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REACHABILITY AND OBSERVABILITY OF LINEAR SYSTEMS OVER MAX–PLUS[1]

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This paper discusses the properties of reachability and observability for linear systems over the max-plus algebra. Working in the event-domain, the concept of asticity is used to develop conditions for weak reachability and weak observability. In the reachability problem, residuation is used to determine if a state is reachable and to generate the required control sequence to reach it. In the observability problem, residuation is used to estimate the state. Finally, as in the continuous-variable case, a duality is shown to exist between the two properties.

1. INTRODUCTION

The max-plus algebra can be used to describe, in a linear fashion, the timing dynamics of systems that are nonlinear in the conventional algebra. Examples of such systems include discrete part manufacturing lines such as automotive assembly lines and electronic circuit board assembly lines, as well as transportation and communication systems. The dynamics of these types of systems are governed by events rather than time as in the more familiar continuous-variable systems. Because of their dependency on events, these systems have become to be known as discrete event dynamic systems (DEDS). For a special class of DEDS that do not contain routing decisions, it is well known that the dynamics of the timing of events can be written over the max-plus algebra [1, 3, 4].

This paper discusses the system properties of reachability and observability of event-index-invariant, linear systems over max-plus without the need of graph-based arguments. Analogous to the time-invariant case for continuous-variable systems, an event-index-invariant system is one in which the system parameters do not change with respect to the event index. Because the properties discussed here are not as strong as those in the continuous-variable case, definitions of weak reachability and weak observability are introduced. Also, necessary and sufficient conditions for a system to be weakly reachable and weakly observable are presented. Definitions and conditions for stronger properties are under development.

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Results presented here are based on [7] which, in turn, is based on an original concept of reachability and observability over max-plus defined in [8] in 1993.

A complementary treatment of reachability and observability in max-plus can be found in [10]. There, working with a different mapping, the authors consider the equivalency of determining reachability of a state in the max-plus algebra to the finding of eigenvectors in the min-plus algebra, and pose an open question regarding when such an equivalency holds.

The paper is organized as follows. Section 2 briefly reviews linear systems over max-plus, Section 3 presents the system properties of weak reachability and weak observability, and Section 4 concludes the paper and discusses further research.

2. MAX-PLUS LINEAR SYSTEMS

The max-plus algebra is both a semi-field and a semi-ring. Also, it is a dioid [3], i. e., addition is idempotent which implies that there are no nontrivial inverses. Because a multiplicative inverse does exist, however, the algebra is a semi-field. The max-plus structure used here is denoted as $\mathbb{R}_{\max}$ and is defined next.

Definition 1. ($\mathbb{R}_{\max}$) With $\mathbb{R}$ representing the set of real numbers, the set $\mathbb{R} \cup \{-\infty\}$ with $\oplus$ defined as maximization and $\otimes$ defined as conventional addition is a dioid and is denoted as $\mathbb{R}_{\max}$. The identity elements for addition and multiplication are $\epsilon = -\infty$ and $e = 0$, respectively.

In a similar fashion, let $\overline{\mathbb{R}}_{\max}$ represent the dioid consisting of the set $\mathbb{R} \cup \{-\infty\} \cup$ $\cup \{\infty\}$, and $\mathbb{R}^{+}_{\max}$ denote $\mathbb{R}^{+} \cup \{-\infty\}$ where $\mathbb{R}^{+}$ represents the set of nonnegative real numbers. A natural order is imposed on two vectors $x, y \in \overline{\mathbb{R}}^{n}_{\max}$ by defining $x \leq y$ if $x \oplus y = y$. The Cayley–Hamilton theorem holds over max-plus as well, although in a slightly different form than in the continuous-variable case. As shown in [9], the characteristic equation, written in the indeterminate variable z, of a $n \times n$ matrix A in $\mathbb{R}^{n \times n}_{\max}$ is given as $p^{+}_{A}(z) = p^{-}_{A}(z)$, where the coefficients of $p^{+}_{A}(z)$ and $p^{-}_{A}(z)$ involve finding the dominant of a matrix which for space consideration will not be detailed here (see [1]). Olsder and Roos show in [9] that the Cayley–Hamilton theorem holds in max-plus, that is $p^{+}_{A}(A) = p^{-}_{A}(A)$.

The general form of a linear, event-index-invariant system is given by

$$\begin{aligned} X(k+1) &= AX(k) \oplus BU(k+1) &(1)\\ Y(k) &= CX(k), &(2) \end{aligned}$$

where $k \in N^{+} = \{1, 2, \ldots\}$ is the event index, $X(k)$ is a $n \times 1$ vector of completion times for the kth event, $Y(k)$ is a $m \times 1$ vector of system output times, and $U(k)$ is a $p \times 1$ vector of part arrival times. The matrices A, B, and C are of the appropriate sizes, are functions of the system service and transportation times, and have entries ranging over $\mathbb{R}_{\max}$. The completion times evolve over the event index k according to (1), and the output times of the system evolve as specified in (2). Cohen et al. term this approach the "dater" representation since the dates or times of events are selected as the quantities of interest [3]. A method to generate the algebraic model given by (1) – (2) directly from a manufacturing system without the need of a graphical construction (such as a Petri net) has been developed by Doustmohammadi and Kamen and is given in [5].

3. SYSTEM PROPERTIES

Working with the system described by (1) – (2), definitions of weak reachability and weak observability are presented along with necessary and sufficient conditions to determine if a system is weakly reachable and if a system is weakly observable.

3.1. Reachability

Reachability refers to the issue of steering a system from the origin to a specified state using the input. For linear time-invariant systems in the continuous-variable case, reachability of a system is determined by the rank of the reachability matrix. If the rank is full, then all states in $\mathbb{R}^n$ can be obtained. For max-plus linear systems, the transfer to any arbitrary state is not possible except in very special cases. Hence, unlike the continuous-variable case, the event-time state space is seldom equal to all of $\mathbb{R}^n$.

Using (1) in a recursive fashion, we can write the state at event index q as $X(q) = A^q X(0) \oplus [B\ AB \cdots A^{q-1}B] \otimes [U^T(q)\ U^T(q-1) \cdots U^T(1)]^T$. By defining the reachability matrix,

$$\Gamma_q := [B \quad AB \quad \cdots \quad A^{q-1}B],$$

and using a shorthand notation for the input sequence, $\boldsymbol{U}_q = [U^T(q)\ U^T(q-1) \cdots U^T(1)]^T$, we can write $X(q) = A^q X(0) \oplus \Gamma_q \boldsymbol{U}_q$. Unlike the continuous-variable case, the contribution from the initial condition $X(0)$ cannot be subtracted out. Consider then, the following definition of a reachable state.

Definition 2. (Reachable State) Given $X(0) \in \mathbb{R}^n_{\max}$, a state $X \in \mathbb{R}^n$ is reachable in q-steps from $X(0)$ if there exists a control sequence $\{U(1), U(2), \cdots, U(q)\}$ over $\mathbb{R}_{\max}$ which achieves $X(q) = X$.

The collection of all such states leads to the following definition.

Definition 3. (Reachable Set) Given $X(0) \in \mathbb{R}^n_{\max}$, and a positive integer q, let $\Omega_{q,X(0)}$ be the set of all states $X \in \mathbb{R}^n$ that can be reached in q steps from $X(0)$, that is,

$$\Omega_{q,X(0)} = \quad \{X \in \mathbb{R}^n : X = A^q X(0) \oplus \Gamma_q \boldsymbol{U}_q, \quad \text{where } \boldsymbol{U}_q \text{ ranges over } \mathbb{R}^{pq}_{\max}\}.$$

The issue of reachability pertains to whether a control sequence $\{U(1), U(2), \cdots \cdots, U(q)\}$ can be found that achieves a desired state. To proceed further, we need a result that deals with the solutions of general equations of the form $A \otimes X = B$, where X and B are column vectors with elements in $\mathbb{R}_{\max}$. The solution involves an operation † that acts like an "inverse," and the min function. The operation †, called conjugation in [4], represents the negation and transpose operations, i.e., for $A = \{a_{ij}\}$, $A^\dagger = \{-a_{ji}\}$. The symbol $\otimes'$ represents multiplication using the min function, the counterpart to $\otimes$. From lattice theory, we have the following two results [1, 4].

Proposition 3.1. Given $A \in \mathbb{R}_{\max}^{m\times n}, B \in \mathbb{R}_{\max}^{m}$, there exists a solution in $\overline{\mathbb{R}}_{\max}^{n}$ to $AX = B$ if and only if

$$Z = A^\dagger \otimes' B \tag{3}$$

is a solution; furthermore, Z is actually the greatest solution.

Proposition 3.2. Given $A \in \mathbb{R}_{\max}^{m\times n}, B \in \mathbb{R}_{\max}^{m}$, the greatest subsolution in $\overline{\mathbb{R}}_{\max}^{n}$ to $AX \le B$ is $Z = A^\dagger \otimes' B$.

We can use these results to obtain a necessary and sufficient condition for a state to be reachable.

Theorem 3.1. Given an initial state $X(0) \in \mathbb{R}_{\max}^{n}$, and a state X, then $X \in \Omega_{q,X(0)}$ if and only if

$$\Gamma_q \otimes (\Gamma_q^\dagger \otimes' X) \oplus A^q X(0) = X, \tag{4}$$

in which case $\boldsymbol{U}_q = \Gamma_q^\dagger \otimes' X$ drives the system state from $X(0)$ to $X(q) = X$.

Proof. If (4) is true, then $\boldsymbol{U}_q = \Gamma_q^\dagger \otimes' X$. If $\boldsymbol{U}_q \in \mathbb{R}^{pq}$, then obviously $X \in \Omega_{q,X(0)}$; otherwise, if for some j, $(\boldsymbol{U}_q)_j = \infty$, then $(\boldsymbol{U}_q)_j$ can be set to any value in $\mathbb{R}_{\max}$ without changing the value of $\Gamma_q \otimes \boldsymbol{U}_q = \Gamma_q \otimes (\Gamma_q^\dagger \otimes' X)$. To see this, we note that each element of the jth row of $\Gamma_q^\dagger$ must equal $-\epsilon$; hence, the jth column of Γ_q must consist entirely of ϵ. Since ϵ is absorbing for any element in $\mathbb{R}_{\max}$, infinite values in $\boldsymbol{U}_q$ can be replaced by any value in $\mathbb{R}_{\max}$. Thus, $X \in \Omega_{q,X(0)}$. If $X \in \Omega_{q,X(0)}$, then, by definition, some $\boldsymbol{U}_q$ exists such that $X = A^q X(0) \oplus \Gamma_q \boldsymbol{U}_q$. Hence, $\Gamma_q \otimes \boldsymbol{U}_q \le X$. By Proposition 3.2, $\boldsymbol{U}_q = \Gamma_q^\dagger \otimes' X$ is the greatest subsolution; hence, $\Gamma_q \otimes (\Gamma_q^\dagger \otimes' X) \le X$. So, $\Gamma_q \otimes \boldsymbol{U}_q \le \Gamma_q \otimes (\Gamma_q^\dagger \otimes' X) \le X$. Adding $A^q X(0)$ to each term, we have, $A^q X(0) \oplus \Gamma_q \otimes \boldsymbol{U}_q \le A^q X(0) \oplus \Gamma_q \otimes (\Gamma_q^\dagger \otimes' X) \le A^q X(0) \oplus X$. Since the first and last terms are equal to X, $\Gamma_q \otimes (\Gamma_q^\dagger \otimes' X) \oplus A^q X(0) = X$, i.e., (4) is satisfied. □

In the continuous-variable case, a system that is reachable ensures that any state in $\mathbb{R}^n$ can be reached from the origin, i.e., the set of reachable states is all of $\mathbb{R}^n$. In the max-plus case, because of the max operation, $A^q X(0) \oplus \Gamma_q \boldsymbol{U}_q$ cannot be equal to states that are less than the unforced terminal state $A^q X(0)$. Also, note that for reachable systems in the continuous-variable case, all components of the state can be set arbitrarily via the input and each component can be modified independently of other components. In the max-plus case, it is not possible to ensure that all components can be set independently other than for a small class of systems [6]. Instead, we focus on systems for which it is possible to reach a state whose components are greater than the unforced terminal state and call such systems weakly reachable. We reserve the term completely reachable for systems for which any state in $\mathbb{R}^n$ can be obtained. Thus, we consider the following definition of a weakly-reachable system.

Definition 4. (q-step weakly reachable) A system is said to be q-step weakly reachable if given any $X(0)$, a control sequence exists such that each component of the terminal state $X(q)$ can be made greater than the unforced terminal state $A^q X(0)$, i. e., there exists $\boldsymbol{U}_q$ such that $(X(q))_j > (A^q X(0))_j$ for $j = 1, 2, \ldots, n$.

Before introducing a weakly reachability condition, we need to introduce a matrix property defined in [4] called asticity.

Definition 5. (Asticity) A $n \times m$ matrix $G = \{g_{ij}\}$ is termed *row-astic* if for each row $i = 1, 2, \ldots, n$, $\bigoplus_{j=1}^{m} g_{ij} \in \mathbb{R}$. *Column-asticity* is similarly defined. A matrix is termed *doubly-astic* if it is both row and column-astic.

Theorem 3.2. A system is q-step weakly reachable if and only if Γ_q is row-astic.

Proof. If Γ_q is row-astic, then with a large enough $\boldsymbol{U}_q$, $(\Gamma_q \boldsymbol{U}_q)_j > (A^q X(0))_j$ for $j = 1, 2, \cdots, n$. Hence, a state can easily be found which is greater than the unforced terminal state. If a system is q-step weakly reachable, then for $j = 1, 2, \cdots, n$, we must have $(\Gamma_q \boldsymbol{U}_q)_j > (A^q X(0))_j$. Thus, $(\Gamma_q \boldsymbol{U}_q)_j$ must be finite for each j, and hence Γ_q must be row-astic. □

While it may not be possible to set all components of the state independently for a q-step weakly-reachable system, it is possible to set one component to an arbitrary value. This result is stated next.

Corollary 3.1. If a system is q-step weakly reachable and $X(0) = \epsilon$, then given any $\beta \in \mathbb{R}$, there exists a control sequence that results in at least one component of the terminal state being set to β. That is, there exists a $\boldsymbol{U}_q$ such that $(X(q))_j = \beta$ for at least one $j \in \{1, 2, \cdots, n\}$.

Proof. See [6]. □

We note that in the scalar case if a system is q-step weakly reachable and $X(0) = \epsilon$, then all states in $\mathbb{R}^n$ can be reached, i. e., the system is completely reachable. The number of components that are able to be selected independently appears to lead to a possible measure of the size of the set of states that can be reached by the system.

Definitions for structural controllability are given in [1]. In contrast to the algebraic notions defined in this work, these definitions pertain to the graphical representations of event graphs. Because of the strong tie between event graphs and the max-plus algebra, however, the results are closely related. In essence, the required row-asticity condition on the reachability matrix Γ_q ensures that a path exists (over q events) from at least one input to each internal transition or state component. Also, Cofer in [2] defines events to be controllable if their execution can be arbitrarily delayed. This is equivalent to ensuring a path exists from some input to the event.

In the continuous-variable case, the Cayley–Hamilton theorem is used to show that only n-steps need be considered to determine the reachability of the system. Here, the Cayley–Hamilton theorem in max-plus can be used to show that if the system is not weakly reachable after n-steps, it won't be weakly reachable for step sizes larger than n. This result is stated next.

Corollary 3.2. If a system is not n-step weakly reachable, then the system will not be weakly reachable for $q \geq n$.

Proof. From the Cayley–Hamilton theorem, we have $p_A^+(A) = p_A^-(A)$ or $A^n \oplus p_{n-1}^+ A^{n-1} \oplus \cdots \oplus p_0^+ E = p_{n-1}^- A^{n-1} \oplus p_{n-2}^- A^{n-2} \oplus \cdots \oplus p_0^- E$, where E is the identity matrix in max-plus and consists of e along the diagonal and ϵ everywhere else, and the coefficients p_i^+, p_i^- are determined from the characteristic equation of A [9]. Since $\otimes$ distributes over $\oplus$ and since scalar multiplication commutes, we have $A^n B \oplus p_{n-1}^+ A^{n-1} B \oplus \cdots \oplus p_0^+ B = p_{n-1}^- A^{n-1} B \oplus p_{n-2}^- A^{n-2} B \oplus \cdots \oplus p_0^- B$. In essence, the right-hand side is the check for the row-asticity of Γ_n and the left-hand side is the check of Γ_{n+1}. Since the asticity checks are equal, the row-asticity of Γ_{n+1} will be the same as that of Γ_n. By assumption, Γ_n is not row-astic. Thus, Γ_{n+1} is not row-astic and the system is not weakly reachable. Proofs for $q > n$ follow directly.□

Unlike the continuous-variable case, where the Cayley–Hamilton theorem can be used to show that the reachable space does not change after n-steps, increasing the number of steps may lead to reaching a state that was not reachable in fewer steps. This result is stated next.

Corollary 3.3. If $X \notin \Omega_{n,X(0)}$, then X may belong to $\Omega_{q,X(0)}$ where $q > n$.

Proof. See [6]. □

This result leads directly to the following realization.

Corollary 3.4. Increasing the number of steps from n may result in reaching a state that was not reachable in n or fewer steps.

Proof. Direct application of Corollary 3.3. □

3.1.1. Feasible systems

The results presented thus far are for general A, B, and C matrices. Now, consider system, input, and output matrices for a real or *feasible* system. A feasible system is one that could be implemented in practice. Thus, in a feasible system, service times are real and nonnegative and release times must be nonnegative and nondecreasing.

Definition 6. (Feasible System) A feasible system is a system where the entries of A, B, and C range over $\mathbb{R}^{+}_{\max}$.

Since ϵ represents a "zero" in max-plus, this definition allows entries of ϵ for convenience.

Because negative release times result in an impractical, noncausal solution, a feasible state must result from a sequence of nonnegative release times. Likewise, for a practical system, release times must be nondecreasing.

Definition 7. (Feasible State) For $X \in \Omega_{q,X(0)}$, X is feasible if the entries of the control sequence $\boldsymbol{U}_q$ that result in X are nonnegative and nondecreasing.

A sufficient condition that determines if a state is not feasible is given next.

Theorem 3.3. For a feasible system, if $\exists j$ such that $(\Gamma^{\dagger}_{q_0} \otimes' X)_j < 0$, and $X \in \Omega_{q_0,X(0)}$, then X will not be feasible. Furthermore, X will not be feasible for $q \geq q_0$.

Proof. Since $X \in \Omega_{q_0,X(0)}$, $\boldsymbol{U}_q = \Gamma^{\dagger}_{q_0} \otimes' X$, and so by assumption, the jth element is negative; hence, X is not feasible. Since $\Gamma_q = \left[\Gamma_{q_0} \; A^{q_0} B \cdots A^{q-1} B\right]$, then $(\Gamma^{\dagger}_q \otimes' X)_j < 0$ and so X will not be feasible after q-steps either. □

3.2. Observability

The ability to determine the states of the system from measurements of the output is reflected in the property of observability. The conditions for state observability in the max-plus case are more restrictive than in the continuous-variable case. One difficulty arises from the lack of an additive inverse. For states that do not directly contribute to the output, only an upper bound on the event-time state can be determined.

As in the continuous-variable case, suppose that we have a sequence of q output values. Using (1) − (2), we can write

$$\begin{bmatrix} Y(k) \\ Y(k+1) \\ \vdots \\ Y(k+q-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} X(k) \oplus \\ \begin{bmatrix} \epsilon & \epsilon & \cdots & \epsilon \\ CB & \epsilon & \cdots & \epsilon \\ \vdots & \ddots & \vdots & \vdots \\ CA^{q-2}B. & \cdots & CAB & CB \end{bmatrix} \begin{bmatrix} U(k+1) \\ U(k+2) \\ \vdots \\ U(k+q-1) \end{bmatrix}. \tag{5}$$

Using a shorthand notation for the output and input sequences $\boldsymbol{Y}_q := [Y^T(k) \, Y^T(k+1) \cdots Y^T(k+q-1)]^T$, $\boldsymbol{U}_q := [U^T(k+1) \;\; U^T(k+2) \;\; \cdots \;\; U^T(k+q-1)]^T$, and defining the q-step observability matrix,

$$\mathcal{O}_q := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix},$$

we have

$$\boldsymbol{Y}_q = \mathcal{O}_q X(k) \oplus H\boldsymbol{U}_q, \tag{6}$$

where H is defined appropriately from (5). To begin, we define an output that has been generated from the system under study as an "observed output".

Definition 8. (Observed Output Sequence) A sequence of observed outputs $\boldsymbol{Y}_q \in \mathbb{R}^{mq}$ is a series of outputs given by $\boldsymbol{Y}_q = \mathcal{O}_q X(k) \oplus H\boldsymbol{U}_q$ where $\boldsymbol{U}_q \in \mathbb{R}_{\max}^{p(q-1)}$ and $X(k) \in \overline{\mathbb{R}}_{\max}^n$.

The collection of all such sequences leads to the following definition.

Definition 9. (Observed Output Sequence Set) Given a positive integer p and $\boldsymbol{U}_q \in \mathbb{R}_{\max}^{p(q-1)}$, let

$$\Sigma_{q,\boldsymbol{U}_q} = \{\boldsymbol{Y}_q \in \mathbb{R}^{mq} : \boldsymbol{Y}_q = \mathcal{O}_q X(k) \oplus H\boldsymbol{U}_q, \quad \text{where } X(k) \text{ ranges over } \overline{\mathbb{R}}_{\max}^n\}.$$

Consider the following necessary and sufficient condition for whether an output sequence is an observed output sequence.

Theorem 3.4. Given a sequence $\boldsymbol{Y}_q \in \mathbb{R}^{mq}$, and an input sequence $\boldsymbol{U}_q \in \mathbb{R}_{\max}^{p(q-1)}$, then $\boldsymbol{Y}_q \in \Sigma_{q,\boldsymbol{U}_q}$ if and only if

$$\mathcal{O}_q(\mathcal{O}_q^\dagger \otimes' \boldsymbol{Y}_q) \oplus H\boldsymbol{U}_q = \boldsymbol{Y}_q. \tag{7}$$

Proof. The proof is similar in nature to that of Theorem 3.1. See [6] for details. □

Because of the nature of the max-plus algebra, specifically because addition is idempotent, determination of the actual system state is often not possible. Instead, we consider whether it is possible to determine the latest state that results in the observed output sequence. The latest state provides an upper bound on the completion times that result in the output sequence. Consider then, the following definition of the latest event-time state.

Definition 10. (Latest Event–Time State) Given a q-length sequence of observed outputs $\boldsymbol{Y}_q$ with a sequence of inputs $\boldsymbol{U}_q$, the latest event-time state $\gamma(k)$ which results in $\boldsymbol{Y}_q$ is

$$\gamma(k) := \max_{X(k)}\{X(k) \in \overline{\mathbb{R}}^n_{\max} : \boldsymbol{Y}_q = \mathcal{O}_q X(k) \oplus H\boldsymbol{U}_q\},$$

where the max is over each component.

Because the latest event-time state may not be finite, $\gamma(k)$ is defined over $\overline{\mathbb{R}}^n_{\max}$. Since infinite event times represent the trivial case and do not provide any information about the state, a definition of observability should exclude this case by requiring the latest event-time state to be finite. This leads to the following definition of weak observability.

Definition 11. (q-step weakly observable) A system is q-step weakly observable if for any q-length sequence of observed outputs $\boldsymbol{Y}_q \in \Sigma_{q,\boldsymbol{U}_q}$, the latest event-time state $\gamma(k)$ is finite and can be computed from $\boldsymbol{Y}_q$.

A necessary and sufficient condition for a system to be q-step weakly observable is given next.

Theorem 3.5. A system is q-step weakly observable if and only if $\mathcal{O}_q$ is column-astic.

Proof. If $\mathcal{O}_q$ is column-astic, then for a finite $\boldsymbol{Y}_q$, $\gamma(k) = \mathcal{O}_q^\dagger \otimes' \boldsymbol{Y}_q$ is finite. Given any $\boldsymbol{Y}_q \in \Sigma_{q,\boldsymbol{U}_q}$, by Theorem 3.4, $\mathcal{O}_q(\mathcal{O}_q^\dagger \otimes' \boldsymbol{Y}_q) \oplus H\boldsymbol{U}_q = \boldsymbol{Y}_q$ and so $\gamma(k)$ results in an observable output sequence. On the other hand, if the system is q-step weakly observable, $\gamma(k)$ must be finite and must be computable from a sequence of observed outputs $\boldsymbol{Y}_q$. By Theorem 3.4, $\mathcal{O}_q(\mathcal{O}_q^\dagger \otimes' \boldsymbol{Y}_q) \oplus H\boldsymbol{U}_q = \boldsymbol{Y}_q$. In order for $\gamma(k) = \mathcal{O}_q^\dagger \otimes' \boldsymbol{Y}_q$ to be finite, $\mathcal{O}_q$ must be column-astic. □

Definitions for structural observability are given in [1]. As mentioned before, the results here are closely related to the graphical constructs given in the cited references. The required column-asticity condition on the observability matrix $\mathcal{O}_q$ ensures that a path exists from each internal transition or state to at least one output.

Similar to the continuous-variable case, the Cayley–Hamilton theorem can be used to show that the column-asticity of the observation matrix will not change by adding rows of higher powers of A. Using the Cayley–Hamilton theorem, we have the following result.

Corollary 3.5. If $\mathcal{O}_n$ is not column-astic, then $\mathcal{O}_q$, where $q > n$, will not be column-astic.

Proof. The proof is similar in nature to that of Theorem 3.2. See [6] for details. □

The direct result of this theorem is that observing more than n output values does not provide more information regarding the latest event-time state.

Corollary 3.6. If a system is not n-step weakly observable, then it won't be q-step weakly observable for $q \geq n$.

Proof. Direct application of Corollary 3.5. □

3.3. Duality

As in the continuous-variable case, there exists a duality between the properties of weak reachability and weak observability.

Theorem 3.6. If the system described by (A, B, C) is q-step weakly reachable (q-step weakly observable), then the dual system (A^T, C^T, B^T) is q-step weakly observable (q-step weakly reachable).

Proof. If the system (A, B, C) is q-step weakly reachable, then Γ_q must be row-astic. In the dual system, $\mathcal{O}_q^{\text{dual}} = \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^T)^{q-1} \end{bmatrix} = (\Gamma_q)^T$. Hence, $\mathcal{O}_q^{\text{dual}}$ is column-astic and the dual system is q-step weakly observable. If (A, B, C) is q-step weakly observable, then $\mathcal{O}_q$ is column-astic. In the dual system, $\Gamma_q^{\text{dual}} = \begin{bmatrix} C^T & A^T C^T & \cdots & (A^T)^{q-1} C^T \end{bmatrix} = (\mathcal{O}_q)^T$. Hence, Γ_q^{dual} is row-astic and the dual system is q-step weakly reachable. □

4. CONCLUSIONS

This paper examined the properties of reachability and observability of linear max-plus systems in an algebraic fashion. Necessary and sufficient conditions were presented for determining if a system is weakly reachable and weakly observable. In addition, a necessary and sufficient condition was given for determining if a state is reachable. Future work centers on further exploring the size and linearity of the reachability set, considering stronger definitions and conditions for reachability and observability, investigating the implications of reachability on state-feedback control, and examining ways to determine a limit on the required number of steps to reach a given state.

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