1\}, -2\delta - \log\{1-2\delta\}\right) \ge \frac{\delta^2}{2}. \tag{2.2}$$

On the other hand, from the inequality $|\log(1+x)| \le 2|x|$ for $|x| \le \frac{1}{2}$, we deduce that for any nonnegative random variable y, $P(2|y-1| \le \epsilon) \le P(|\log y| \le \epsilon)$ for $\epsilon \le 1/2$ and

$$P\left\{\left|\log\left(\frac{\tilde{C}(d)}{C(d)}\right)\right| > \epsilon\right\} \le P\left\{\left|\frac{\tilde{C}(d) - C(d)}{C(d)}\right| > \frac{\epsilon}{2}\right\}$$

and thus $\sup_{\Theta_1}|T(d)| \xrightarrow{p} 0$ if

a) $\sup_{\Theta_1} \left| \frac{\tilde{C}(d) - C(d)}{C(d)} \right| = o_p(1)$

b) $\sup_{\Theta_1} \left| \frac{2(d-d_1)+1}{m-l} \sum_{l+1}^{m} (\frac{j}{m-l})^{2(d-d_1)} - 1 \right| = o(1)$

c) $\left| \frac{1}{m-l} \sum_{l+1}^{m} \log j - \log(m-l) + 1 \right| = o(1).$

If $d_1 \leq d_2$ ($l = 0$) the left hand sides of b) and c) are $O(m^{-1-2(\Delta-d_1)})$ and $O(\frac{\log m}{m})$ respectively from Lemmas 1 and 2 in Robinson [18]. If $d_1 > d_2$ ($l \to \infty$) the left hand sides of b) and c) are $O((\frac{l}{m})^{1+2(\Delta-d_1)})$ and $O(\frac{l \log m}{m})$ respectively from Lemmas 2 and 3 in the Appendix. Since $1 + 2(\Delta - d_1) > 0$ in Θ_1, condition A.4 implies that b) and c) hold.

In order to prove a) write,

$$\frac{\tilde{C}(d) - C(d)}{C(d)} = \frac{A(d)}{B(d)}$$

where

$$A(d) = \frac{2(d-d_1)+1}{m-l} \sum_{l+1}^{m} \left(\frac{j}{m-l}\right)^{2(d-d_1)} \left(\frac{I_j}{g_j} - 1\right)$$

$$B(d) = \frac{2(d-d_1)+1}{m-l} \sum_{l+1}^{m} \left(\frac{j}{m-l}\right)^{2(d-d_1)}$$

for $g_j = C\lambda_j^{-2d_1}$. Since $B(d) + |B(d) - 1| \geq 1$ it follows that

$$\inf_{\Theta_1} B(d) \geq 1 - \sup_{\Theta_1} |B(d) - 1| \geq \frac{1}{2} \tag{2.3}$$

for all sufficiently large m using Lemma 2. Now, by summation by parts, $A(d)$ is bounded in absolute value by,

$$\frac{3}{m-l} \left| \sum_{r=l+1}^{m-1} \left\{ \left(\frac{r}{m-l}\right)^{2(d-d_1)} - \left(\frac{r+1}{m-l}\right)^{2(d-d_1)} \right\} \sum_{j=l+1}^{r} \left(\frac{I_j}{g_j} - 1\right) \right| \tag{2.4}$$

$$+ \frac{3}{m-l} \left(\frac{m}{m-l}\right)^{2(d-d_1)} \left| \sum_{j=l+1}^{m} \left(\frac{I_j}{g_j} - 1\right) \right|. \tag{2.5}$$

Using the mean value theorem we have that for $r \geq 1$,

$$\left| \left(1 + \frac{1}{r}\right)^{2(d-d_1)} - 1 \right| \leq \frac{2|d-d_1|}{r} \max_{r \geq 1} \left(1 + \frac{1}{r}\right)^{2(d-d_1)-1} \leq \frac{4}{r}$$

in Θ. Thus the supremum in Θ_1 of (2.4) is bounded by

$$\begin{aligned} \sup_{\Theta_1} \quad & \frac{3}{m-l}\left|\sum_{r=l+1}^{m-1}\left(\frac{r}{m-l}\right)^{2(d-d_1)}\left\{1-\left(\frac{r+1}{r}\right)^{2(d-d_1)}\right\}\sum_{j=l+1}^{r}\left(\frac{I_j}{g_j}-1\right)\right| \\ \leq \quad & 12\left(\frac{m}{m-l}\right)^{2(\Delta_2-d_1)+1}\sum_{r=l+1}^{m-1}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{1}{r^2}\left|\sum_{j=l+1}^{r}\left(\frac{I_j}{g_j}-1\right)\right|. \end{aligned}$$

Since $(\frac{m}{m-l})^{\alpha}\to 1$ for all α we focus on the analysis of

$$\sum_{r=l+1}^{m-1}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{1}{r^2}\left|\sum_{j=l+1}^{r}\left(\frac{I_j}{g_j}-1\right)\right|. \tag{2.6}$$

Now,

$$\frac{I_j}{g_j}-1=\left(1-\frac{g_j}{f_j}\right)\frac{I_j}{g_j}+\frac{1}{f_j}[I_j-|\alpha_j|^2 I_{\epsilon j}]+(2\pi I_{\epsilon j}-1) \tag{2.7}$$

where $I_{\epsilon j}=I_\epsilon(\omega+\lambda_j)=|W_\epsilon(\omega+\lambda_j)|^2$, $W_\epsilon(\lambda)=\frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n}\epsilon_t e^{it\lambda}$, $f_j=f(\omega+\lambda_j)$, $I_j=I_n(\omega+\lambda_j)$ and $\alpha_j=\alpha(\omega+\lambda_j)=\sum_{k=0}^{\infty}\alpha_k e^{ik(\omega+\lambda_j)}$. Assumption A.1 implies that

$$\left|1-\frac{g_j}{f_j}\right|=O(\lambda_j^{\alpha}). \tag{2.8}$$

Assumptions A.1 and A.2 and Lemma 1 imply that for n sufficiently large,

$$E\left|\frac{I_j}{g_j}\right|=1+O\left(\left(\frac{j}{n}\right)^{\alpha}+\frac{n^{2(d_2-d_1)}\log j}{j^{1+2(d_2-d_1)}}\right). \tag{2.9}$$

Thus

$$\begin{aligned} & E\left\{\sum_{r=l+1}^{m-1}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{1}{r^2}\left|\sum_{j=l+1}^{r}\left(1-\frac{g_j}{f_j}\right)\frac{I_j}{g_j}\right|\right\} \\ = \; & O\left(\sum_{l+1}^{m}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{1}{r^2}\sum_{l+1}^{r}\left(\frac{j}{n}\right)^{\alpha}\left(1+\left(\frac{j}{n}\right)^{\alpha}+\frac{n^{2(d_2-d_1)}\log j}{j^{1+2(d_2-d_1)}}\right)\right) \\ = \; & O\left(\left(\frac{m}{n}\right)^{\alpha}\left(1+\left(\frac{m}{n}\right)^{\alpha}+\frac{n^{2(d_2-d_1)}(\log m)^2}{ml^{2(d_2-d_1)}}+\frac{n^{2(d_2-d_1)}\log m}{l^{1+2(d_2-d_1)}}\right)\right) \\ = \; & O\left(\left(\frac{m}{n}\right)^{\alpha}\right)=o(1) \end{aligned}$$

in Θ_1 under A.4.

Since $||a|^2-|b|^2|=|Re\{(a-b)(\bar{a}+\bar{b})\}|\leq|(a-b)(\bar{a}+\bar{b})|\leq|a-b|\,|a+b|$, applying the Cauchy-Schwarz inequality we have that $E|I_j-|\alpha_j|^2 I_{\epsilon j}|$ is bounded by

$$\begin{aligned} & \{EI_j-\alpha_j EW_{\epsilon j}\bar{W}_j-\bar{\alpha}_j E\bar{W}_{\epsilon j}W_j+|\alpha_j|^2 EI_{\epsilon j}\}^{\frac{1}{2}} \\ & \quad \{EI_j+\alpha_j EW_{\epsilon j}\bar{W}_j+\bar{\alpha}_j E\bar{W}_{\epsilon j}W_j+|\alpha_j|^2 EI_{\epsilon j}\}^{\frac{1}{2}}. \end{aligned} \tag{2.10}$$

From Lemma 1 we have that as $n \to \infty$

$$EI_j = f_j + O\left(\frac{\log j}{j}\lambda_j^{-2d_2}\right), \quad EW_j\bar{W}_{\epsilon j} = \frac{\alpha_j}{2\pi} + O\left(\frac{\log j}{j}\lambda_j^{-2d_2}\right)$$

because $I_{\epsilon j} = \frac{1}{2\pi n}\sum_{t=1}^n \sum_{s=1}^n \epsilon_t \epsilon_s e^{i(t-s)(\omega+\lambda_j)}$ and under A.3 $EI_{\epsilon j} = \frac{1}{2\pi}$. Thus under A.4 and $d_2 > d_1$, (2.10) is $O\left(\left(\frac{\log j}{j}\right)^{\frac{1}{2}} \lambda_j^{-(d_2+d_1)}\right)$. Now

$$E\left\{\sum_{r=l+1}^{m-1}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{1}{r^2}\left|\sum_{j=l+1}^{r}\frac{1}{f_j}(I_j - |\alpha_j|^2 I_{\epsilon j})\right|\right\}$$

$$= O\left(\sum_{l+1}^{m}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{1}{r^2}\sum_{l+1}^{r}\left(\frac{\log j}{j}\right)^{\frac{1}{2}}\left(\frac{n}{j}\right)^{d_2-d_1}\right). \quad (2.11)$$

We distinguish the cases $d_2 - d_1 > 1/2, = 1/2, < 1/2$. When $d_2 - d_1 < 1/2$, (2.11) is

$$O\left(\frac{n^{d_2-d_1}(\log m)^{\frac{1}{2}}}{m^{2(\Delta-d_1)+1}}\sum_{l+1}^{m} r^{2(\Delta-d_1)-(d_2-d_1)-\frac{1}{2}}\right) = O\left(\frac{n^{d_2-d_1}(\log m)^{\frac{3}{2}}}{l^{\frac{1}{2}+d_2-d_1}}\right) = o(1) \quad (2.12)$$

in Θ_1 because of A.4. Now if $d_2 - d_1 = 1/2$, (2.11) is

$$O\left(\frac{\sqrt{n}(\log m)^{\frac{1}{2}}}{m^{2(\Delta-d_1)+1}}\sum_{l+1}^{m} r^{2(\Delta-d_1)-1}(\log r)\right) = O\left(\frac{\sqrt{n}(\log m)^{\frac{3}{2}}}{l})\right) = o(1) \quad (2.13)$$

in Θ_1 and because of A.4. Finally when $d_2 - d_1 > 1/2$, (2.11) is

$$O\left(\frac{n^{d_2-d_1}(\log m)^{\frac{1}{2}}}{m^{2(\Delta-d_1)+1}} l^{\frac{1}{2}-(d_2-d_1)}\sum_{l+1}^{m} r^{2(\Delta-d_1)-1}\right) = O\left(\frac{n^{d_2-d_1}(\log m)^{\frac{3}{2}}}{l^{\frac{1}{2}+d_2-d_1}}\right) \quad (2.14)$$

and consequently (2.11) is $o(1)$ in Θ_1 under A.4.

The final contribution to (2.6) comes from the term involving $2\pi I_{\epsilon j} - 1$. Write

$$\begin{aligned} 2\pi I_{\epsilon j} - 1 &= \frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n}\epsilon_t\epsilon_s e^{i(t-s)(\omega+\lambda_j)} - 1 \\ &= \frac{1}{n}\sum_{t=1}^{n}(\epsilon_t^2 - 1) + \frac{1}{n}\sum\sum_{t\neq s}\cos\{(t-s)(\omega+\lambda_j)\}\epsilon_t\epsilon_s \end{aligned}$$

using $\sin(\lambda) = -\sin(-\lambda)$. Thus

$$\sum_{r=l+1}^{m-1}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{1}{r^2}\left|\sum_{j=l+1}^{r}(2\pi I_{\epsilon j} - 1)\right|$$

$$\leq \left|\frac{1}{n}\sum_{t=1}^{n}(\epsilon_t^2-1)\right|\sum_{l+1}^{m-1}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{r-l}{r^2} \tag{2.15}$$

$$+\sum_{l+1}^{m-1}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{1}{r^2}\frac{1}{n}\sum_{l+1}^{r}\left|\sum\sum_{t\neq s}\cos\{(t-s)(\omega+\lambda_j)\}\,\epsilon_t\epsilon_s\right|. \tag{2.16}$$

Under A.3, $\frac{1}{n}\sum_{t=1}^{n}(\epsilon_t^2)\xrightarrow{p}1$ from Theorem 1 in Heyde and Seneta [13] and

$$\sum_{r=l+1}^{m-1}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{r-l}{r^2}=O\left(\sum_{l+1}^{m}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{1}{r}\right)=O(1)$$

and thus (2.15) is $o_p(1)$ in Θ_1. Assumption A.3 also implies that

$$\begin{aligned}
&E\left[\sum_{s}\sum_{t\neq s}\epsilon_t\epsilon_s\sum_{j=l+1}^{r}\cos\{(t-s)(\omega+\lambda_j)\}\right]^2\\
&= 2\sum_{j=l+1}^{r}\sum_{k=l+1}^{r}\left[\sum_{t=1}^{n}\sum_{s=1}^{n}\cos\{(t-s)(\omega+\lambda_j)\}\cos\{(t-s)(\omega+\lambda_k)\}-n\right]\\
&= (r-l)\,n^2-2(r-l)^2n
\end{aligned} \tag{2.17}$$

for r such that $0<\omega+\lambda_r<\pi$. To prove (2.17) write

$$\cos\{(t-s)(\omega+\lambda_j)\}\cos\{(t-s)(\omega+\lambda_k)\}=a_{ts}+b_{ts}+c_{ts}+d_{ts}$$

where

$$\begin{aligned}
a_{ts} &= \cos[s(\omega+\lambda_j)]\cos[s(\omega+\lambda_k)]\cos[t(\omega+\lambda_j)]\cos[t(\omega+\lambda_k)]\\
b_{ts} &= \cos[s(\omega+\lambda_j)]\sin[s(\omega+\lambda_k)]\cos[t(\omega+\lambda_j)]\sin[t(\omega+\lambda_k)]\\
c_{ts} &= \sin[s(\omega+\lambda_j)]\cos[s(\omega+\lambda_k)]\sin[t(\omega+\lambda_j)]\cos[t(\omega+\lambda_k)]\\
d_{ts} &= \sin[s(\omega+\lambda_j)]\sin[s(\omega+\lambda_k)]\sin[t(\omega+\lambda_j)]\sin[t(\omega+\lambda_k)].
\end{aligned} \tag{2.18}$$

Now

$$\begin{aligned}
\sum_{t=1}^{n}\sum_{s=1}^{n}a_{ts}=\sum_{t=1}^{n}\sum_{s=1}^{n}d_{ts} &= \frac{n^2}{4}\quad\text{if } k=j\\
&= 0\quad\text{otherwise}
\end{aligned}$$

and

$$\sum_{t=1}^{n}\sum_{s=1}^{n}b_{ts}=\sum_{t=1}^{n}\sum_{s=1}^{n}c_{ts}=0$$

which proves (2.17), and thus (2.16) is

$$O_p\left(\sum_{l+1}^{m}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{(r-l)^{\frac{1}{2}}}{r^2}\right)=O_p\left(\frac{1}{m^{2(\Delta-d_1)+1}}\sum_{l+1}^{m}r^{2(\Delta-d_1)-\frac{1}{2}}\right)$$

$$=O_p\left(\frac{\log m}{\sqrt{m}}+\frac{1}{\sqrt{l}}\left(\frac{l}{m}\right)^{2(\Delta-d_1)+1}\right)=o_p(1)$$

as $n \to \infty$. Thus we have proved that $\sup_{\Theta_1}$ (2.4) $= o_p(1)$.

Now $\sup_{\Theta_1}$ (2.5) is bounded by

$$\frac{3}{m-l}\left(\frac{m}{m-l}\right)^{2(\Delta_2-d_1)}\left|\sum_{j=l+1}^{m}\left(\frac{I_j}{g_j}-1\right)\right|. \tag{2.19}$$

Because $(\frac{m}{m-l})^\alpha \to 1$ for all α as $n \to \infty$ we focus on $\frac{1}{m}\left|\sum_{l+1}^m(\frac{I_j}{g_j}-1)\right|$ and use (2.7) to show in the same manner as above that $\sup_{\Theta_1}$ (2.5) is $o_p(1)$. Since $d_1 < d_2$,

$$\begin{aligned} & E\left\{\frac{1}{m}\sum_{l+1}^{m}\left|1-\frac{g_j}{f_j}\right|\left|\frac{I_j}{g_j}\right|\right\} \\ = \; & O\left(\frac{1}{m}\sum_{l+1}^{m}\left(\frac{j}{n}\right)^\alpha\left(1+\left(\frac{j}{n}\right)^\alpha+\frac{n^{2(d_2-d_1)}\log j}{j^{1+2(d_2-d_1)}}\right)\right)=O\left(\frac{m^\alpha}{n^\alpha}\right)=o(1)\end{aligned}$$

under A.4. On the other hand,

$$\begin{aligned} & E\left\{\frac{1}{m}\sum_{l+1}^{m}\frac{1}{f_j}\left|I_j-|\alpha_j|^2 I_{\epsilon j}\right|\right\}=O\left(\frac{1}{m}\sum_{l+1}^{m}\frac{n^{d_2-d_1}(\log j)^{\frac{1}{2}}}{j^{\frac{1}{2}+d_2-d_1}}\right) \\ = \; & O\left(\frac{n^{d_2-d_1}(\log m)^{\frac{1}{2}}}{l^{\frac{1}{2}+d_2-d_1}}\right)=o(1)\end{aligned}$$

under A.4, and finally $\frac{1}{m}\sum_{l+1}^m|2\pi I_{\epsilon j}-1|$ is bounded by

$$\begin{aligned} & \frac{m-l}{m}\left|\frac{1}{n}\sum_{t=1}^{n}(\epsilon_t^2-1)\right|+\left|\frac{1}{m}\sum_{l+1}^{m}\frac{1}{n}\sum_{t}\sum_{s\neq t}\cos\{(t-s)(\omega+\lambda_j)\}\,\epsilon_t\epsilon_s\right| \\ = \; & o_p(1)+O_p\left(\frac{1}{\sqrt{m}}\right)=o_p(1).\end{aligned}$$

We have shown that $\sup_{\Theta_1}|A(d)| \xrightarrow{p} 0$ and thus

$$\sup_{\Theta_1}\left|\frac{\tilde{C}(d)}{C(d)}-1\right|=\sup_{\Theta_1}\left|\frac{A(d)}{B(d)}\right|\leq\frac{\sup_{\Theta_1}|A(d)|}{\inf_{\Theta_1}|B(d)|}\xrightarrow{p}0$$

and the proof is completed in the case $d_1 < \Delta_1+\frac{1}{2}$. But if $d_1 \geq \Delta_1+\frac{1}{2}$, Θ_2 is not an empty set and $P(\inf_{\Theta_2} S(d) \leq 0)$ may be different from zero. However we will see that in fact $P(\inf_{\Theta_2} S(d) \leq 0) \to 0$ as $n \to \infty$. Write $p = p_m = \exp\left(\frac{1}{m-l}\sum_{l+1}^{m}\log j\right)$ and $S(d) = \log\left\{\frac{\hat{D}(d)}{\hat{D}(d_1)}\right\}$ where $\hat{D}(d) = \frac{1}{m-l}\sum_{l+1}^{m}\left(\frac{j}{p}\right)^{2(d-d_1)} j^{2d_1} I_j$. Since $l+1 \leq p \leq$

m then,

$$\begin{array}{llll} \inf_{\Theta_2}\left(\frac{j}{p}\right)^{2(d-d_1)} & \geq & \left(\frac{j}{p}\right)^{2(\Delta-d_1)} & \text{for } l+1 \leq j \leq p \\ \inf_{\Theta_2}\left(\frac{j}{p}\right)^{2(d-d_1)} & \geq & \left(\frac{j}{p}\right)^{2(\Delta_1-d_1)} & \text{for } p < j \leq m. \end{array}$$

It follows that

$$\inf_{\Theta_2} \hat{D}(d) \geq \frac{1}{m-l}\sum_{l+1}^{m} a_j j^{2d_1} I_j$$

where

$$a_j = \begin{cases} \left(\frac{j}{p}\right)^{2(\Delta-d_1)} & \text{for} \quad l+1 \leq j \leq p \\ \left(\frac{j}{p}\right)^{2(\Delta_1-d_1)} & \text{for} \quad p < j \leq m \end{cases}.$$

Thus

$$P\left(\inf_{\Theta_2} S(d) \leq 0\right) \leq P\left(\frac{1}{m-l}\sum_{l+1}^{m}(a_j-1)\, j^{2d_1} I_j \leq 0\right).$$

Under A.4,

$$\begin{aligned} p &\sim \exp\left(\frac{1}{m-l}\{m[\log m - 1] - l[\log l - 1]\}\right) \\ &= \exp(-1+\log m)\exp\left(\frac{l\log m}{m-l} - \frac{l\log l}{m-l}\right) \\ &= \frac{m}{e}(1+o(1)) \sim \frac{m}{e}, \end{aligned}$$

and

$$\sum_{j=l+1}^{p} a_j \sim p^{2(d_1-\Delta)}\int_l^p x^{2(\Delta-d_1)}dx = \frac{p}{2(\Delta-d_1)+1} - \frac{l^{2(\Delta-d_1)+1}}{(2(\Delta-d_1)+1)\,p^{2(\Delta-d_1)}}.$$

Thus

$$\begin{aligned} & \frac{1}{m-l}\sum_{l+1}^{m}(a_j-1) \geq \frac{1}{m-l}\sum_{l+1}^{p} a_j - 1 \\ \sim\ & \frac{1}{e(2(\Delta-d_1)+1)}\left(1+\frac{l}{m-l} - \frac{(le)^{2(\Delta-d_1)+1}}{(m-l)m^{2(\Delta-d_1)}}\right) - 1 \\ =\ & \frac{1}{e(2(\Delta-d_1)+1)} - 1 + o(1). \end{aligned}$$

Choosing $\Delta < d_1 - 1/2 + 1/(4e)$ (which can be done without loss of generality because $d_1 - 1/2 \geq \Delta_1$ in Θ_2) we have that for m sufficiently large and $\frac{l}{m} \to 0$ as

$m \to \infty$, $\frac{1}{m-l}\sum_{l+1}^{m}(a_j - 1) \geq 1$ and

$$P\left(\frac{1}{m-l}\sum_{l+1}^{m}(a_j - 1)\, j^{2d_1} I_j \leq 0\right) = P\left(\frac{1}{m-l}\sum_{l+1}^{m}(a_j - 1)\frac{\lambda_j^{2d_1} I_j}{C} + 1 \leq 1\right)$$

$$\leq P\left(\frac{1}{m-l}\sum_{l+1}^{m}(a_j - 1)\frac{I_j}{g_j} + 1 \leq \frac{1}{m}\sum_{l+1}^{m}(a_j - 1)\right)$$

$$\leq P\left(\left|\frac{1}{m-l}\sum_{l+1}^{m}(a_j - 1)\left(\frac{I_j}{g_j} - 1\right)\right| \geq 1\right).$$

Since $\sum_{l+1}^{m} a_j \sim p^{2(d_1-\Delta)}\int_l^p x^{2(\Delta-d_1)}dx + p^{2(d_1-\Delta_1)}\int_p^m x^{2(\Delta_1-d_1)}dx = O(m)$ it follows that

$$\left|\frac{1}{m-l}\sum_{l+1}^{m}(a_j - 1)\left(1 - \frac{g_j}{f_j}\right)\frac{I_j}{g_j}\right|$$

$$= O_p\left(\frac{1}{m}\sum_{l+1}^{m}(a_j + 1)\left(\frac{j}{n}\right)^{\alpha}\left(1 + \left(\frac{j}{n}\right)^{\alpha} + \frac{n^{2(d_2-d_1)}\log j}{j^{1+2(d_2-d_1)}}\right)\right)$$

$$= O_p\left(\left(\frac{m}{n}\right)^{\alpha}\right) = o_p(1)$$

as $n \to \infty$, under A.4 and because $\alpha > 0$. On the other hand,

$$\left|\frac{1}{m-l}\sum_{l+1}^{m}(a_j - 1)\frac{1}{f_j}[I_j - |\alpha_j|^2 I_{\epsilon j}]\right|$$

$$= O_p\left(\frac{1}{m-l}\sum_{l+1}^{m}(a_j - 1)\frac{n^{d_2-d_1}(\log j)^{\frac{1}{2}}}{j^{\frac{1}{2}+d_2-d_1}}\right)$$

$$= O_p\left(\frac{n^{d_2-d_1}(\log m)^{\frac{1}{2}}}{l^{\frac{1}{2}+d_2-d_1}}\right) = o_p(1)$$

as $n \to \infty$ under A.4, and finally

$$\frac{1}{m-l}\sum_{l+1}^{m}(a_j - 1)(2\pi I_{\epsilon j} - 1)$$

$$= \frac{1}{n}\sum_{t=1}^{n}(\epsilon_t^2 - 1)\frac{1}{m-l}\sum_{l+1}^{m}(a_j + 1) \qquad (2.20)$$

$$+ \frac{1}{n}\sum_{t}\sum_{s\neq t}\frac{1}{m-l}\sum_{l+1}^{m}(a_j - 1)\cos\{(t-s)(\omega + \lambda_j)\}\,\epsilon_t\epsilon_s. \qquad (2.21)$$

Since $\frac{1}{m-l}\sum(a_j - 1) = O(1)$, (2.20) is $o_p(1)$. Now (2.21) has variance

$$\frac{2}{n^2}\frac{1}{(m-l)^2}\sum_{j=l+1}^{m}(a_j - 1)\sum_{k=l+1}^{m}(a_k - 1)\left(\sum_{t=1}^{n}\sum_{s=1}^{n}(a_{ts} + b_{ts} + c_{ts} + d_{ts}) - n\right)$$

where a_{ts}, b_{ts}, c_{ts} and d_{ts} are defined in (2.18). Thus the variance of (2.21) is

$$\frac{2}{n^2}\frac{1}{(m-l)^2}\sum_{l+1}^{m}(a_j-1)^2\frac{n^2}{2}-\frac{2}{n(m-l)^2}\left(\sum_{l+1}^{m}(a_j-1)\right)^2. \tag{2.22}$$

The second term of (2.22) is $O(n^{-1})$ because $\sum_{l+1}^{m}(a_j-1)=O(m)$. Now

$$\begin{aligned}\sum_{l+1}^{m}a_j^2 &= \sum_{l+1}^{p}\left(\frac{j}{p}\right)^{4(\Delta-d_1)}+\sum_{p+1}^{m}\left(\frac{j}{p}\right)^{4(\Delta_1-d_1)}\\ &= O\left(p\log p+l\left(\frac{l}{p}\right)^{4(\Delta-d_1)}+m\log m\left(\frac{m}{p}\right)^{4(\Delta_1-d_1)}\right)\\ &= O\left(m\log m+l\left(\frac{l}{m}\right)^{4(\Delta-d_1)}\right)\end{aligned}$$

and thus

$$\frac{1}{(m-l)^2}\sum_{l+1}^{m}a_j^2=O\left(\frac{\log m}{m}+\frac{l^{4(\Delta-d_1)+1}}{m^{4(\Delta-d_1)+2}}\right)=o(1)$$

because $4(\Delta-d_1)+2>0$. Thus (2.21) is $o_p(1)$ and consequently $P(\inf_{\Theta_2}S(d)\leq 0)\to 0$ and the proof is completed. □

3. ASYMPTOTIC DISTRIBUTION

In this section we show that under some conditions stronger than those needed for consistency but milder than the assumptions imposed in the log-periodogram regression in the sense that Gaussianity is not needed,

$$\sqrt{m}(\tilde{d}_1-d_1)\xrightarrow{d}N\left(0,\frac{1}{4}\right).$$

The constancy of the asymptotic variance of $\tilde{d}_1$ makes easy the use of approximate rules of inference. We also observe the gain in efficiency with respect to the log-periodogram regression where the asymptotic variance has an upper bound of $\pi^2/24$ and a lower bound 1/4 but this lower bound is not attainable by that class of estimates (see Robinson [17] and Arteche [1]).

We introduce the following assumptions:

A.2'. In a neighbourhood $(-\delta,0)\cup(0,\delta)$ of ω $\alpha(\lambda)=\sum_{k=0}^{\infty}\alpha_k e^{ik\lambda}$ is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\alpha(\omega\pm\lambda)=O\left(\frac{|\alpha(\omega\pm\lambda)|}{\lambda}\right)\quad\text{as }\lambda\to 0^+.$$

A.3'. Assumption A.3 holds and

$$E(\epsilon_t^3|F_{t-1}) = \mu_3 \quad \text{and} \quad E(\epsilon_t^4|F_{t-1}) = \mu_4, \quad t = 0, \pm 1, \ldots$$

for finite constants μ_3 and μ_4.

A.4'. If $d_1 \geq d_2$,

$$\frac{1}{m} + \frac{m^{1+2\alpha}(\log m)^2}{n^{2\alpha}} \to 0 \quad \text{as } n \to \infty,$$

and if $d_1 < d_2$

$$\frac{(\log m)^3}{l^2} + \frac{l^3}{m}(\log m)^4 + \frac{n^{2(d_2-d_1)}}{l^{1+2(d_2-d_1)}} \log m + \frac{m^{1+2\alpha}}{n^{2\alpha}}(\log m)^2 \to 0$$

as $n \to \infty$.

Assumption A.2' implies A.2 because $f(\lambda) = \frac{\sigma^2}{2\pi}|\alpha(\lambda)|^2$. A.3' implies that x_t is fourth order stationary and holds if the ϵ_t are independent and identically distributed with finite fourth conditional moments. A.4' is Assumption A.4' in Robinson [18] if $d_1 \geq d_2$ but when $d_1 < d_2$ a strong trimming is needed as in the proof of the consistency. Taking $m \sim n^\theta$ and $l \sim n^\phi$ we have that in case $d_1 \geq d_2$, $\theta < 2\alpha/(1+2\alpha)$ suffices, but when $d_2 > d_1$, A.4' can only be satisfied if $d_2 - d_1 < \alpha/(3+4\alpha)$. For instance when $\alpha = 2$, $d_2 - d_1$ has to be smaller than $2/11$. However we can relax A.4' by strengthening A.3'. We thus can consider:

A.5. The fourth cumulant of ϵ_t is zero for all t.

A.6. If $d_1 \geq d_2$ A.4' holds and when $d_1 < d_2$

$$\frac{(\log m)^3}{l^2} + \frac{l^2}{m}(\log m)^2 + \frac{n^{2(d_2-d_1)}}{l^{1+2(d_2-d_1)}} \log m + \frac{m^{1+2\alpha}}{n^{2\alpha}}(\log m)^2 \to 0$$

as $n \to \infty$.

Assumption A.5 is implied by Gaussianity and A.6 entails $(d_2 - d_1) < \alpha/2(1+\alpha)$, where the upper bound is $1/3$ when $\alpha = 2$. This requirement is not much stronger than $d_2 - d_1 < 1/2$ which is satisfied if there is both a left and right (stationary) spectral pole at ω.

Theorem 2. Under A.1, A.2', A.3' and either A.4' or A.5 and A.6,

$$\sqrt{m}(\tilde{d}_1 - d_1) \xrightarrow{d} N\left(0, \frac{1}{4}\right) \quad \text{as} \quad n \to \infty.$$

Proof. Like in the proof of consistency we focus on the case $d_2 > d_1$, the proof with $d_2 \leq d_1$ is a straightforward extension of that in Robinson [18]. Since $\tilde{d}_1$ is consistent under the conditions in Theorem 2, then with probability approaching 1 as $n \to \infty$ $\tilde{d}_1$ satisfies

$$0 = \frac{\mathrm{d}R(\tilde{d}_1)}{\mathrm{d}d} = \frac{\mathrm{d}R(d_1)}{\mathrm{d}d} + \frac{\mathrm{d}^2 R(\bar{d})}{\mathrm{d}d^2}(\tilde{d}_1 - d_1) \tag{3.1}$$

where $|\bar{d} - d_1| \leq |\tilde{d}_1 - d_1|$. Write

$$\tilde{C}_k(d) = \frac{1}{m-l}\sum_{j=l+1}^{m} \lambda_j^{2d}(\log\lambda_j)^k I_j.$$

Then

$$\frac{\mathrm{d}R(d)}{\mathrm{d}d} = 2\frac{\tilde{C}_1(d)}{\tilde{C}(d)} - \frac{2}{m-l}\sum_{l+1}^{m}\log\lambda_j, \quad \frac{\mathrm{d}^2R(d)}{\mathrm{d}d^2} = \frac{4\{\tilde{C}_2(d)\tilde{C}(d) - \tilde{C}_1^2(d)\}}{\tilde{C}^2(d)}.$$

Define also

$$\tilde{F}_k = \frac{1}{m-l}\sum_{j=l+1}^{m}(\log j)^k \lambda_j^{2d} I_j, \quad \tilde{E}_k(d) = \frac{1}{m-l}\sum_{j=l+1}^{m}(\log j)^k j^{2d} I_j,$$

thus

$$\frac{\mathrm{d}^2R(d)}{\mathrm{d}d^2} = \frac{4\{\tilde{F}_2(d)\tilde{F}_0(d) - \tilde{F}_1^2(d)\}}{\tilde{F}_0^2(d)} = \frac{4\{\tilde{E}_2(d)\tilde{E}_0(d) - \tilde{E}_1^2(d)\}}{\tilde{E}_0^2(d)}.$$

Fix $\zeta > 0$ such that $2\zeta < (\log m)^2$. On the set $M = \{d : (\log m)^3\,|d - d_1| \leq \zeta\}$

$$\begin{aligned}|\tilde{E}_k(d) - \tilde{E}_k(d_1)| &\leq \frac{1}{m-l}\sum_{l+1}^{m}|j^{2(d-d_1)} - 1|j^{2d_1}(\log j)^k I_j \\ &\leq 2e|d - d_1|\tilde{E}_{k+1}(d_1) \leq 2e\zeta(\log m)^{k-2}\tilde{E}_0(d_1)\end{aligned}$$

where the second inequality comes from the fact that

$$\frac{|j^{2(d-d_1)} - 1|}{2|d - d_1|} \leq (\log j)m^{2|d-d_1|} \leq m^{\frac{1}{\log m}}\log j = e\log j$$

on M. Thus for $\eta > 0$,

$$\begin{aligned}&P\left(|\tilde{E}_k(\bar{d}) - \tilde{E}_k(d_1)| > \eta\left(\frac{2\pi}{n}\right)^{-2d_1}\right) \\ \leq\ & P\left(2e\zeta(\log m)^{k-2}\tilde{E}_0(d_1) > \eta\left(\frac{2\pi}{n}\right)^{-2d_1} \mid \bar{d} \in M\right) \\ +\ & P\left(|\tilde{E}_k(\bar{d}) - \tilde{E}_k(d_1)| > \eta\left(\frac{2\pi}{n}\right)^{-2d_1} \mid \bar{d} \notin M\right) \\ \leq\ & P(\tilde{C}(d_1) > \frac{\eta}{2e\zeta}(\log m)^{2-k}) + P((\log m)^3|\bar{d} - d_1| > \zeta).\end{aligned} \tag{3.2}$$

Since from the proof of Theorem 1, $\tilde{C}(d_1) \xrightarrow{p} C \in (0,\infty)$, the first probability in (3.2) tends to zero for ζ sufficiently small and $k = 0, 1, 2$. The second probability is bounded by

$$\begin{aligned} & P\left((\log m)^3 |\tilde{d}_1 - d_1| > \zeta\right) \qquad (3.3) \\ \leq\ & P\left(\inf_{\Theta_1 \cap N_\delta \cap \bar{M}} S(d) \leq 0\right) + P\left(\inf_{\Theta_1 \cap \bar{N}_\delta} S(d) \leq 0\right) + P\left(\inf_{\Theta_2} S(d) \leq 0\right) \end{aligned}$$

where $\bar{M} = (-\infty, \infty) - M$. We have already shown in the proof of Theorem 1 that the last two probabilities in (3.3) tend to zero. The first probability is bounded by

$$P\left(\sup_{\Theta_1 \cap N_\delta} |T(d)| \geq \inf_{\Theta_1 \cap N_\delta \cap \bar{M}} U(d)\right). \qquad (3.4)$$

As in the proof of Theorem 1,

$$\inf_{\Theta_1 \cap N_\delta \cap \bar{M}} U(d) \geq \frac{\zeta^2}{(\log m)^6}.$$

Call $\gamma = 2(\Delta - d_1) + 1$. On Θ_1, $\gamma > 0$. Consider $\left(\frac{l}{m}\right)^\gamma (\log m)^6 = \frac{l^\gamma}{m^{\gamma - a}} \frac{(\log m)^6}{m^a}$ where $0 < a < \gamma$. Now $\frac{(\log m)^6}{m^a} \to 0$ as $m \to \infty$ and

$$\left(\frac{l^\gamma}{m^{\gamma - a}}\right)^{\frac{2}{\gamma}} = \frac{l^2}{m} \frac{1}{m^{1 - \frac{2a}{\gamma}}}.$$

Under A.6 (and of course A.4'), $\frac{l^2}{m} \to 0$ as $n \to \infty$. Choose $a < \frac{\gamma}{2}$, which can always be done because $\gamma > 0$. Then $\left(\frac{l}{m}\right)^\gamma (\log m)^6 \to 0$ as $n \to \infty$. Thus noting the form of $T(d)$ and the orders of magnitude obtained in Lemmas 2 and 3 it follows that under A.4 or A.6 (3.4) tends to 0 if

$$\sup_{\Theta_1 \cap N_\delta} \left| \frac{\tilde{C}(d) - C(d)}{C(d)} \right| = o_p\left((\log m)^{-6}\right).$$

Using the notation in the proof of Theorem 1, and because of (2.3),

$$\inf_{\Theta_1 \cap N_\delta} B(d) \geq \inf_{\Theta_1} B(d) \geq \frac{1}{2}$$

for all large enough m. Thus it remains to prove that $\sup_{\Theta_1 \cap N_\delta} |A(d)| = o_p\left((\log m)^{-6}\right)$. Now

$$\begin{aligned} & \sup_{\Theta_1 \cap N_\delta} |A(d)| \\ \leq\ & \sup_{\Theta_1 \cap N_\delta} \left\{ 12 \left(\frac{m}{m-l}\right)^{2(d-d_1)+1} \sum_{r=l+1}^{m-1} \left(\frac{r}{m}\right)^{2(d-d_1)+1} \frac{1}{r^2} \left| \sum_{j=l+1}^{r} \left(\frac{I_j}{g_j} - 1\right) \right| \right. \\ & \left. + \frac{3}{m-l} \left(\frac{m}{m-l}\right)^{2(d-d_1)} \left| \sum_{j=l+1}^{m} \left(\frac{I_j}{g_j} - 1\right) \right| \right\}. \end{aligned}$$

Since $\left(\frac{m}{m-l}\right)^{\alpha} \to 1$ for all α we focus on

$$\sup_{\Theta_1 \cap N_\delta} \left\{ \sum_{l+1}^{m-1} \left(\frac{r}{m}\right)^{2(d-d_1)+1} \frac{1}{r^2} \left| \sum_{l+1}^{r} \left(\frac{I_j}{g_j} - 1\right) \right| + \frac{1}{m} \left| \sum_{l+1}^{m} \left(\frac{I_g}{g_j} - 1\right) \right| \right\}$$

$$\leq \sum_{r=l+1}^{m} \left(\frac{r}{m}\right)^{1-2\delta} \frac{1}{r^2} \left| \sum_{j=l+1}^{r} \left(\frac{I_j}{g_j} - 1\right) \right| + \frac{1}{m} \left| \sum_{l+1}^{m} \left(\frac{I_j}{g_j} - 1\right) \right|. \tag{3.5}$$

Now, using Lemmas 4 and 6, the first part of (3.5) is

$$\sum_{l+1}^{m} \left(\frac{r}{m}\right)^{1-2\delta} \frac{1}{r^2} \left| \sum_{l+1}^{r} \left(\frac{I_j}{g_j} - 2\pi I_{\epsilon j} + 2\pi I_{\epsilon j} - 1\right) \right|$$

$$= O_p\left(\sum_{l+1}^{m} \left(\frac{r}{m}\right)^{1-2\delta} \frac{1}{r^2} \left(\frac{r^{\alpha+1}}{n^\alpha} + l^{\frac{3}{4}} r^{\frac{1}{4}} + r^{\frac{1}{2}} \right) \right)$$

$$= O_p\left(\left(\frac{m}{n}\right)^{\alpha} + \left(\frac{l}{m}\right)^{\frac{3}{4}} \log m + \left(\frac{l}{m}\right)^{1-2\delta} + \frac{\log m}{\sqrt{m}} \right) \tag{3.6}$$

under A.4' or A.6. Since $\delta < \frac{1}{4}$ then (3.6) is $o_p((\log m)^{-6})$. Similarly the second part of (3.5) is

$$O_p\left(\frac{1}{m}(\sqrt{m} + \frac{m^{\alpha+1}}{n^\alpha} + l^{\frac{3}{4}} m^{\frac{1}{4}}) \right) = o_p\left((\log m)^{-6}\right)$$

under A.4' or A.6. Thus $P\left(\inf_{\Theta_1 \cap N_\delta \cap \bar{M}} S(d) \leq 0\right) \to 0$ and

$$P\left(\left| \tilde{E}_k(\bar{d}) - \tilde{E}_k(d_1) \right| > \eta \left(\frac{2\pi}{n}\right)^{-2d_1} \right) \to 0$$

as $n \to \infty$. Consequently $\frac{\mathrm{d}^2 R(\bar{d})}{\mathrm{d}d^2}$ is

$$\frac{4[\{\tilde{E}_2(d_1) + o_p(n^{2d_1})\}\{\tilde{E}_0(d_1) + o_p(n^{2d_1})\} - \{\tilde{E}_1(d_1) + o_p(n^{2d_1})\}^2]}{\{\tilde{E}_0(d_1) + o_p(n^{2d_1})\}^2}$$

$$= \frac{4[\tilde{F}_2(d_1)\tilde{F}_0(d_1) - \tilde{F}_1^2(d_1)]}{\tilde{F}_0^2(d_1)} + o_p(1) \qquad \text{as} \qquad n \to \infty. \tag{3.7}$$

Now for $k = 0, 1, 2$,

$$\left| \tilde{F}_k(d_1) - C\frac{1}{m-l} \sum_{l+1}^{m} (\log j)^k \right|$$

$$= \left| \frac{C}{m-l} \sum_{l+1}^{m} (\log j)^k \left(\frac{I_j}{g_j} - 1\right) \right| \leq (\log m)^k \frac{C}{m-l} \sum_{l+1}^{m} \left| \frac{I_j}{g_j} - 1 \right|$$

$$= O_p\left(\frac{(\log m)^2}{\sqrt{m}} + \left(\frac{m}{n}\right)^{\alpha} (\log m)^2 + \left(\frac{l}{m}\right)^{\frac{3}{4}} (\log m)^2 \right) = o_p(1) \tag{3.8}$$

under A.4' or A.6. Thus from (3.7) and (3.8)

$$\frac{d^2R(\bar{d})}{dd^2}=4\left\{\frac{1}{m-l}\sum_{l+1}^{m}(\log j)^2-\left(\frac{1}{m-l}\sum_{l+1}^{m}\log j\right)^2\right\}(1+o_p(1))+o_p(1)\xrightarrow{p}4$$

as $n\to\infty$. Now since $\tilde{C}(d_1)\xrightarrow{p}C$

$$\begin{aligned}\sqrt{m}\frac{dR(d_1)}{dd} &= 2\frac{\sqrt{m}}{m-l}\sum_{l+1}^{m}\left(\frac{\lambda_j^{2d_1}I_j\log\lambda_j}{\tilde{C}(d_1)}-\log\lambda_j\right)\\ &= 2\frac{\sqrt{m}}{m-l}\frac{1}{C+o_p(1)}\sum_{l+1}^{m}\left(\log\lambda_j-\frac{1}{m-l}\sum_{l+1}^{m}\log\lambda_j\right)\lambda_j^{2d_1}I_j\\ &= \frac{2}{\sqrt{m}}\sum_{l+1}^{m}v_j\left(\frac{I_j}{g_j}-2\pi I_{\epsilon j}\right)(1+o_p(1)) \qquad (3.9)\\ &+ \frac{2}{\sqrt{m}}\sum_{l+1}^{m}v_j 2\pi I_{\epsilon_j}(1+o_p(1)) \qquad (3.10)\end{aligned}$$

where $v_j=\log j-\frac{1}{m-l}\sum_{l+1}^{m}\log j$ satisfies $\sum_{l+1}^{m}v_j=0$. Since $|v_j|=O(\log m)$, using Lemma 6 we have that (3.9) is

$$O_p\left(\frac{m^{\alpha+\frac{1}{2}}}{n^\alpha}\log m+\frac{l^{\frac{3}{4}}}{m^{\frac{1}{4}}}\log m\right)$$

under A.4' and

$$O_p\left(\frac{m^{\alpha+\frac{1}{2}}}{n^\alpha}\log m+\frac{l}{\sqrt{m}}\log m\right)$$

under A.5 and A.6. In both cases (3.9) is $o_p(1)$. Apart from the $o_p(1)$ terms, (3.10) is

$$\begin{aligned}&\frac{2}{\sqrt{m}}\frac{1}{n}\sum_{l+1}^{m}v_j\sum_{t=1}^{n}\sum_{s=1}^{n}\epsilon_t\epsilon_s e^{i(t-s)(\omega+\lambda_j)}\\ =\;&\frac{2}{\sqrt{m}}\frac{1}{n}\sum_{l+1}^{m}v_j 2\sum_{t=2}^{n}\sum_{s=1}^{t-1}\epsilon_t\epsilon_s\cos\{(t-s)(\omega+\lambda_j)\}=2\sum_{t=1}^{n}z_t\end{aligned}$$

where $z_1=0$ and

$$\begin{aligned}z_t &= \epsilon_t\sum_{s=1}^{t-1}\epsilon_s c_{t-s} \quad \text{for} \quad t=2,3,\dots,n,\\ c_s &= \frac{2}{n}\frac{1}{\sqrt{m}}\sum_{l+1}^{m}v_j\cos\{s(\omega+\lambda_j)\}. \qquad (3.11)\end{aligned}$$

The z_t form a zero-mean martingale difference array and from a standard martingale CLT (Hall and Heyde [10], section 3.2) $\sum_{t=1}^{n} z_t$ converges in distribution to a $N(0,1)$ random variable if

$$\text{a)}\quad \sum_{1}^{n} E(z_t^2|F_{t-1}) - 1 \xrightarrow{p} 0$$

$$\text{b)}\quad \sum_{1}^{n} E(z_t^2 I(|z_t| > \delta)) \to 0 \qquad \text{for all } \delta > 0.$$

To prove a) write $\sum_1^n E(z_t^2|F_{t-1}) - 1$ as

$$\sum_2^n E\left[\epsilon_t^2\left(\sum_{s=1}^{t-1}\epsilon_s c_{t-s}\right)^2 |F_{t-1}\right] - 1 = \sum_2^n\left(\sum_{s=1}^{t-1}\epsilon_s c_{t-s}\right)^2 - 1$$

$$= \left\{\sum_{t=2}^{n}\sum_{s=1}^{t-1}\epsilon_s^2 c_{t-s}^2 - 1\right\} + \sum_{t=2}^{n}\sum\sum_{r\neq s}^{t-1}\epsilon_r\epsilon_s c_{t-r}c_{t-s}.$$

The term in braces is

$$\left\{\sum_{t=1}^{n-1}(\epsilon_t^2 - 1)\sum_{s=1}^{n-t}c_s^2\right\} + \left\{\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}c_s^2 - 1\right\}. \tag{3.12}$$

Now $\sum_{t=1}^{n-1}\sum_{s=1}^{n-t} c_s^2$ is equal to

$$\frac{4}{n^2}\frac{1}{m}\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\left(\sum_{l+1}^{m} v_j\cos\{s(\omega+\lambda_j)\}\right)^2$$

$$= \frac{4}{mn^2}\sum_{j=l+1}^{m} v_j^2\sum_{1}^{n-1}\sum_{1}^{n-t}\cos^2\{s(\omega+\lambda_j)\} \tag{3.13}$$

$$+ \frac{2}{mn^2}\sum_{l+1}^{m}\sum_{j\neq k} v_j v_k\sum_{1}^{n-1}\sum_{1}^{n-t}[\cos\{s(2\omega+\lambda_j+\lambda_k)\}+\cos\{s(\lambda_j-\lambda_k)\}]. \tag{3.14}$$

From formula (4.18) in Robinson [18], namely,

$$\sum_{r=1}^{q-1}\sum_{t=1}^{q-r}\cos(\theta t) = \frac{\cos\theta - \cos(q\theta)}{4\sin^2\frac{\theta}{2}} - \frac{q-1}{2} \tag{3.15}$$

for $\theta \neq 0, \mathrm{mod}(2\pi)$, we have that for j such that $0 < \omega + \lambda_j < \pi$ (which holds for n large enough),

$$\sum_{t=1}^{n}\sum_{s=1}^{n-t}\cos^2\{s(\omega+\lambda_j)\} = \frac{1}{2}\sum_{1}^{n-1}\sum_{1}^{n-t}(1+\cos\{2s(\omega+\lambda_j)\})$$

$$= \frac{1}{2}\sum_{t=1}^{n-1}(n-t) + \frac{1}{2}\frac{\cos\{2(\omega+\lambda_j)\} - \cos\{2n(\omega+\lambda_j)\}}{4\sin^2(\omega+\lambda_j)} - \frac{n-1}{4}$$
$$= \frac{(n-1)^2}{4} + O(1).$$

Since

$$\frac{1}{m-l}\sum_{l+1}^{m} v_j^2 = \frac{1}{m-l}\sum_{l+1}^{m}(\log j)^2 - \left(\frac{1}{m-l}\sum_{l+1}^{m}\log j\right)^2 = 1 + O\left(\frac{(\log m)^2}{m}\right)$$

we have that (3.13) is

$$4\frac{m-l}{mn^2}\left(1+O\left(\frac{(\log m)^2}{m}\right)\right)\left(\frac{(n-1)^2}{4}+O(1)\right)$$
$$= \frac{4}{n^2}\left(\frac{(n-1)^2}{4} + O\left(\frac{n^2}{m}(\log m)^2\right)\right) \to 1 \quad \text{as } n\to\infty.$$

Now for j, k such that $0 < 2\omega + \lambda_j + \lambda_k < 2\pi$ (which always holds for a large enough n) and $j \neq k$ we can again apply formula (3.15) and we get that

$$\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}[\cos\{s(2\omega+\lambda_j+\lambda_k)\} + \cos\{s(\lambda_j-\lambda_k)\}] = -n + O(1)$$

so that (3.14) is

$$O\left(\frac{1}{n}\sum_{l+1}^{m} v_j^2\right) = O\left(\frac{m}{n}\right) = o(1).$$

Thus the second term in (3.12) tends to zero as $n \to \infty$. The first term has mean zero and its variance is

$$O\left(\sum_{t=1}^{n-1}\left(\sum_{s=1}^{n-t} c_s^2\right)^2\right).$$

Now

$$|c_s| \leq \frac{2}{\sqrt{m}}\frac{1}{n}\sum_{l+1}^{m}|v_j| = O\left(\frac{\sqrt{m}\log m}{n}\right) \tag{3.16}$$

and for $1 \leq s \leq n/2$, by summation by parts, $|c_s|$ is

$$\left|\frac{2}{n\sqrt{m}}\sum_{l+1}^{m} v_j\cos\{s(\omega+\lambda_j)\}\right|$$
$$= \left|\frac{2}{n\sqrt{m}}\sum_{r=l+1}^{m-1}(v_r - v_{r+1})\sum_{j=l+1}^{r}\cos\{s(\omega+\lambda_j)\} + \frac{2}{n\sqrt{m}}v_m\sum_{l+1}^{m}\cos\{s(\omega+\lambda_j)\}\right|$$
$$= O\left(\frac{1}{n\sqrt{m}}\sum_{r=l+1}^{m-1}\log\left(1+\frac{1}{r}\right)\frac{n}{s} + \frac{1}{n\sqrt{m}}\log m\frac{n}{s}\right) = O\left(\frac{\log m}{s\sqrt{m}}\right) \tag{3.17}$$

because $\sum_{l+1}^{m} \cos\{s(\omega+\lambda_j)\} = O(ns^{-1})$ for $1 \le s \le n/2$ (see proof of Lemma 4) and $\left|\log\left(1+\frac{1}{r}\right)\right| \le 1/r$ for $r \ge 1$. The bound in (3.17) is at least as good as that in (3.16) for $n/m \le s \le n/2$. Consider ω a harmonic frequency (which can always be done for n sufficiently large), then $c_s = c_{n-s}$ and from (3.16) and (3.17)

$$\sum_{s=1}^{n} c_s^2 = O\left(\frac{n}{m}\frac{m(\log m)^2}{n^2} + \frac{(\log m)^2}{m}\sum_{s>\frac{n}{m}} s^{-2}\right) = O\left(\frac{(\log m)^2}{n}\right) \tag{3.18}$$

and the variance of the first part of (3.12) is $O\left(\frac{(\log m)^4}{n}\right)$. Thus (3.12) is $o_p(1)$. In order to prove a) it remains to show that

$$\sum_{t=2}^{n}\sum\sum_{r\neq s}^{t-1} \epsilon_r\epsilon_s c_{t-r}c_{t-s} = o_p(1). \tag{3.19}$$

The left hand side of (3.19) has mean zero and variance

$$\begin{aligned}
&\sum_{t=2}^{n}\sum_{u=2}^{n}\sum\sum_{r\neq s}^{t-1}\sum\sum_{p\neq q}^{u-1} E[\epsilon_r\epsilon_s\epsilon_p\epsilon_q]\, c_{t-r}c_{t-s}c_{u-p}c_{u-q}\\
= \;& 2\sum_{t=2}^{n}\sum_{u=2}^{n}\sum\sum_{r\neq s}^{\min(t-1,u-1)} c_{t-r}c_{t-s}c_{u-r}c_{u-s}\\
= \;& 2\sum_{t=2}^{n}\sum\sum_{r\neq s}^{t-1} c_{t-r}^2 c_{t-s}^2 + 4\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum\sum_{r\neq s}^{u-1} c_{t-r}c_{t-s}c_{u-r}c_{u-s}.
\end{aligned} \tag{3.20}$$

The first part of (3.20) is $O\left((\log m)^4 n^{-1}\right)$ from (3.18). The second part is bounded in absolute value by

$$4\sum_{t=3}^{n}\sum_{u=2}^{t-1}\left(\sum_{r=1}^{u-1} c_{t-r}^2 \sum_{s=1}^{u-1} c_{u-s}^2\right) \le 4\left(\sum_{1}^{n} c_t^2\right)\left(\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum_{t-u+1}^{t-1} c_r^2\right). \tag{3.21}$$

Now

$$\begin{aligned}
&\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum_{t-u+1}^{t-1} c_r^2 = \sum_{j=1}^{n-2} j(n-j-1)c_{j+1}^2 \le n\sum_{1}^{n} jc_{j+1}^2\\
&\le n\sum_{2}^{[nm^{-\frac{2}{3}}]} jc_j^2 + n\sum_{[nm^{-\frac{2}{3}}]+1}^{n} jc_j^2 = O\left(\frac{n(\log m)^2}{m^{\frac{1}{3}}}\right)
\end{aligned}$$

using (3.16) and (3.17). Thus noting (3.18) we see that (3.21) is

$$O\left(\frac{n(\log m)^2}{m^{\frac{1}{3}}}\frac{(\log m)^2}{n}\right) = O\left(\frac{(\log m)^4}{m^{\frac{1}{3}}}\right) = o(1)$$

and (3.19), and thus a), are proved.

In order to prove b) we check the sufficient condition

$$\sum_{t=1}^{n} E[z_t^4] \to 0 \quad \text{as} \quad n \to \infty.$$

Now

$$\sum_{1}^{n} E[z_t^4] = \sum_{2}^{n} E\left[\epsilon_t \sum_{s=1}^{t-1} \epsilon_s c_{t-s}\right]^4 = \mu_4 \sum_{2}^{n} E\left[\sum_{1}^{t-1} \epsilon_s c_{t-s}\right]^4$$

$$\leq \mu_4 \sum_{2}^{n} E\left[\sum_{s}\sum_{r}\sum_{p}\sum_{q=1}^{t-1} \epsilon_s \epsilon_r \epsilon_p \epsilon_q c_{t-s} c_{t-r} c_{t-p} c_{t-q}\right]$$

$$\leq \mu_4^2 \sum_{2}^{n} \left(\sum_{s=1}^{n} c_s^4\right) + 3\mu_4 \mu_2^2 \sum_{t=2}^{n} \sum_{s} \sum_{r=1}^{t-1} c_{t-s}^2 c_{t-r}^2$$

$$= O\left(n\left(\sum_{1}^{n} c_t^2\right)^2\right) = O\left(\frac{(\log m)^4}{n}\right) = o(1)$$

in view of (3.18) and this concludes the proof of the theorem. □

4. APPENDIX: TECHNICAL LEMMAS

Let $v_j = \frac{W(\omega+\lambda_j)}{C_1^{\frac{1}{2}} \lambda_j^{-d_1}}$ be a scaled discrete Fourier transform and denote $\bar{v}_j$ the complex conjugate of $v(\lambda)$.

Lemma 1. Let assumptions A.1 and A.2 hold and let $k = k(n)$ and $j = j(n)$ be two sequences of positive integers such that $j > k$ and $\frac{j}{n} \to 0$ as $n \to \infty$. Then as $n \to \infty$:

a) $E|v_j|^2 = 1 + O\left(\frac{\log j}{j} \lambda_j^{-2(d_i - d_1)} + \left(\frac{j}{n}\right)^\alpha\right)$

b) $E v_j v_j = O\left(\frac{\log j}{j} \lambda_j^{-2(d_i - d_1)}\right)$

c) $E v_j \bar{v}_k = O\left(\frac{\log j}{\sqrt{jk}} \lambda_j^{-(d_i - d_1)} \lambda_k^{-(d_i - d_1)}\right)$

d) $E v_j v_k = O\left(\frac{\log j}{\sqrt{jk}} \lambda_j^{-(d_i - d_1)} \lambda_k^{-(d_i - d_1)}\right)$

where $d_i = \max(d_1, d_2)$.

Proof. See Arteche [1]. □

Lemma 2. For $\epsilon \in (0,1]$ and $\kappa \in (\epsilon, \infty)$, when $l \to \infty$ and $\frac{l}{m} \to 0$,

$$\sup_{\epsilon \le \gamma \le \kappa} \left| \frac{\gamma}{m-l} \sum_{j=l+1}^{m} \left(\frac{j}{m-l}\right)^{\gamma-1} - 1 \right| = O\left(\left(\frac{l}{m}\right)^{\epsilon}\right). \tag{4.1}$$

Proof. For $\gamma > 0$

$$\begin{aligned}
&\left| \frac{\gamma}{m-l} \sum_{l+1}^{m} \left(\frac{j}{m-l}\right)^{\gamma-1} - 1 \right| \\
\le\ & \gamma \int_0^{\frac{1}{m-l}} \left\{ \left(\frac{l+1}{m-l}\right)^{\gamma-1} + x^{\gamma-1} \right\} \mathrm{d}x + \gamma \sum_{l+2}^{m} \left| \int_{\frac{j-l-1}{m-l}}^{\frac{j-l}{m-l}} \left\{ \left(\frac{j}{m-l}\right)^{\gamma-1} - x^{\gamma-1} \right\} \mathrm{d}x \right| \\
\le\ & \frac{\gamma}{m-l}\left(\frac{l+1}{m-l}\right)^{\gamma-1} + \frac{1}{(m-l)^{\gamma}} + \frac{l\gamma|\gamma-1|}{(m-l)^2} \sum_{l+1}^{m} \left(\frac{j}{m-l}\right)^{\gamma-2} \qquad (4.2)
\end{aligned}$$

using the mean value theorem. The first term is $O(m^{-\gamma} l^{\gamma-1})$, the second is $O(m^{-\gamma})$ and the third is $O(m^{-1} l)$ for $\gamma > 1$, zero for $\gamma = 1$ and $O(m^{-\gamma} l^{\gamma})$ if $\gamma < 1$. Thus (4.2) is $O\left(\left(\frac{l}{m}\right)^{\gamma} + \frac{l}{m}\right)$ and the left hand side of (4.1) is $O\left(\left(\frac{l}{m}\right)^{\epsilon}\right)$ because $\epsilon \in (0,1]$. □

Lemma 3. Let $l \to \infty$ and $\frac{l}{m} \to 0$ as $m \to \infty$. Then,

$$\left| \frac{1}{m-l} \sum_{j=l+1}^{m} \log j - \log(m-l) + 1 \right| = O\left(\frac{l}{m} \log m\right).$$

Proof.

$$\begin{aligned}
&\left| \frac{1}{m-l} \sum_{l+1}^{m} \log j - \log(m-l) + 1 \right| \\
=\ & \left| \frac{1}{m-l} \sum_{l+2}^{m} \int_{j-l-1}^{j-l} \log\left(\frac{j}{x}\right) \mathrm{d}x + \frac{1}{m-l} + \frac{1}{m-l} \log(l+1) \right| \\
\le\ & \frac{1}{m-l} \sum_{l+2}^{m} \int_{j-l-1}^{j-l} |x-j| \frac{1}{j-l-1} \mathrm{d}x + \frac{1}{m-l} + \frac{1}{m-l} \log(l+1) \\
\le\ & \frac{|l+1|}{m-l} \sum_{1}^{m-l-1} \frac{1}{j} + \frac{1}{m-l}(1 + \log(l+1)) = O\left(\frac{l}{m} \log m\right).
\end{aligned}$$

□

Lemma 4. Let $r > l$ and $I_{\epsilon j}$ defined in (2.7) and A.3' hold. Then

$$\sum_{j=l+1}^{r} (2\pi I_{\epsilon j} - 1) = O_p(r^{\frac{1}{2}}).$$

Proof. Write

$$\begin{aligned} 2\pi I_{\epsilon j} &= 2\pi |W_\epsilon(\omega + \lambda_j)|^2 = \frac{1}{n}\left|\sum_{t=1}^{n} \epsilon_t e^{it(\omega+\lambda_j)}\right|^2 \\ &= \frac{1}{n}\sum_{t=1}^{n} \epsilon_t^2 + \frac{2}{n}\sum_{t=2}^{n} \epsilon_t \sum_{s=1}^{t-1} \epsilon_s \cos\{(\omega + \lambda_j)(t-s)\}. \end{aligned}$$

Thus

$$\sum_{j=l+1}^{r} (2\pi I_{\epsilon j} - 1) = \frac{r-l}{n}\sum_{t=1}^{n}(\epsilon_t^2 - 1) + \sum_{t=2}^{n} \epsilon_t \sum_{s=1}^{t-1} \epsilon_s d_{t-s} \tag{4.3}$$

where $d_s = \frac{2}{n}\sum_{l+1}^{r} \cos\{(\omega+\lambda_j)s\}$. If ω is a harmonic frequency of the form $\omega = \frac{2\pi w}{n}$, where w is an integer then $d_s = d_{n-s}$. Since in our analysis $n \to \infty$, we can express any frequency $\omega \in (0, \pi]$ as a harmonic frequency for a large enough n. Also $|d_s| \le \frac{2r}{n}$ and for $1 \le s \le n/2$, $|d_s| \le \frac{6}{\pi s} + \frac{6}{n}$. This last inequality can be proved in the following way. Write

$$d_s = \frac{2}{n}\sum_{l+1}^{r} \cos(\omega s)\cos(s\lambda_j) - \frac{2}{n}\sum_{l+1}^{r} \sin(\omega s)\sin(s\lambda_j),$$

then

$$|d_s| \le \frac{2}{n}\left|\sum_{l+1}^{r}\cos(s\lambda_j)\right| + \frac{2}{n}\left|\sum_{l+1}^{r}\sin(s\lambda_j)\right|.$$

By formulae (5.10) and (5.11) in Zygmund [20], Chapter 2),

$$\left|\frac{1}{2} + \sum_{v=1}^{r}\cos tv - \frac{1}{2}\cos rt\right| \le \frac{1}{t} \quad \text{for} \quad 0 < t \le \pi$$

and

$$\left|\sum_{v=1}^{r}\sin tv - \frac{1}{2}\sin rt\right| \le \frac{2}{t}.$$

Thus

$$\begin{aligned} \left|\sum_{l+1}^{r}\cos(\lambda_j s)\right| &\le \left|\sum_{1}^{r}\cos(\lambda_j s)\right| + \left|\sum_{1}^{l}\cos(\lambda_j s)\right| \\ &\le \left|\frac{1}{2} + \sum_{1}^{r}\cos(\lambda_j s) - \frac{1}{2}\cos(\lambda_r s)\right| + \left|\frac{1}{2} - \frac{1}{2}\cos(\lambda_r s)\right| \end{aligned}$$

$$+ \left|\frac{1}{2} + \sum_1^l \cos(\lambda_j s) - \frac{1}{2}\cos(\lambda_l s)\right| + \left|\frac{1}{2} - \frac{1}{2}\cos(\lambda_l s)\right|$$
$$\leq \frac{n}{\pi s} + 2 \quad \text{for} \quad 1 \leq s \leq \frac{n}{2}$$

and similarly $\left|\sum_{l+1}^r \sin(\lambda_j s)\right|$ is bounded by

$$\left|\sum_1^r \sin(\lambda_j s) - \frac{1}{2}\sin(\lambda_r s)\right| + \left|\frac{1}{2}\sin(\lambda_r s)\right|$$
$$+ \left|\sum_1^l \sin(\lambda_j s) - \frac{1}{2}\sin(\lambda_l s)\right| + \left|\frac{1}{2}\sin(\lambda_l s)\right| \leq \frac{2n}{\pi s} + 1$$

and thus $|d_s| \leq \frac{6}{\pi s} + \frac{6}{n}$.

Both terms on the right hand side of (4.3) have zero mean and variance respectively $O\left(\frac{r^2}{n}\right)$ and

$$O\left(n\sum_1^n d_s^2\right) = O\left(n\sum_1^{[\frac{n}{r}]}\left(\frac{r}{n}\right)^2 + n\sum_{[\frac{n}{r}]}^n\left(\frac{1}{\pi s} + \frac{1}{n}\right)^2\right) = O(r)$$

which concludes the proof. □

Lemma 5. Let j be a sequence of integers such that $\frac{j}{n} \to 0$ as $n \to \infty$. Then under A.1 and A.2',

$$\begin{aligned}\int_{-\pi}^{\pi}\left|\frac{\alpha(\lambda)}{\alpha(\omega+\lambda_j)} - 1\right|^2 K(\lambda-\lambda_j-\omega)\,\mathrm{d}\lambda &= O\left(\frac{1}{j}\right) \quad \text{if } d_1 \geq d_2 \\ &= O\left(\frac{1}{j}\left[\frac{n}{j}\right]^{2(d_2-d_1)}\right) \quad \text{if } d_1 < d_2.\end{aligned}$$

P r o o f. A.1 and A.2' imply that we can pick $\delta \in (2\lambda_j, \pi)$ such that for some $C < \infty$,

$$|\alpha(\omega+\lambda)| \leq C\lambda^{-d_1}, \quad |\alpha(\omega-\lambda)| \leq C\lambda^{-d_2}$$

and

$$|\alpha'(\omega+\lambda)| \leq C\lambda^{-d_1-1}, \quad |\alpha'(\omega-\lambda)| \leq C\lambda^{-d_2-1}$$

for $0 < \lambda < \delta$. Now split the integral up into,

$$\int_{-\pi}^{\omega-\delta} + \int_{\omega-\delta}^{\omega-\frac{\lambda_j}{2}} + \int_{\omega-\frac{\lambda_j}{2}}^{\omega+\frac{\lambda_j}{2}} + \int_{\omega+\frac{\lambda_j}{2}}^{\omega+2\lambda_j} + \int_{\omega+2\lambda_j}^{\omega+\delta} + \int_{\omega+\delta}^{\pi}.$$

Write $\alpha_j = \alpha(\omega+\lambda_j)$ and $f_j = f(\omega+\lambda_j)$. The first integral is equal to

$$\frac{1}{|\alpha_j|^2}\int_{-\pi}^{\omega-\delta}\left\{|\alpha(\lambda)|^2 - \alpha(\lambda)\bar{\alpha}_j - \alpha_j\bar{\alpha}(\lambda) + |\alpha_j|^2\right\} K(\lambda-\lambda_j-\omega)\,\mathrm{d}\lambda$$

and this is bounded in absolute value by

$$\begin{aligned}
&\frac{1}{f_j}\left\{\max_{-\pi\le\lambda\le\omega-\delta} K(\lambda-\lambda_j-\omega)\right\}\left\{\int_{-\pi}^{\pi} f(\lambda)\,\mathrm{d}\lambda + \frac{|\bar{\alpha}_j|}{2\pi}\int_{-\pi}^{\pi}|\alpha(\lambda)|\,\mathrm{d}\lambda\right.\\
&\qquad\left. + \frac{|\alpha_j|}{2\pi}\int_{-\pi}^{\pi}\bar{\alpha}(\lambda)\,\mathrm{d}\lambda\right\} + \int_{-\pi}^{\omega-\delta} K(\lambda-\lambda_j-\omega)\,\mathrm{d}\lambda\\
&= O\left(\frac{j^{2d_1}}{n^{1+2d_1}} + \frac{j^{d_1}}{n^{1+d_1}} + n^{-1}\right) = O(j^{-1})
\end{aligned}$$

using well known properties of the Fejer's Kernel (see formulae (4.5) – (4.7) in Arteche [1]). Similarly

$$\left|\int_{\omega+\delta}^{\pi}\right| = O(j^{-1}).$$

The integral over $[\omega-\delta, \omega-\lambda_j/2]$ has an absolute value bounded by

$$\begin{aligned}
&\frac{1}{f_j}\left\{\max_{\frac{\lambda_j}{2}\le\lambda\le\delta}\frac{f(\omega-\lambda)}{\lambda^{\frac{1}{2}-d_2}}\right\}\int_{\frac{\lambda_j}{2}}^{\delta}\lambda^{\frac{1}{2}-d_2}K(-\lambda-\lambda_j)\,\mathrm{d}\lambda\\
&\quad + \frac{|\bar{\alpha}_j|}{|\alpha_j|^2}\left\{\max_{\frac{\lambda_j}{2}\le\lambda\le\delta}\frac{|\alpha(\omega-\lambda)|}{\lambda^{\frac{1}{2}-d_2}}\right\}\int_{\frac{\lambda_j}{2}}^{\delta}\lambda^{\frac{1}{2}-d_2}K(-\lambda-\lambda_j)\,\mathrm{d}\lambda\\
&\quad + \frac{|\alpha_j|}{|\alpha_j|^2}\left\{\max_{\frac{\lambda_j}{2}\le\lambda\le\delta}\frac{|\bar{\alpha}(\omega-\lambda)|}{\lambda^{\frac{1}{2}-d_2}}\right\}\int_{\frac{\lambda_j}{2}}^{\delta}\lambda^{\frac{1}{2}-d_2}K(-\lambda-\lambda_j)\,\mathrm{d}\lambda\\
&\quad + \int_{\frac{\lambda_j}{2}}^{\delta}K(-\lambda-\lambda_j)\,\mathrm{d}\lambda\\
&= O\left(\lambda_j^{2d_1}n^{-1}\lambda_j^{-1-2d_2} + \lambda_j^{d_1}n^{-1}\lambda_j^{-1-d_2} + n^{-1}\lambda_j^{-1}\right)\\
&= O\left(\frac{1}{j}\right) \quad \text{if} \quad d_1 \ge d_2\\
&= O\left(\frac{1}{j}\left[\frac{n}{j}\right]^{2(d_2-d_1)}\right) \quad \text{if} \quad d_1 < d_2.
\end{aligned}$$

Proceeding similarly we get that

$$\left|\int_{\omega+2\lambda_j}^{\omega+\delta}\right| = O(j^{-1}).$$

Now the integral over $\left[\omega \pm \frac{\lambda_j}{2}\right]$ is bounded in modulus by

$$\max_{-\frac{\lambda_j}{2}\le\lambda\le\frac{\lambda_j}{2}}|K(\lambda-\lambda_j)|\left\{\frac{1}{f_j}\int_{-\frac{\lambda_j}{2}}^{\frac{\lambda_j}{2}} f(\omega+\lambda)\,\mathrm{d}\lambda\right.$$

$$
+ \frac{|\alpha_j|}{|\alpha_j|^2} \int_{-\frac{\lambda_j}{2}}^{\frac{\lambda_j}{2}} |\bar{\alpha}(\omega+\lambda)| \, \mathrm{d}\lambda + \frac{|\bar{\alpha}_j|}{|\alpha_j|^2} \int_{-\frac{\lambda_j}{2}}^{\frac{\lambda_j}{2}} |\alpha(\omega+\lambda)| \, \mathrm{d}\lambda + \lambda_j \Bigg\}
$$

$$
= O\left(n^{-1}\lambda_j^{-2}\left[\lambda_j^{2d_1}\lambda_j^{1-2d_i} + \lambda_j^{d_1}\lambda_j^{1-d_i} + \lambda_j\right]\right) \tag{4.4}
$$

where $i = 1$ if $d_1 \geq d_2$ and $i = 2$ if $d_2 > d_1$. Thus (4.4) is $O(j^{-1})$ if $d_1 \geq d_2$ and $O\left(\frac{1}{j}\left[\frac{n}{j}\right]^{2(d_2-d_1)}\right)$ if $d_2 > d_1$.

Finally using the mean value theorem

$$
\begin{aligned}
\left|\int_{\omega+\frac{\lambda_j}{2}}^{\omega+2\lambda_j}\right| &= \frac{1}{|\alpha_j|^2}\left|\int_{\frac{\lambda_j}{2}}^{2\lambda_j} |\alpha(\omega+\lambda) - \alpha(\omega+\lambda_j)|^2 K(\lambda-\lambda_j)\,\mathrm{d}\lambda\right| \\
&\leq \frac{1}{|\alpha_j|^2}\left\{\max_{\frac{\lambda_j}{2}\leq\lambda\leq 2\lambda_j}\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}\alpha(\omega+\lambda)\right|^2\right\}\int_{\frac{\lambda_j}{2}}^{2\lambda_j} |\lambda-\lambda_j|^2 K(\lambda-\lambda_j)\,\mathrm{d}\lambda \\
&= O\left(\lambda_j^{2d_1}\lambda_j^{-2-2d_1}n^{-1}\lambda_j\right) = O\left(j^{-1}\right)
\end{aligned}
$$

which concludes the proof. □

Lemma 6. Let $0 < l < r \leq m$. Let A.1, A.2' and A.3' hold and $d_2 > d_1$. Then

$$
\sum_{j=l+1}^{r}\left(\frac{I_j}{g_j} - 2\pi I_{\epsilon j}\right) = O_p\left(\frac{r^{\alpha+1}}{n^\alpha} + l^{\frac{3}{4}}r^{\frac{1}{4}}\right) \text{ under C.4} \tag{4.5}
$$

$$
= O_p\left(\frac{r^{\alpha+1}}{n^\alpha} + l\right) \text{ under C.5 and C.6} \tag{4.6}
$$

where I_j is the periodogram of $x_t = Ex_1 + \sum_{j=0}^{\infty}\alpha_j\epsilon_{t-j}$ at $(\omega+\lambda_j)$, $I_{\epsilon j}$ is the periodogram of ϵ_t at frequency $(\omega+\lambda_j)$ and $g_j = C\lambda_j^{-2d_1}$.

P r o o f. From Lemma 1 and A.1,

$$
E\left|\sum_{l+1}^{r}\left(\frac{I_j}{g_j} - \frac{I_j}{f_j}\right)\right| = E\left|\sum_{l+1}^{r}\left(1-\frac{g_j}{f_j}\right)\left(\frac{I_j}{g_j}\right)\right|
$$

$$
\begin{aligned}
&= O\left(\sum_{l+1}^{r}\left(\frac{j}{n}\right)^\alpha\left(1 + (\frac{j}{n})^\alpha + \frac{n^{2(d_2-d_1)}\log j}{j^{1+2(d_2-d_1)}}\right)\right) \\
&= O\left(\frac{r^{\alpha+1}}{n^\alpha} + \frac{r^{\alpha+1}}{n^\alpha}\frac{n^{2(d_2-d_1)}\log r}{l^{1+2(d_2-d_1)}}\right) = O\left(\frac{r^{\alpha+1}}{n^\alpha}\right)
\end{aligned}
$$

under A.4' or A.6.

Write $u_j = \sqrt{2\pi}\frac{W_j}{|\alpha_j|}$ and $v_j = \sqrt{2\pi}W_{\epsilon j}$ where W_j and $W_{\epsilon j}$ are discrete Fourier transforms of x_t and ϵ_t respectively at frequency $\omega+\lambda_j$, and $\alpha_j = \alpha(\omega+\lambda_j) =$

$\sum_{k=0}^{\infty} \alpha_k e^{ik(\omega+\lambda_j)}$. Then

$$E\left\{\sum_{l+1}^{r}\left(\frac{I_j}{f_j}-2\pi I_{\epsilon j}\right)\right\}^2 = E\left\{\sum_{l+1}^{r}(|u_j|^2-|v_j|^2)\right\}^2 = a+b$$

where

$$\begin{aligned} a &= \sum_{j=l+1}^{r}\left(E|u_j|^4+E|v_j|^4-2E|u_jv_j|^2\right) \\ b &= 2\sum_{j=l+1}^{r}\sum_{k>j}\left(E|u_ju_k|^2-E|u_jv_k|^2-E|u_kv_j|^2+E|v_jv_k|^2\right). \end{aligned}$$

Since for any zero mean random variables u, v, w, z,

$$E(uvwz) = E(uv)\,E(wz) + E(uw)\,E(vz) + E(uz)\,E(vw) + cum(u,v,w,z)$$

where $cum(u,v,w,z)$ is the joint cumulant of u, v, w and z; we can decompose a and b into a_1+a_2 and b_1+b_2 where

$$\begin{aligned} a_1 &= \sum_{l+1}^{r}\{2(E|u_j|^2)^2+|E(u_j^2)|^2-2|E(u_jv_j)|^2 \\ &\quad -2|E(u_j\bar{v}_j)|^2-2E|u_j|^2E|v_j|^2+2(E|v_j|^2)^2+|E(v_j^2)|^2\} \\ a_2 &= \sum_{l+1}^{r}\{cum(u_j,u_j,\bar{u}_j,\bar{u}_j)-2cum(u_j,v_j,\bar{u}_j,\bar{v}_j)+cum(v_j,v_j,\bar{v}_j,\bar{v}_j)\} \\ b_1 &= 2\sum_{j=l+1}^{r}\sum_{k>j}\{E|u_j|^2E|u_k|^2+|E(u_ju_k)|^2+|E(u_j\bar{u}_k)|^2-E|u_j|^2E|v_k|^2 \\ &\quad -|E(u_jv_k)|^2-|E(u_j\bar{v}_k)|^2-E|u_k|^2E|v_j|^2-|E(u_kv_j)|^2-|E(u_k\bar{v}_j)|^2 \\ &\qquad +E|v_j|^2E|v_k|^2+|E(v_jv_k)|^2+|E(v_j\bar{v}_k)|^2\} \\ b_2 &= 2\sum_{j=l+1}^{r}\sum_{k>j}\{cum(u_j,u_k,\bar{u}_j,\bar{u}_k)-cum(u_j,v_k,\bar{u}_j,\bar{v}_k) \\ &\qquad -cum(u_k,v_j,\bar{u}_k,\bar{v}_j)+cum(v_j,v_k,\bar{v}_j,\bar{v}_k)\}. \end{aligned}$$

Now because $E|v_j|^2=1$ and Lemma 1,

$$\begin{aligned} a_1 &= \sum_{l+1}^{r}\{2(E|u_j|^2-1)^2+2(E|u_j|^2-1)+|E(u_j^2)|^2-2|E(u_jv_j)|^2 \\ &\qquad -2|E(u_j\bar{v}_j)-1|^2-2(E(u_j\bar{v}_j-1)-2(E(\bar{u}_jv_j)-1)+|E(v_j^2)|^2\} \\ &= O\left(\sum_{l+1}^{r}\frac{n^{2(d_2-d_1)}\log j}{j^{1+2(d_2-d_1)}}\right) = O\left(\frac{n^{2(d_2-d_1)}}{l^{2(d_2-d_1)}}\log r\right) \end{aligned}$$

$$\begin{aligned} b_1 &= 2\sum_{l+1}^{r}\sum_{k>j}\{(E|u_j|^2-1)(E|u_k|^2-1)+|E(u_ju_k)|^2+|E(u_j\bar{u}_k)|^2-|E(u_jv_k)|^2 \\ &\qquad -|E(u_j\bar{v}_k)|^2-|E(u_kv_j)|^2-|E(u_k\bar{v}_j)|^2+|E(v_jv_k)|^2+|E(v_j\bar{v}_k)|^2\} \\ &= O\left(\sum_{j=l+1}^{r}\sum_{k>j}\frac{n^{4(d_2-d_1)}(\log k)^2}{k^{1+2(d_2-d_1)}j^{1+2(d_2-d_1)}}\right) = O\left(\frac{n^{4(d_2-d_1)}}{l^{4(d_2-d_1)}}(\log r)^2\right) \end{aligned}$$

and under A.4' or A.6, a_1 is $O(l)$ and b_1 is $O(l^2)$. Now applying formula (2.6.3) in Brillinger [3],

$$cum(u_j, v_k, \bar{u}_j, \bar{v}_k) = \iiint_{-\pi}^{\pi} f_{u_j,v_k,\bar{u}_j,\bar{v}_k}(\lambda,\mu,\zeta)\,\mathrm{d}\lambda\mathrm{d}\mu\mathrm{d}\zeta$$

where $f_{u_j,v_k,\bar{u}_j,\bar{v}_k}$ is the fourth order cumulant spectrum, and by formula (2.10.3) in Brillinger [3], we have that

$$cum(u_j, v_k, \bar{u}_j, \bar{v}_k) = \iiint_{-\pi}^{\pi} \frac{\kappa}{(2\pi)^3} A_{u_j}(-\lambda-\mu-\zeta)A_{v_k}(\lambda)A_{\bar{u}_j}(\mu)A_{\bar{v}_k}(\zeta)\,\mathrm{d}\lambda\mathrm{d}\mu\mathrm{d}\zeta$$

where κ is the fourth cumulant of ϵ_t, $\kappa = \mu_4 - 3$, and $A_{u_j}, A_{v_k}, A_{\bar{u}_j}, A_{\bar{v}_k}$ are transfer functions of the filters implied in the definition of u_j and v_j,

$$\begin{aligned} u_j &= \frac{1}{|\alpha_j|}\frac{1}{\sqrt{n}}\sum_{t=1}^{n} e^{it(\omega+\lambda_j)}\sum_{k=0}^{\infty}\alpha_k\epsilon_{t-k} \\ v_j &= \frac{1}{\sqrt{n}}\sum_{t=1}^{n} e^{it(\omega+\lambda_j)}\epsilon_t \end{aligned}$$

so that if $\alpha(\lambda) = \sum_{k=0}^{\infty}\alpha_k e^{ik\lambda}$,

$$\begin{aligned} A_{u_j}(\lambda) &= \frac{1}{|\alpha_j|}\frac{1}{\sqrt{n}}\alpha(-\lambda)\sum_{t=1}^{n} e^{it(\omega+\lambda_j+\lambda)} \\ A_{\bar{u}_j}(\lambda) &= \frac{1}{|\alpha_j|}\frac{1}{\sqrt{n}}\alpha(-\lambda)\sum_{t=1}^{n} e^{it(\lambda-\omega-\lambda_j)} \\ A_{v_k}(\lambda) &= \frac{1}{\sqrt{n}}\sum_{t=1}^{n} e^{it(\omega+\lambda_k+\lambda)} \\ A_{\bar{v}_k}(\lambda) &= \frac{1}{\sqrt{n}}\sum_{t=1}^{n} e^{it(\lambda-\omega-\lambda_k)}. \end{aligned}$$

Since $\kappa = 0$ under A.5, then $a_2 = b_2 = 0$, and (4.6) follows. In any other case $cum(u_j, v_k, \bar{u}_j, \bar{v}_k)$ is equal to

$$\frac{\kappa}{(2\pi)^3}\frac{1}{n^2}\iiint_{-\pi}^{\pi}\frac{\alpha(\lambda+\mu+\zeta)\alpha(-\mu)}{|\alpha_j|^2}E_{jk}(\lambda,\mu,\zeta)\,\mathrm{d}\lambda\mathrm{d}\mu\mathrm{d}\zeta \qquad (4.7)$$

where $E_{jk}(\lambda,\mu,\zeta) = D(\omega+\lambda_j-\lambda-\mu-\zeta)D(\omega+\lambda_k+\lambda)\,D(\mu-\omega-\lambda_j)\,D(\zeta-\omega-\lambda_k)$ and $D(\lambda)=\sum_{t=1}^{n} e^{it\lambda}$ is Dirichlet's kernel. Doing the same with the other cumulants in b_2 we see that the summand of $(2\pi)^3 b_2$ is

$$\frac{\kappa}{n^2}\iiint_{-\pi}^{\pi}\left\{\frac{\alpha(\lambda+\mu+\zeta)\alpha(-\mu)}{|\alpha_j|^2}-1\right\}\left\{\frac{\alpha(-\lambda)\alpha(-\zeta)}{|\alpha_k|^2}-1\right\}E_{jk}(\lambda,\mu,\zeta)\,\mathrm{d}\lambda\mathrm{d}\mu\mathrm{d}\zeta. \tag{4.8}$$

Since

$$(c_1c_2-1)(c_3c_4-1)=\prod_{j=1}^{4}(c_j-1)+\sum_{i=1}^{4}\prod_{j\neq i}(c_j-1)+\sum_{i=1}^{2}\sum_{j-1}^{2}(c_i-1)(c_{j+2}-1)$$

then (4.8) has components of three types. The first one is

$$\frac{\kappa}{n^2}\iiint_{-\pi}^{\pi}\left\{\frac{\alpha(\lambda+\mu+\zeta)}{\alpha_j}-1\right\}\left\{\frac{\alpha(-\mu)}{\bar\alpha_j}-1\right\}\left\{\frac{\alpha(-\lambda)}{\alpha_k}-1\right\}\left\{\frac{\alpha(-\zeta)}{\bar\alpha_k}-1\right\}E_{jk}(\lambda,\mu,\zeta)\,\mathrm{d}\lambda\mathrm{d}\mu\mathrm{d}\zeta. \tag{4.9}$$

Proceeding as in Robinson [18] we have that because of the Schwarz inequality and by periodicity, (4.8) is bounded in absolute value by $\kappa(2\pi)^3P_jP_k$ where

$$P_j=\int_{-\pi}^{\pi}\left|\frac{\alpha(\lambda)}{\alpha_j}-1\right|^2 K(\lambda-\omega-\omega_j)\,\mathrm{d}\lambda$$

and $K(\lambda)=\frac{|D(\lambda)|^2}{2\pi n}$ is Fejer's kernel. The second component is

$$\frac{\kappa}{n^2}\iiint_{-\pi}^{\pi}\left\{\frac{\alpha(\lambda+\mu+\zeta)}{\alpha_j}-1\right\}\left\{\frac{\alpha(-\mu)}{\bar\alpha_j}-1\right\}\left\{\frac{\alpha(-\lambda)}{\alpha_k}-1\right\}E_{jk}(\lambda,\mu,\zeta)\,\mathrm{d}\lambda\mathrm{d}\mu\mathrm{d}\zeta. \tag{4.10}$$

As before, (4.10) is bounded in absolute value by $\kappa(2\pi)^3P_jP_k^{\frac{1}{2}}$. An example of the third type component is

$$\begin{aligned}&\frac{\kappa}{n^2}\iiint_{-\pi}^{\pi}\left\{\frac{\alpha(\lambda+\mu+\zeta)}{\alpha_j}-1\right\}\left\{\frac{\alpha(-\lambda)}{\alpha_k}-1\right\}E_{jk}(\lambda,\mu,\zeta)\,\mathrm{d}\lambda\mathrm{d}\mu\mathrm{d}\zeta\\ &=\frac{\kappa 2\pi}{n^2}\iint_{-\pi}^{\pi}\left\{\frac{\alpha(\theta)}{\alpha_j}-1\right\}\left\{\frac{\alpha(-\lambda)}{\alpha_k}-1\right\}\\ &\qquad\times D(\omega+\lambda_j-\theta)\,D(\omega+\lambda_k+\lambda)D(\theta-2\omega-\lambda-\lambda_j-\lambda_k)\,\mathrm{d}\lambda\mathrm{d}\theta\end{aligned} \tag{4.11}$$

because

$$\int_{-\pi}^{\pi}D(u+\lambda)\,D(v-\lambda)\,\mathrm{d}\lambda=2\pi D(u+v).$$

Thus the absolute value of (4.11) is bounded by $\frac{\kappa(2\pi)^3}{\sqrt{n}}P_j^{\frac{1}{2}}P_k^{\frac{1}{2}}$.

Now since the summand of a_2 is that of b_2 with $j=k$, applying Lemma 5 we have when $d_1<d_2$,

$$a_2 \;=\; O\left(\sum_{l+1}^{r}\left\{\frac{n^{4(d_2-d_1)}}{j^{2+4(d_2-d_1)}}+\frac{n^{3(d_2-d_1)}}{j^{\frac{3}{2}+3(d_2-d_1)}}+\frac{n^{2(d_2-d_1)}-\frac{1}{2}}{j^{1+2(d_2-d_1)}}\right\}\right)$$

$$= O\left(\frac{n^{4(d_2-d_1)}}{l^{1+4(d_2-d_1)}} + \frac{n^{3(d_2-d_1)}}{l^{\frac{1}{2}+3(d_2-d_1)}} + \frac{n^{2(d_2-d_1)} - \frac{1}{2}}{l^{2(d_2-d_1)}}\right) = O(l)$$

$$\begin{aligned} b_2 &= O\left(\sum_{j=l+1}^{r}\sum_{k=l+1}^{r}\left\{\frac{n^{4(d_2-d_1)}}{j^{1+2(d_2-d_1)}k^{1+2(d_2-d_1)}}\right.\right. \\ &\quad + \left.\left.\frac{n^{3(d_2-d_1)}}{j^{1+2(d_2-d_1)}k^{\frac{1}{2}+(d_2-d_1)}} + \frac{1}{\sqrt{n}}\frac{n^{2(d_2-d_1)}}{j^{\frac{1}{2}+(d_2-d_1)}k^{\frac{1}{2}+(d_2-d_1)}}\right\}\right) \\ &= O\left(\frac{n^{4(d_2-d_1)}}{l^{4(d_2-d_1)}} + \frac{n^{3(d_2-d_1)}\log r}{l^{2(d_2-d_1)}r^{-\frac{1}{2}+(d_2-d_1)}} + \frac{n^{3(d_2-d_1)}}{l^{-\frac{1}{2}+3(d_2-d_1)}}\right. \\ &\quad + \left.\frac{1}{\sqrt{n}}\frac{n^{2(d_2-d_1)}}{l^{-1+2(d_2-d_1)}} + \frac{1}{\sqrt{n}}\frac{n^{2(d_2-d_1)}}{r^{-1+2(d_2-d_1)}\log r}\right) = O\left(l^{\frac{3}{2}}r^{\frac{1}{2}}\right) \end{aligned}$$

under A.4' which completes the proof of the lemma. □

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REFERENCES

[1] J. Arteche: Log-periodogram regression in seasonal/cyclical long memory time series. Working Paper, Biltoki 98.17, University of the Basque Country, 1998.

[2] J. Arteche and P. M. Robinson: Seasonal and cyclical long memory. In: Asymptotics, Nonparametrics and Time Series (S. Ghosh, ed.), Marcel Dekker Inc., New York 1999, pp. 115–148.

[3] D. R. Brillinger: Time Series: Data Analysis and Theory. Holden–Day, San Francisco 1975.

[4] J. B. Carlin and A. P. Dempster: Sensitivity analysis of seasonal adjustments: empirical case studies. J. Amer. Statist. Assoc. *84* (1989), 6–20.

[5] R. Dahlhaus: Efficient parameter estimation for self-similar processes. Ann. Statist. *17* (1989), 1749–1766.

[6] R. Fox and M. S. Taqqu: Large sample properties of parameter estimates for strongly dependent stationary Gaussian time series. Ann. Statist. *14* (1986), 517–532.

[7] J. Geweke and S. Porter–Hudak: The estimation and application of long–memory time series models. J. Time Ser. Anal. *4* (1983), 221–238.

[8] L. Giraitis and D. Surgailis: A central limit theorem for quadratic forms in strongly dependent linear variables and its application to asymptotic normality of Whittle's estimate. Probab. Theory Related Fields *86* (1990), 87–104.

[9] H. L. Gray, N. F. Zhang and W. A. Woorward: On generalized fractional processes. J. Time Ser. Anal. *10* (1989), 233–257.

[10] P. Hall and C. C. Heyde: Martingale Limit Theory and its Application in Probability and Mathematical Statistics. Academic Press, New York 1980.

[11] U. Hassler: (Mis)specification of long-memory in seasonal time series. J. Time Ser. Anal. *15* (1994), 19–30.

[12] C. C. Heyde and R. Gay: Smoothed periodogram asymptotics and estimation for processes and fields with possible long–range dependence. Stoch. Process. Appl. *45* (1993), 169–182.

[13] C. C. Heyde and E. Seneta: Estimation theory for growth and immigration rates in multiplicative process. J. Appl. Probab. *9* (1972), 235–256.

[14] A. J. Jonas: Persistent Memory Random Processes. Ph.D. Thesis. Department of Statistics, Harvard University, 1983.
[15] H. R. Kunsch: Statistical aspects of self-similar processes. In: Proc. First World Congress Bernoulli Soc. (Yu. Prohorov and V. V. Sazanov, eds.), VNU Science Press, Utrecht, *1* (1987), 67–74.
[16] P. M. Robinson: Efficient tests of non-stationary hypothesis. J. Amer. Statist. Assoc. *89* (1994), 1420–1437.
[17] P. M. Robinson: Log–periodogram regression of time series with long-range dependence. Ann. Statist. *23* (1995), 3, 1048–1072.
[18] P. M. Robinson: Gaussian semiparametric estimation of long–range dependence. Ann. Statist. *23* (1995), 1630–1661.
[19] P. Whittle: Estimation and information in stationary time series. Ark. Mat. *2* (1953), 423–434.
[20] A. Zygmund: Trigonometric Series. Cambridge University Press, Cambridge, UK 1977.

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