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ON THE RAO–BLACKWELL THEOREM FOR FUZZY RANDOM VARIABLES[1]

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In a previous paper, conditions have been given to compute iterated expectations of fuzzy random variables, irrespectively of the order of integration. In another previous paper, a generalized real-valued measure to quantify the absolute variation of a fuzzy random variable with respect to its expected value have been introduced and analyzed. In the present paper we combine the conditions and generalized measure above to state an extension of the basic Rao–Blackwell Theorem. An application of this extension is carried out to construct a proper unbiased estimator of the expected value of a fuzzy random variable in the random sampling with replacement from a finite population.

1. INTRODUCTION

Fuzzy random variables, as intended by Puri and Ralescu [12], were introduced as an operational mathematical model for a quantification process in a random experiment, which associates a fuzzy value with each experimental outcome.

To describe the central tendency of a fuzzy random variable, the (fuzzy) expected value has been defined by Puri and Ralescu [12]. To measure the "variability" of a fuzzy random variable with respect to its fuzzy expected value, a generalized measure has been recently stated by Lubiano et al [10] (see also Lubiano [8], Lubiano and Körner [11] for a more general definition).

In a previous paper (López–Díaz and Gil [7]) we have examined the problem of computing expectations of fuzzy random variables from product probability spaces by means of iterated expectations, and conditions have been given under which the order in iterated fuzzy expectation does not matter.

In this paper, we are going to employ the results by López–Díaz and Gil [7] to obtain from a given fuzzy unbiased estimator a new fuzzy unbiased estimator (the unbiasedness being understood in Puri and Ralescu's expected value sense). On the other hand, the use of the generalized real-valued measure of absolute variation above mentioned will allow us to guarantee that the new fuzzy unbiased estimator

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is "more precise" than the first one. In other words, we are going to develop an extension of the basic version of the well-known Rao–Blackwell Theorem to fuzzy random variables.

Finally, an example illustrating this extended result is presented.

2. PRELIMINARIES

Fuzzy random variables have been presented by Puri and Ralescu in connection with a Euclidean space of an arbitrary finite dimension. For purposes of operativeness, a convexity condition for variable values is often added.

Let $\mathcal{K}(\mathbb{R}^k)$ $(\mathcal{K}_c(\mathbb{R}^k))$ denote the class of nonempty compact (respectively, nonempty compact convex) subsets of $\mathbb{R}^k$, with $k \in \mathbb{N}$.

Let $\mathcal{F}(\mathbb{R}^k)$ $(\mathcal{F}_c(\mathbb{R}^k))$ denote the class of the upper semicontinuous elements $\widetilde{V}$ of $[0,1]^{\mathbb{R}^k}$ such that the α-level sets $\widetilde{V}_\alpha$ belong to $\mathcal{K}(\mathbb{R}^k)$ (respectively, to $\mathcal{K}_c(\mathbb{R}^k)$) for all $\alpha \in [0,1]$, with $\widetilde{V}_\alpha = \{x \in \mathbb{R}^k \mid \widetilde{V}(x) \geq \alpha\}$ for $\alpha \in (0,1]$, and $\widetilde{V}_0 = \overline{\text{co}}\{x \in \mathbb{R}^k \mid \widetilde{V}(x) > 0\}$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. A mapping $\mathcal{X} : \Omega \to \mathcal{F}(\mathbb{R}^k)$ is said to be a *fuzzy random variable* (also referred to in the literature as a *random fuzzy set*, or simply an $\mathcal{F}(\mathbb{R}^k)$*-valued random element*) associated with the measurable space $(\Omega, \mathcal{A})$ if, and only if, the section $\mathcal{X}_\alpha : \Omega \to \mathcal{K}(\mathbb{R}^k)$, which is called the *$\alpha$-level function* and is defined by $\mathcal{X}_\alpha(\omega) = (\mathcal{X}(\omega))_\alpha$ for all $\omega \in \Omega$, is a compact random set (or a compact convex random set if it is $\mathcal{K}_c(\mathbb{R}^k)$-valued) for all $\alpha \in [0,1]$.

A fuzzy random variable $\mathcal{X}$ is said to be an *integrably bounded fuzzy random variable* associated with the probability space $(\Omega, \mathcal{A}, P)$ if, and only if, $\|\mathcal{X}_0\| \in L^1(\Omega, \mathcal{A}, P)$, where $\|\mathcal{X}_0(\omega)\| = d_H(\{0\}, \mathcal{X}_0(\omega))$ for all $\omega \in \Omega$, d_H being the well-known Hausdorff metric on $\mathcal{K}(\mathbb{R}^k)$.

If $\mathcal{X}$ is an integrably bounded fuzzy random variable associated with the probability space $(\Omega, \mathcal{A}, P)$, the *expected value* of $\mathcal{X}$ (Puri and Ralescu [12]) is the unique element in $\mathcal{F}(\mathbb{R}^k)$, $\widetilde{E}(\mathcal{X}|P)$, with the property $\left(\widetilde{E}(\mathcal{X}|P)\right)_\alpha = E(\mathcal{X}_\alpha|P)$ for all $\alpha \in [0,1]$, where $E(\mathcal{X}_\alpha|P)$ means the Aumann integral of $\mathcal{X}_\alpha$ in Ω, or expected value of the random set $\mathcal{X}_\alpha$ with respect to P (i.e., $E(\mathcal{X}_\alpha|P) = \{\, E(f|P) \mid f : \Omega \to \mathbb{R}^k, f \in L^1(\Omega, \mathcal{A}, P), f \in \mathcal{X}_\alpha \text{ a.s.} [P] \,\}$, $E(f|P)$ being the Bochner integral of f over Ω with respect to P – see Aumann [1]). $\widetilde{E}(\mathcal{X}|P)$ can be proven to belong to $\mathcal{F}(\mathbb{R}^k)$ (see Puri and Ralescu [12]).

In particular, when $\mathcal{X}$ is $\mathcal{F}_c(\mathbb{R})$-valued, then $\inf \mathcal{X}_\alpha$ and $\sup \mathcal{X}_\alpha$ are real-valued random variables, and $\left(\widetilde{E}(\mathcal{X}|P)\right)_\alpha = \left[E(\inf \mathcal{X}_\alpha|P), E(\sup \mathcal{X}_\alpha|P)\right]$ for all $\alpha \in [0,1]$.

In accordance with López-Díaz and Gil [7], we can state the following result:

Assume that $(\Omega, \mathcal{A}, P)$ is a probability space, and $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R}^k)$, $\mathcal{Y} : \Omega \to \mathcal{F}_c(\mathbb{R}^k)$ are two integrably bounded fuzzy random variables associated with it. Let $\sigma_{\mathcal{X}}$ and $\sigma_{\mathcal{Y}}$ be the σ-fields in $\mathcal{F}_c(\mathbb{R}^k)$ induced from $\mathcal{A}$ by $\mathcal{X}$ and $\mathcal{Y}$, respectively (that is, $\sigma_{\mathcal{X}} = \{B \subset \mathcal{F}_c(\mathbb{R}^k) \mid \mathcal{X}^{-1}(B) \in \mathcal{A}\}$, $\sigma_{\mathcal{Y}} = \{B \subset \mathcal{F}_c(\mathbb{R}^k) \mid \mathcal{Y}^{-1}(B) \in \mathcal{A}\}$, and let $P_{\mathcal{X}}$ and $P_{\mathcal{Y}}$ be the probability measures induced from P by $\mathcal{X}$ and $\mathcal{Y}$, respectively. Consider the product probability space $(\mathcal{F}_c(\mathbb{R}^k) \times \mathcal{F}_c(\mathbb{R}^k), \sigma_{\mathcal{X}} \otimes \sigma_{\mathcal{Y}}, P_{\mathcal{X}} \otimes P_{\mathcal{Y}})$, and let $\mathcal{X}^* : \mathcal{F}_c(\mathbb{R}^k) \times \mathcal{F}_c(\mathbb{R}^k) \to \mathcal{F}_c(\mathbb{R}^k)$ be the integrably bounded fuzzy random variable such that $\mathcal{X}^*(\tilde{x},\tilde{y}) = \tilde{x}$, for all $\tilde{x}$, $\tilde{y} \in \mathcal{F}_c(\mathbb{R}^k)$. Assume that when $\mathcal{Y} = \tilde{y}$ the conditional probability distribution induced by $\mathcal{X}$ is given by a regular conditional probability distribution on $(\mathcal{F}_c(\mathbb{R}^k), \sigma_X)$ denoted by $P_{\tilde{y}}$ (see, for instance, Breiman [3, pp. 67–81]). Then, if we identify $\widetilde{E}(\mathcal{X}|\mathcal{Y} = \tilde{y}) = \widetilde{E}(\mathcal{X}^*_{\tilde{y}}|P_{\tilde{y}})$ and $\widetilde{E}\left(\widetilde{E}(\mathcal{X}|\mathcal{Y})|P_{\mathcal{Y}}\right) = \widetilde{E}\left(\widetilde{E}(\mathcal{X}|\mathcal{Y} = \tilde{y})|P_{\mathcal{Y}}\right)$, we obtain that

$$\widetilde{E}(\mathcal{X}|P) = \widetilde{E}\left(\widetilde{E}(\mathcal{X}|\mathcal{Y})|P_{\mathcal{Y}}\right).$$

On the other hand, if $\mathcal{X}$ is an $\mathcal{F}_c(\mathbb{R})$-valued random element, then the *central S-mean squared dispersion* of $\mathcal{X}$ is given (Lubiano [8], Lubiano et al [10], Lubiano and Körner [11]) by

$$\Delta^2_S(\mathcal{X}|P) = \int_\Omega \left[D_S\left(\mathcal{X}(\omega), \widetilde{E}(\mathcal{X})\right)\right]^2 \mathrm{d}P(\omega),$$

where D_S is the metric on $\mathcal{F}_c(\mathbb{R})$ defined by Bertoluzza et al [2] so that for all $\widetilde{A}$, $\widetilde{B} \in \mathcal{F}_c(\mathbb{R})$

$$D_S\left(\widetilde{A},\widetilde{B}\right) = \sqrt{\int_{(0,1]} \left[\mathrm{d}_S\left(\widetilde{A}_\alpha, \widetilde{B}_\alpha\right)\right]^2 \mathrm{d}\alpha},$$

with

$$d_S\left(\widetilde{A}_\alpha,\widetilde{B}_\alpha\right) = \sqrt{\int_{[0,1]} \left[f_{\widetilde{A}}(\alpha,\lambda) - f_{\widetilde{B}}(\alpha,\lambda)\right]^2 \mathrm{d}S(\lambda)},$$

and

$$f_{\widetilde{A}}(\alpha,\lambda) = \lambda \sup \widetilde{A}_\alpha + (1-\lambda)\inf \widetilde{A}_\alpha,$$

S being a normalized weight measure on $([0,1], \mathcal{B}_{[0,1]})$ which can be expressed as the sum of a term being absolutely continuous with respect to the Lebesgue measure m on $[0,1]$, and another term corresponding to a weighted finite distribution on a finite set $\{\lambda_1,\ldots,\lambda_L\}$, that is,

$$\mathrm{d}S = g(\lambda)\,\mathrm{d}\lambda,$$

with

$$g(\lambda) = \overline{g}(\lambda) + \sum_{l=1}^{L} k_l \delta|\lambda - \lambda_l|,$$

where $g(0) > 0$, $g(1) > 0$, $\lambda_1 = 0$, $\lambda_L = 1$, $\overline{g}$ is a Lebesgue-measurable function and δ denotes the Dirac distribution (that is, $\delta|\lambda - \lambda_l| = 1$ if $\lambda = \lambda_l$, $= 0$ otherwise).

The central S-mean squared dispersion have been proved to satisfy suitable properties for a measure of the absolute variation of a fuzzy random variable, and it preserves the most valuable features from the real-valued case (see Lubiano [8], Lubiano and Gil [9], Lubiano et al [10], Lubiano and Körner [11]).

3. RAO–BLACKWELLIZATION PROCESS FOR FUZZY RANDOM VARIABLES

In this section we are going to present a result on the basis of which we can obtain a fuzzy unbiased estimator from another given one and having lower central mean squared dispersion than it. This method extends the ideas in the Rao–Blackwell Theorem (see, for instance, Dudewicz and Mishra [6], Casella and Berger [4]).

Theorem. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\mathcal{X} : \Omega \longrightarrow \mathcal{F}_c(\mathbb{R})$ and $\mathcal{Y} : \Omega \longrightarrow \mathcal{F}_c(\mathbb{R})$ be two integrably bounded fuzzy random variables. Let $\sigma_{\mathcal{X}}$ and $\sigma_{\mathcal{Y}}$ be the σ-fields in $\mathcal{F}_c(\mathbb{R})$ induced from $\mathcal{A}$ by $\mathcal{X}$ and $\mathcal{Y}$, respectively, and let $P_{\mathcal{X}}$ and $P_{\mathcal{Y}}$ be the probability measures induced from P by $\mathcal{X}$ and $\mathcal{Y}$, respectively.

Consider the product probability space $(\mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}), \sigma_{\mathcal{X}} \otimes \sigma_{\mathcal{Y}}, P_{\mathcal{X}} \otimes P_{\mathcal{Y}})$, and let $\mathcal{Y}^* : \mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}) \longrightarrow \mathcal{F}_c(\mathbb{R})$ be the integrably bounded fuzzy random variable such that $\mathcal{Y}^*(\tilde{x}, \tilde{y}) = \tilde{y}$ for all $\tilde{x}, \tilde{y} \in \mathcal{F}_c(\mathbb{R})$. Assume that when $\mathcal{X} = \tilde{x}$ the conditional probability distribution induced by $\mathcal{Y}$ is given by a regular conditional probability distribution on $(\mathcal{F}_c(\mathbb{R}), \sigma_{\mathcal{Y}})$ denoted by $P_{\tilde{x}}$, that is,

- $P_{\tilde{x}}$ is a probability measure on $(\mathcal{F}_c(\mathbb{R}), \sigma_{\mathcal{Y}})$ for each $\tilde{x} \in \mathcal{X}(\Omega)$, and
- for each $B \in \sigma_{\mathcal{Y}}$, the mapping $g_B : \mathcal{X}(\Omega) \longrightarrow [0,1]$ such that $g_B(\tilde{x}) = P_{\tilde{x}}(B)$ is a real-valued random variable associated with the measurable space $(\mathcal{F}_c(\mathbb{R}), \sigma_{\mathcal{X}})$, and satisfying that for all $A \in \sigma_{\mathcal{X}}$

$$P(\mathcal{X} \in A,\ \mathcal{Y} \in B) = \int_A P_{\tilde{x}}(B)\,\mathrm{d}P_{\mathcal{X}}.$$

Assume that $\widetilde{V} \in \mathcal{F}_c(\mathbb{R})$ is a fuzzy parameter and $\widetilde{E}(\mathcal{Y}|P) = \widetilde{V}$ (that is, $\mathcal{Y}$ is a fuzzy unbiased estimator of $\widetilde{V}$), and such that $\Delta_S^2(\mathcal{Y}|P) < \infty$. Let $\varphi : \mathcal{X}(\Omega) \longrightarrow \mathcal{F}_c(\mathbb{R})$ be defined so that $\varphi(\tilde{x}) = \widetilde{E}(\mathcal{Y}|\mathcal{X} = \tilde{x}) = \widetilde{E}(\mathcal{Y}^*_{\tilde{x}}|P_{\tilde{x}})$ for all $\tilde{x} \in \mathcal{X}(\Omega)$. Then, $\widetilde{E}(\varphi(\mathcal{X})|P) = \widetilde{V}$, and $\Delta_S^2(\varphi(\mathcal{X})|P) \le \Delta_S^2(\mathcal{Y}|P)$, with equality if, and only if, $f_{\mathcal{Y}(\omega)}(\alpha, \lambda) = f_{\varphi(\mathcal{X}(\omega))}(\alpha, \lambda)$ a. e. $[m \otimes S \otimes P]$.

Proof. Indeed, in accordance with the results stated by López–Díaz and Gil [7] in connection with the computation of iterated expectations of fuzzy random variables, irrespectively of the order of integration, we have that

$$\widetilde{E}\left(\widetilde{E}\left(\mathcal{Y}^*_{\tilde{y}}|P_{\tilde{y}}\right)|P_{\mathcal{Y}}\right) = \widetilde{E}\left(\widetilde{E}\left(\mathcal{Y}^*_{\tilde{x}}|P_{\tilde{x}}\right)|P_{\mathcal{X}}\right),$$

and

$$\widetilde{E}(\mathcal{Y}|P) = \widetilde{E}\left(\widetilde{E}\left(\mathcal{Y}^*_{\tilde{y}}|P_{\tilde{y}}\right)|P_{\mathcal{Y}}\right).$$

If $\varphi(\tilde{x}) = \widetilde{E}\left(\mathcal{Y}^*_{\tilde{x}}|P_{\tilde{x}}\right)$, then

$$\widetilde{V} = \widetilde{E}(\mathcal{Y}|P) = \widetilde{E}\left(\widetilde{E}\left(\mathcal{Y}^*_{\tilde{x}}|P_{\tilde{x}}\right)|P_{\mathcal{X}}\right) = \widetilde{E}(\varphi(\mathcal{X})|P).$$

On the other hand,

$$\Delta_S^2(\mathcal{Y}|P) = \int_\Omega \int_{(0,1]} \int_{[0,1]} \left[f_{\mathcal{Y}(\omega)}(\alpha,\lambda) - f_{\widetilde{V}}(\alpha,\lambda)\right]^2 \, \mathrm{d}S(\lambda)\, \mathrm{d}\alpha\, \mathrm{d}P(\omega),$$

and, in virtue of the classical Fubini Theorem, we have that

$$\begin{aligned}
\Delta_S^2(\mathcal{Y}|P) &= \int_{(0,1]} \int_{[0,1]} \left(\int_\Omega \left[f_{\mathcal{Y}(\omega)}(\alpha,\lambda) - f_{\widetilde{V}}(\alpha,\lambda)\right]^2 \mathrm{d}P(\omega)\right) \mathrm{d}S(\lambda)\,\mathrm{d}\alpha \\
&= \int_{(0,1]} \int_{[0,1]} \left(\int_\Omega \left[f_{\mathcal{Y}(\omega)}(\alpha,\lambda) - f_{\varphi(\tilde{x})}(\alpha,\lambda)\right]^2 \mathrm{d}P(\omega)\right) \mathrm{d}S(\lambda)\,\mathrm{d}\alpha \\
&\quad + \int_{(0,1]} \int_{[0,1]} \left(\int_\Omega \left[f_{\varphi(\tilde{x})}(\alpha,\lambda) - f_{\widetilde{V}}(\alpha,\lambda)\right]^2 \mathrm{d}P(\omega)\right) \mathrm{d}S(\lambda)\,\mathrm{d}\alpha \\
&\quad + 2 \int_{(0,1]} \int_{[0,1]} \left(\int_\Omega \left[f_{\mathcal{Y}(\omega)}(\alpha,\lambda) - f_{\varphi(\tilde{x})}(\alpha,\lambda)\right] \right. \\
&\quad \left. \cdot \left[f_{\varphi(\tilde{x})}(\alpha,\lambda) - f_{\widetilde{V}}(\alpha,\lambda)\right] \mathrm{d}P(\omega)\right) \mathrm{d}S(\lambda)\,\mathrm{d}\alpha.
\end{aligned}$$

Since $\varphi(\tilde{x}) = \widetilde{E}\left(\mathcal{Y}^*_{\tilde{x}}|P_{\tilde{x}}\right)$, then for all $\alpha \in (0,1]$ and $\lambda \in [0,1]$

$$\begin{aligned}
&\int_\Omega \left[f_{\mathcal{Y}(\omega)}(\alpha,\lambda) - f_{\varphi(\tilde{x})}(\alpha,\lambda)\right] \left[f_{\varphi(\tilde{x})}(\alpha,\lambda) - f_{\widetilde{V}}(\alpha,\lambda)\right] \mathrm{d}P(\omega) \\
&= \int_{\mathcal{X}(\Omega)} \left[f_{\varphi(\tilde{x})}(\alpha,\lambda) - f_{\widetilde{V}}(\alpha,\lambda)\right] \left(\int_{\mathcal{Y}(\Omega)} \left[f_{\tilde{y}}(\alpha,\lambda) - f_{\varphi(\tilde{x})}(\alpha,\lambda)\right] \mathrm{d}P_{\tilde{x}}(\tilde{y})\right) \mathrm{d}P_{\mathcal{X}}(\tilde{x}),
\end{aligned}$$

whence for all $\alpha \in (0,1]$ and $\lambda \in [0,1]$ we have that

$$\begin{aligned}
&\int_{\mathcal{Y}(\Omega)} \left[f_{\tilde{y}}(\alpha,\lambda) - f_{\varphi(\tilde{x})}(\alpha,\lambda)\right] \mathrm{d}P_{\tilde{x}}(\tilde{y}) = \int_{\mathcal{Y}(\Omega)} f_{\tilde{y}}(\alpha,\lambda)\,\mathrm{d}P_{\tilde{x}}(\tilde{y}) - f_{\varphi(\tilde{x})}(\alpha,\lambda) \\
&= \int_{\mathcal{Y}(\Omega)} \left[\lambda \sup \tilde{y}_\alpha + (1-\lambda)\inf \tilde{y}_\alpha\right] \mathrm{d}P_{\tilde{x}}(\tilde{y}) \\
&\quad - \left[\lambda \sup \left(\widetilde{E}\left(\mathcal{Y}^*_{\tilde{x}}|P_{\tilde{x}}\right)\right)_\alpha + (1-\lambda)\inf \left(\widetilde{E}\left(\mathcal{Y}^*_{\tilde{x}}|P_{\tilde{x}}\right)\right)_\alpha\right].
\end{aligned}$$

Since

$$\left(\widetilde{E}\left(\mathcal{Y}^*_{\tilde{x}}|P_{\tilde{x}}\right)\right)_\alpha = \left[\int_{\mathcal{Y}(\Omega)} \inf \tilde{y}_\alpha\,\mathrm{d}P_{\tilde{x}}(\tilde{y}), \int_{\mathcal{Y}(\Omega)} \sup \tilde{y}_\alpha\,\mathrm{d}P_{\tilde{x}}(\tilde{y})\right],$$

then for all $\alpha \in (0,1]$ and $\lambda \in [0,1]$ we can conclude that

$$\int_{\mathcal{Y}(\Omega)} \left[f_{\tilde{y}}(\alpha,\lambda) - f_{\varphi(\tilde{x})}(\alpha,\lambda)\right] \mathrm{d}P_{\tilde{x}}(\tilde{y}) = 0$$

whatever $\tilde{x} \in \mathcal{X}(\Omega)$ may be.

Consequently,

$$\Delta_S^2(\mathcal{Y}|P) = \int_{(0,1]}\int_{[0,1]}\left(\int_\Omega \left[f_{\mathcal{Y}(\omega)}(\alpha,\lambda) - f_{\varphi(\bar{x})}(\alpha,\lambda)\right]^2 \mathrm{d}P(\omega)\right)\mathrm{d}S(\lambda)\,\mathrm{d}\alpha$$
$$+\int_{(0,1]}\int_{[0,1]}\left(\int_\Omega \left[f_{\varphi(\bar{x})}(\alpha,\lambda) - f_{\widetilde{V}}(\alpha,\lambda)\right]^2 \mathrm{d}P(\omega)\right)\mathrm{d}S(\lambda)\,\mathrm{d}\alpha.$$

Furthermore,

$$\Delta_S^2(\varphi(\mathcal{X})|P) = \int_{(0,1]}\int_{[0,1]}\left(\int_\Omega \left[f_{\varphi(\bar{x})}(\alpha,\lambda) - f_{\widetilde{V}}(\alpha,\lambda)\right]^2 \mathrm{d}P(\omega)\right)\mathrm{d}S(\lambda)\,\mathrm{d}\alpha$$

so that

$$\Delta_S^2(\mathcal{Y}|P) \geq \Delta_S^2(\varphi(\mathcal{X})|P),$$

and hence $\varphi(\mathcal{X})$ is "better" than $\mathcal{Y}$ to estimate $\widetilde{V}$. In addition, we have that $\Delta_S^2(\mathcal{Y}|P) = \Delta_S^2(\varphi(\mathcal{X})|P)$ if, and only if, $f_{\mathcal{Y}(\omega)}(\alpha,\lambda) = f_{\varphi(\mathcal{X}(\omega))}(\alpha,\lambda)$ a. e. $[m \otimes S \otimes P]$. □

The theorem above can be particularized in an obvious way to the case in which one of the involved variables is real-valued. In the example enclosed in the following section we will find this particular situation.

4. ILLUSTRATIVE EXAMPLE

The result we have established in the preceding section can be illustrated by means of the following example, which refers to the estimation of the expected value of a fuzzy random variable in the random sampling with replacement from finite populations.

Example. Let $\mathcal{X}$ be a fuzzy random variable which is defined on a population Ω of N sampling units, $\omega_1, \ldots, \omega_N$.

Assume that a sample of size n is chosen at random and with replacement from Ω. In Lubiano and Gil [9] (see Lubiano [8] for a more general proof), it has been proved that the fuzzy sample mean $\overline{\mathcal{X}}_n$ which associates with a random sample v of size n from Ω with units $\omega_{v1}, \ldots, \omega_{vn}$, the fuzzy expected value of the variable taking on the values of $\mathcal{X}(\omega_{v1}), \ldots, \mathcal{X}(\omega_{vn})$, with probabilities $1/n$, that is, for all $\alpha \in [0,1]$

$$\left(\overline{\mathcal{X}}_n(v)\right)_\alpha = \left[\frac{1}{n}\sum_{i=1}^n \inf \mathcal{X}_\alpha(\omega_{vi}), \frac{1}{n}\sum_{i=1}^n \sup \mathcal{X}_\alpha(\omega_{vi})\right],$$

defines a fuzzy unbiased estimator of the fuzzy expected value $\widetilde{E}(\mathcal{X})$ of $\mathcal{X}$ over Ω, and $\Delta_S^2(\overline{\mathcal{X}}_n) = \Delta_S^2(\mathcal{X})/n$.

The use of the extension of the Rao–Blackwell Theorem developed in Section 3, allows us to construct an unbiased estimator of $\widetilde{E}(\mathcal{X})$ with a lower central S-mean squared dispersion.

More precisely, consider the probability space $(\Upsilon_n^w, \mathcal{P}(\Upsilon_n^w), p^w)$ (Υ_n^w being the space of the $CR_{N,n} = \binom{N+n-1}{n}$ distinct possible random samples with replacement of size n from Ω, and $p^w[v]$ being the probability of choosing the sample $v \in \Upsilon_n^w$). $\overline{\mathcal{X}}_n$ is a fuzzy random variable associated with $(\Upsilon_n^w, \mathcal{P}(\Upsilon_n^w), p^w)$.

Let Υ_k be the space of the $C_{N,k} = \binom{N}{k}$ distinct possible random samples without replacement of size k from Ω. Consider also an arbitrary ordering on the set $\bigcup_{k=1}^n \Upsilon_k$ of all simple random samples from Ω of size lower than or equal to n. Let $M = \operatorname{card}(\bigcup_{k=1}^n \Upsilon_k) = \sum_{i=1}^n \binom{N}{i}$, and let $I\!M = \{1, \ldots, M\}$. One can state a real-valued random variable $Y : \Upsilon_n^w \to I\!M$ associated with the probability space $(\Upsilon_n^w, \mathcal{P}(\Upsilon_n^w), p^w)$ and defined so that, for each sample $v \in \Upsilon_n^w$, $Y(v)$ means the rank of the simple random sample of the distinct units in v in the ordering considered on Ω.

If $m \in Y(\Upsilon_n^w)$ and m is the rank corresponding to a simple random sample y_m having k distinct units $\omega_1^*(y_m), \ldots, \omega_k^*(y_m)$ the (conditional given m) probability of $\omega_i^*(y_m)$ to belong to a sample in Υ_n^w for which Y takes on the value m equals $1/k$, $i = 1, \ldots, k$, so that for all $\alpha \in [0,1]$

$$\left(\widetilde{E}(\overline{\mathcal{X}}_n | Y = m)\right)_\alpha = \left[\frac{1}{k}\sum_{i=1}^k \inf \mathcal{X}_\alpha(\omega_i^*(y_m)), \frac{1}{k}\sum_{i=1}^k \sup \mathcal{X}_\alpha(\omega_i^*(y_m))\right].$$

Consequently, $\widetilde{E}(\overline{\mathcal{X}}_n | Y = m)$ is equivalent to the (conditional given m) expected value of the fuzzy estimator $\overline{\mathcal{X}}_\nu$ associating with each $v \in \Upsilon_n^w$ the expected value of $\mathcal{X}$ over the units in the sample associated with $Y(v)$. This estimator is a fuzzy random variable defined on the space $(\Upsilon_n^w, \mathcal{P}(\Upsilon_n^w), p^w)$, and whose distribution depends on the real-valued random variable *effective sample size* ν (see, for instance, Thompson [13]). Then,

$$\widetilde{E}(\overline{\mathcal{X}}_\nu | Y = m) = \widetilde{E}(\overline{\mathcal{X}}_n | Y = m).$$

Therefore, if we consider $\varphi(Y)$ such that $\varphi(m) = \widetilde{E}(\overline{\mathcal{X}}_n | Y = m) = \widetilde{E}(\overline{\mathcal{X}}_\nu | Y = m)$, and $\# y_m$ denotes the number of distinct units in y_m, we have that

$$\begin{aligned}\left(\widetilde{E}(\varphi(Y)) | p^w\right)_\alpha &= \sum_{k=1}^n \left[\sum_{m \in Y(\Upsilon_n^w) \mid \# y_m = k} \left(\varphi(m)\right)_\alpha P(Y = m | \nu = k)\right] P_\nu(k) \\ &= \sum_{k=1}^n \left(\widetilde{E}(\overline{\mathcal{X}}_\nu | \nu = k)\right)_\alpha P_\nu(k),\end{aligned}$$

(which coincides with $\left(\widetilde{E}(\overline{\mathcal{X}}_\nu) | p^w\right)_\alpha$ for all $\alpha \in [0,1]$ (see López–Díaz and Gil [7]).

Since, in virtue of the results by Lubiano [8], and Lubiano and Gil [9], we have that $\widetilde{E}(\overline{\mathcal{X}}_\nu | \nu = k) = \widetilde{E}(\mathcal{X})$ for $k = 1, \cdots, n$, then

$$\widetilde{E}(\varphi(Y) | p^w) = \widetilde{E}(\mathcal{X} | p^w).$$

Moreover, Theorem 3.1 guarantees that

$$\Delta_S^2(\varphi(Y) | p^w) \leq \Delta_S^2\left(\overline{\mathcal{X}}_\nu | p^w\right),$$

and, on the basis of the results in Lubiano [8], Lubiano et al [10], Lubiano and Körner [11],

$$\begin{aligned}
\Delta_S^2\left(\overline{\mathcal{X}}_\nu|p^w\right) &= \int_{(0,1]}\int_{[0,1]} \mathrm{Var}\left[f_{\overline{\mathcal{X}}_\nu}(\alpha,\lambda)|p^w\right]\,\mathrm{d}S(\lambda)\,\mathrm{d}\alpha \\
&= \int_{(0,1]}\int_{[0,1]} \left[\sum_{k=1}^{n} \mathrm{Var}\left[f_{\overline{\mathcal{X}}_\nu}(\alpha,\lambda)|\nu=k\right] P_\nu(k)\right. \\
&\quad \left. +\mathrm{Var}\left(E\left[f_{\overline{\mathcal{X}}_\nu}(\alpha,\lambda)|\nu\right]\right)\right]\,\mathrm{d}S(\lambda)\,\mathrm{d}\alpha \\
&= \int_{(0,1]}\int_{[0,1]} \left[\sum_{k=1}^{n} \mathrm{Var}\left(\overline{(f_{\mathcal{X}}(\alpha,\lambda))}_\nu|\nu=k\right) P_\nu(k)\right. \\
&\quad \left. +\mathrm{Var}\left(E\left(\overline{(f_{\mathcal{X}}(\alpha,\lambda))}_\nu|\nu\right)\right)\right]\,\mathrm{d}S(\lambda)\,\mathrm{d}\alpha,
\end{aligned}$$

where $\overline{(f_{\mathcal{X}}(\alpha,\lambda))}_\nu$ represents the sample mean of the real-valued random variable $f_{\mathcal{X}}(\alpha,\lambda)$ for the distinct units in the sample. Following the conclusions in Sampling Theory for real-valued random variables (see, for instance, Raj and Khamis [5], Thompson [13, pp. 20, 90]), $\overline{(f_{\mathcal{X}}(\alpha,\lambda))}_\nu$ is an unbiased estimator of $\overline{f_{\mathcal{X}(\alpha,\lambda)}} = f_{\overline{\mathcal{X}}}(\alpha,\lambda)$ for any value of ν, whence $\mathrm{Var}\left(E\left(\overline{(f_{\mathcal{X}}(\alpha,\lambda))}_\nu|\nu\right)\right) = 0$.

On the other hand, and also in virtue of the results in Sample Theory for real-valued random variables, we can conclude that

$$\mathrm{Var}\left(\overline{(f_{\mathcal{X}}(\alpha,\lambda))}_\nu|\nu=k\right) = \left(\frac{1}{k}-\frac{1}{N}\right)\mathrm{Var}\left(f_{\mathcal{X}}(\alpha,\lambda)\right),$$

and hence

$$\begin{aligned}
\Delta_S^2\left(\overline{\mathcal{X}}_\nu|p^w\right) &= \int_{(0,1]}\int_{[0,1]} \left[\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{N}\right)\mathrm{Var}\left(f_{\mathcal{X}}(\alpha,\lambda)|p^w\right)P_\nu(k)\right]\,\mathrm{d}S(\lambda)\,\mathrm{d}\alpha \\
&= \left[E\left(\frac{1}{\nu}\right)-\frac{1}{N}\right]\frac{N\Delta_S^2(\mathcal{X})}{N-1}.
\end{aligned}$$

Since, in accordance with Raj and Kharmis [5], we have that $E\left(\frac{1}{\nu}\right) \leq \frac{1}{N}+\frac{N-1}{nN}$, with equality if, and only if, $n=2$, then

$$\Delta_S^2(\varphi(Y)|p^w) \leq \Delta_S^2\left(\overline{\mathcal{X}}_n|p^w\right),$$

with equality if, and only if, $n=2$.

5. CONCLUDING REMARKS

The results in this paper can be easily particularized to the case in which the imprecisely-valued random elements correspond to random compact convex sets, and especially to the case in which we deal with grouped data.

An interesting open problem connected with the result above stated is that of using it to reduce the S-mean squared dispersion of a fuzzy unbiased estimator by using a sufficient statistic (which is sometimes known in the real-valued case as the Rao–Blackwell Improvement Theorem – see, for instance, Dudewicz and Mishra [6]). The main inconvenience to face this problem is that concerning the formalization of the notion of sufficiency for fuzzy random variables and parameters.

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