Commentationes Mathematicae Universitatis Carolinae

Benjamin Cahen

Weyl quantization for the semidirect product of a compact Lie group and a vector space

Commentationes Mathematicae Universitatis Carolinae, Vol. 50 (2009), No. 3, 325--347

Persistent URL: http://dml.cz/dmlcz/134907

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 2009

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

Weyl quantization for the semidirect product of a compact Lie group and a vector space

Benjamin Cahen

Abstract. Let G be the semidirect product $V \rtimes K$ where K is a semisimple compact connected Lie group acting linearly on a finite-dimensional real vector space V. Let $\mathcal{O}$ be a coadjoint orbit of G associated by the Kirillov-Kostant method of orbits with a unitary irreducible representation π of G. We consider the case when the corresponding little group H is the centralizer of a torus of K. By dequantizing a suitable realization of π on a Hilbert space of functions on $\mathbb{C}^n$ where $n = \dim(K/H)$, we construct a symplectomorphism between a dense open subset of $\mathcal{O}$ and the symplectic product $\mathbb{C}^{2n} \times \mathcal{O}'$ where $\mathcal{O}'$ is a coadjoint orbit of H. This allows us to obtain a Weyl correspondence on $\mathcal{O}$ which is adapted to the representation π in the sense of [B. Cahen, *Quantification d'une orbite massive d'un groupe de Poincaré généralisé*, C.R. Acad. Sci. Paris t. 325, série I (1997), 803–806].

Keywords: Weyl quantization, Berezin quantization, semidirect product, coadjoint orbits, unitary representations

Classification: 81S10, 22E46, 22E99, 32M10

1. Introduction

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let π be a unitary irreducible representation of G on a Hilbert space H. Assume that the representation π is associated to a coadjoint orbit $\mathcal{O}$ of G by the Kirillov-Kostant method of orbits [19], [20], [21]. In [5] and [6] we introduced the notion of adapted Weyl correspondence on $\mathcal{O}$ in order to generalize the usual quantization rules directly [1], [15].

Definition 1.1. An *adapted Weyl correspondence* is an isomorphism W from a vector space $\mathcal{A}$ of complex-valued (or real-valued) smooth functions on the orbit $\mathcal{O}$ (called symbols) to a vector space $\mathcal{B}$ of (not necessarily bounded) linear operators on H satisfying the following properties:

(1) the elements of $\mathcal{B}$ preserve a fixed dense domain D of H;
(2) the constant function 1 belongs to $\mathcal{A}$, the identity operator I belongs to $\mathcal{B}$ and $W(1) = I$;
(3) $A \in \mathcal{B}$ and $B \in \mathcal{B}$ implies $AB \in \mathcal{B}$;
(4) for each f in $\mathcal{A}$ the complex conjugate $\bar{f}$ of f belongs to $\mathcal{A}$ and the adjoint of $W(f)$ is an extension of $W(\bar{f})$ (in the real case: for each f in $\mathcal{A}$ the operator $W(f)$ is symmetric);

(5) the elements of D are C^∞-vectors for the representation π, the functions $\tilde{X}$ ($X \in \mathfrak{g}$) defined on $\mathcal{O}$ by $\tilde{X}(\xi) = \langle \xi, X\rangle$ are in $\mathcal{A}$ and $W(i\tilde{X})v = d\pi(X)v$ for each $X \in \mathfrak{g}$ and each $v \in D$.

For example, if G is a connected simply-connected nilpotent Lie group then each coadjoint orbit $\mathcal{O}$ of G is diffeomorphic to $\mathbb{R}^{2n}$ where $n = 1/2 \dim \mathcal{O}$, the unitary irreducible representation of G associated with $\mathcal{O}$ can be realized in the Hilbert space $L^2(\mathbb{R}^n)$ and the usual Weyl correspondence gives an adapted symbol calculus on $\mathcal{O}$ [2], [28]. It is also known that the Berezin calculus on an integral coadjoint orbit $\mathcal{O}$ of a semisimple compact connected Lie group G provides an adapted symbol calculus on $\mathcal{O}$ [5] (see also [12] and, for a similar result for the discrete series representations of a semisimple noncompact Lie group, [11]). By combining the usual Weyl correspondence and the Berezin calculus, we have obtained an adapted Weyl correspondence on the principal series coadjoint orbits of a connected semisimple noncompact Lie group [5], [10] and on the integral coadjoint orbits of the semidirect product $V \rtimes K$ where K is a connected semisimple noncompact Lie group acting linearly on a finite-dimensional real vector space V, under the condition that the little group is a maximal compact subgroup of K [9].

In fact, an adapted Weyl correspondence provides a prequantization map in the sense of [16, Definition 1]. In [9], we briefly described the relationship between adapted Weyl correspondences and the notion of quantization introduced by Mark Gotay (see [16]). Our original motivation for constructing adapted Weyl correspondences was to obtain covariant star-products on coadjoint orbits [5]. More recently, it has been established that adapted Weyl correspondences are useful to study contractions of Lie group representations in the setting of the Kirillov-Kostant method of orbits [14], [7], [8].

In the present paper, we continue the study of the adapted Weyl correspondences for semidirect products started in [9]. We consider here the case of the semidirect product $G = V \rtimes K$ where K is a semisimple compact connected Lie group acting linearly on a real vector space V. Let $\mathcal{O}$ be an integral coadjoint orbit of G whose little group H is the centralizer of a torus of K and let π be a unitary irreducible representation of G associated with $\mathcal{O}$. The representation π is usually realized on a space of square integrable sections of a Hermitian G-homogeneous vector bundle over K/H or, equivalently, on a space of square integrable functions on K/H with values in the space of the corresponding little group representation. Here we use a parametrization of a dense open subset of the generalized flag manifold K/H in order to obtain a realization of π in a space of square integrable functions on $\mathbb{C}^n$ where $n = \dim K/H$ (Section 3). We calculate the corresponding derived representation $d\pi$ (Section 4) and we dequantize $d\pi$ by using the usual Weyl correspondence on $\mathbb{C}^{2n} \simeq \mathbb{R}^{4n}$ and the Berezin calculus on the little group coadjoint orbit $\mathcal{O}'$ associated with $\mathcal{O}$ (Section 5). Then we obtain a symplectomorphism from the symplectic product $\mathbb{C}^{2n} \times \mathcal{O}'$ onto a dense open subset of $\mathcal{O}$ (Section 6). This allows us to construct an adapted Weyl correspondence on the orbit $\mathcal{O}$ (Section 6). In particular, these results can be applied to the case when V is the Lie algebra of K and the action of K on V is the adjoint action (Section 7).

2. Preliminaries

The coadjoint orbits of a semidirect product were described by J.H. Rawnsley in [23] (see also [3] for a detailed analysis of the geometrical structure of these orbits).

Let K be a semisimple compact connected Lie group with Lie algebra $\mathfrak{k}$. Let σ be a representation of K on a finite-dimensional real vector space V. For k in K and v in V we write $k.v$ instead of $\sigma(k)v$. We denote also by $(k,p)\to k.p$ the representation of K on V^* which is contragredient to σ and by $(A,v)\to A.v$ and $(A,p)\to A.p$ the corresponding derived representations of $\mathfrak{k}$ on V and V^*, respectively. For v in V and p in V^* we define $v\wedge p\in\mathfrak{k}^*$ by $(v\wedge p)(A)=p(A.v)=-(A.p)(v)$ for $A\in\mathfrak{k}$. Note that $\mathrm{Ad}^*(k)(v\wedge p)=k.p\wedge k.v$ for $k\in K$, $v\in V$ and $p\in V^*$.

We consider the semidirect product $G=V\rtimes K$. The group law of G is

$$(v,k).(v',k')=(v+k.v',kk')$$

for v, v' in V and k, k' in K. The Lie algebra $\mathfrak{g}$ of G is the space $V\times\mathfrak{k}$ with the Lie bracket

$$[(a,A),\,(a',A')]=(A.a'-A'.a,\,[A,A'])$$

for a, a' in V and A, A' in $\mathfrak{k}$. We identify the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ to $V^*\times\mathfrak{k}^*$. The coadjoint action of G on $\mathfrak{g}^*$ is then given by

$$(v,k).(p,f)=(k.p,\mathrm{Ad}^*(k)f+v\wedge k.p)$$

for $(v,k)\in G$ and $(p,f)\in\mathfrak{g}^*$. We identify K-equivariantly $\mathfrak{k}$ to its dual $\mathfrak{k}^*$ by using the Killing form of $\mathfrak{k}$ defined by $\langle A,B\rangle=\mathrm{Tr}(\mathrm{ad}\,A\,\mathrm{ad}\,B)$ for A and B in $\mathfrak{k}$. Then $\mathfrak{g}^*$ can be identified to $V^*\times\mathfrak{k}$.

Now we consider the orbit $\mathcal{O}(\xi_0)$ of the element $\xi_0=(p_0,f_0)$ of $\mathfrak{g}^*\simeq V^*\times\mathfrak{k}$ under the coadjoint action of G on $\mathfrak{g}^*$. Henceforth we assume that the little group $H:=\{k\in K\,:\,k.p_0=p_0\}$ is the centralizer of a torus T_1 of K. Let $\mathfrak{h}$ denote the Lie algebra of H. Let $Z(p_0)$ be the orbit of p_0 under the action of K on V^*. Then $Z(p_0)$ is diffeomorphic to the generalized flag manifold K/H.

Let us describe how to endow $Z(p_0)\simeq K/H$ with a complex structure. Let T be a maximal torus of K containing T_1. Clearly $T\subset H$. Let $\mathfrak{t}$ be the Lie algebra of T. Let Δ be the root system of K relative to T and let Δ_1 be the root system of H relative to T. We can simultaneously choose a Weyl chamber P of T relative to K and a Weyl chamber P_1 of T relative to H so that if Δ^+ and Δ_1^+ are, respectively, the positive roots of Δ and Δ_1 relative to P and P_1 then

(1) $\Delta^+ \cap \Delta_1 = \Delta_1^+$ and

(2) if $\alpha \in \Delta^+ \setminus \Delta_1^+$, $\beta \in \Delta_1$ and $\alpha + \beta \in \Delta$ then $\alpha + \beta \in \Delta^+ \setminus \Delta_1^+$.

Moreover, if Δ^s is the set of simple roots of Δ relative to P and if Δ_1^s is the set of simple roots of Δ_1 relative to P_1, then $\Delta_1^s \subset \Delta^s$ (see [27, 6.2.8]).

Let $\mathfrak{k}^c$, $\mathfrak{h}^c$ and $\mathfrak{t}^c$ be the complexifications of $\mathfrak{k}$, $\mathfrak{h}$ and $\mathfrak{t}$, respectively. Let K^c, H^c and T^c be the connected complex Lie groups whose Lie algebras are $\mathfrak{k}^c$, $\mathfrak{h}^c$ and $\mathfrak{t}^c$, respectively. Let $\mathfrak{k}^c = \mathfrak{t}^c \oplus \sum_{\alpha\in\Delta} \mathfrak{k}_\alpha$ be the root space decomposition of $\mathfrak{k}^c$. We set $\mathfrak{n}^+ = \sum_{\alpha\in\Delta^+\setminus\Delta_1^+} \mathfrak{k}_\alpha$ and $\mathfrak{n}^- = \sum_{\alpha\in\Delta^+\setminus\Delta_1^+} \mathfrak{k}_{-\alpha}$. Then, by [27, 6.2.1], $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent Lie algebras satisfying $[\mathfrak{h}^c, \mathfrak{n}^\pm] \subset \mathfrak{n}^\pm$. We also have

$$\mathfrak{k}^c = \mathfrak{h}^c \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-, \qquad \mathfrak{h}^c = \mathfrak{t}^c \oplus \sum_{\alpha\in\Delta_1^+} \mathfrak{k}_\alpha \oplus \sum_{\alpha\in\Delta_1^+} \mathfrak{k}_{-\alpha}. \tag{2.1}$$

We denote by N^+ and N^- the analytic subgroups of K^c with Lie algebras $\mathfrak{n}^+$ and $\mathfrak{n}^-$, respectively. A complex structure on K/H is then defined by the diffeomorphism $K/H \simeq K^c/H^cN^-$ [27, 6.2.11]. This complex structure depends on the choice of P and P_1.

The natural projection $K^c \to K^c/H^cN^-$ induces a projection $\tau : K^c \to Z(p_0)$. The natural action of K^c on K^c/H^cN^- induces an action of K^c on $Z(p_0)$; we denote by kp the action of $k \in K^c$ on $p \in Z(p_0)$. Of course, if $k \in K$ then kp is the natural action $k.p$ of $k \in K$ on $p \in V^*$.

Now we introduce a parametrization of a dense open subset of $Z(p_0) \simeq K/H$. Recall that (1) each k in a dense open subset of K^c has a unique Gauss decomposition $k = n^+hn^-$ where $n^+ \in N^+$, $h \in H^c$ and $n^- \in N^-$ and (2) the map $\gamma : Z \to \tau(\exp Z)$ is a holomorphic diffeomorphism from $\mathfrak{n}^+$ onto a dense open subset of $Z(p_0)$ (see [17, Chapter VIII]). Then the action of K^c on $Z(p_0)$ induces an action (defined almost everywhere) of K^c on $\mathfrak{n}^+$. We denote by $k\cdot Z$ the action of $k \in K^c$ on $Z \in \mathfrak{n}^+$. Using the diffeomorphism $K/H \simeq K^c/H^cN^-$ again, we see that for each $Z \in \mathfrak{n}^+$ there exists an element $k_Z \in K$ for which $\tau(k_Z) = \tau(\exp Z)$ or, equivalently, $k_Z \cdot 0 = Z$.

Following [22], we introduce the projections $\kappa : N^+H^cN^- \to H^c$ and $\zeta : N^+H^cN^- \to N^+$. Then, for $k \in K^c$ and $Z \in \mathfrak{n}^+$ we have $k \cdot Z = \log \zeta(k \exp Z)$. We set $(X+iY)^* = -X+iY$ for X, $Y \in \mathfrak{k}$ and we denote by $k \to k^*$ the involutive anti-automorphism of K^c which is obtained by exponentiating $X+iY \to (X+iY)^*$ to K^c. Also, let θ be the conjugation of $\mathfrak{k}^c$ with respect to $\mathfrak{k}$ and let $\tilde{\theta}$ be the automorphism of K^c for which $d\tilde{\theta} = \theta$. Then we have $\theta(X) = -X^*$ for $X \in \mathfrak{k}^c$ and $\tilde{\theta}(k) = (k^*)^{-1}$ for $k \in K^c$.

In the rest of the paper, we fix a Cartan-Weyl basis for $\mathfrak{k}^c$, $(E_\alpha)_{\alpha\in\Delta} \cup (H_\alpha)_{\alpha\in\Delta_s}$, as in [20, Chapter 5]. In particular, $\mathfrak{k}$ is spanned by the elements $E_\alpha - E_{-\alpha}$, $i(E_\alpha + E_{-\alpha})$ for $\alpha \in \Delta^+$ and iH_α for $\alpha \in \Delta_s$ and we have the property $E_\alpha^* = E_{-\alpha}$ for $\alpha \in \Delta$.

Now we describe the K-invariant measure on $Z(p_0)$. Let $d\mu_L(Z)$ be the Lebesgue measure on $\mathfrak{n}^+$ defined as follows. Let $(\alpha_k)_{1\leq k\leq n}$ be an enumeration of $\Delta^+\setminus\Delta_1^+$. Then $(E_{\alpha_k})_{1\leq k\leq n}$ is a basis for $\mathfrak{n}^+$ and we denote by $z_1=x_1+iy_1, z_2=x_2+iy_2,\ldots,z_n=x_n+iy_n$ the coordinates of $Z\in\mathfrak{n}^+$ in this basis. Then we set $d\mu_L(Z)=dx_1dy_1dx_2dy_2\cdots dx_ndy_n$. Now a K-invariant measure on $\mathfrak{n}^+$ is given by $d\mu(Z)=\chi_\Lambda(\kappa(\exp Z^*\exp Z))\,d\mu_L(Z)$ where $\chi_\Lambda(h)=\mathrm{Det}_{\mathfrak{n}^+}\mathrm{Ad}(h)$ is the character of H^c corresponding to the weight $\Lambda=\sum_{\alpha\in\Delta^+\setminus\Delta_1^+}\alpha$, that is, $\Lambda=d\chi_\Lambda|_{\mathfrak{t}^c}$ (see for instance [22] or [12]). Hence, a K-invariant measure on $Z(p_0)$ is $d\tilde{\mu}=\gamma^*(d\mu)$.

The two next lemmas will be needed later. First, we reformulate [23, Lemma 1] as follows.

Lemma 2.1. *The space $\mathcal{V}:=\{v\wedge p_0\,:\,v\in V\}\subset\mathfrak{k}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{k}$. We also have $\mathcal{V}=\{Y+\theta(Y)\,:\,Y\in\mathfrak{n}^-\}$.*

Proof: The first assertion of the lemma follows from the equality $(v\wedge p_0)(A)=p_0(A.v)=-(A.p_0)(v)$ for $A\in\mathfrak{k}$ and $v\in V$. To prove the second assertion, we note that $\mathfrak{h}$ is spanned by the elements $E_\alpha-E_{-\alpha}$, $i(E_\alpha+E_{-\alpha})$ for $\alpha\in\Delta_1^+$ and iH_α for $\alpha\in\Delta_s$. On the other hand, the space $\{Y+\theta(Y)\,:\,Y\in\mathfrak{n}^-\}$ is spanned by the elements $E_\alpha+\theta(E_\alpha)=E_\alpha-E_{-\alpha}$ and $iE_\alpha+\theta(iE_\alpha)=i(E_\alpha+E_{-\alpha})$ for $\alpha\in\Delta^+\setminus\Delta_1^+$. Recalling that $\langle E_\alpha,E_\beta\rangle=\delta_{\alpha,-\beta}$ and $\langle E_\alpha,H_\beta\rangle=0$, the result then follows. □

Observe that, for $v\in V$, one has $(v,e).(p_0,f_0)=(p_0,f_0+v\wedge p_0)$ where e denotes the identity element of K. Then, by Lemma 2.1, we may assume without loss of generality that $\xi_0=(p_0,\varphi_0)$ with $\varphi_0\in\mathfrak{h}$. We shall denote by $\mathcal{O}(\varphi_0)\subset\mathfrak{h}$ the orbit of $\varphi_0\in\mathfrak{h}$ under the adjoint action of H.

Lemma 2.2. (1) *For $k\in N^+H^cN^-$, we have*

$$\kappa(\zeta(k)^*\zeta(k))=(\kappa(k)^*)^{-1}\kappa(k^*k)\kappa(k)^{-1}$$

(2) *For $Z\in\mathfrak{n}^+$, we have $\kappa(\exp Z^*\exp Z)=\kappa(k_Z^{-1}\exp Z)^*\kappa(k_Z^{-1}\exp Z)$.*

Proof: (1) Write $k=zhy$ where $z\in N^+$, $h\in H^c$ and $y\in N^-$. Then $k^*k=y^*h^*z^*zhy$. Hence $\kappa(k^*k)=h^*\kappa(z^*z)h$. This gives the desired result.

(2) Applying (1) to $k=k_Z=\exp Zhy$ where $h\in H^c$ and $y\in N^-$, we get $\kappa(\exp Z^*\exp Z)=(h^*)^{-1}h^{-1}$. Now $k_Z^{-1}\exp Z=y^{-1}h^{-1}=h^{-1}(hy^{-1}h^{-1})$ gives $\kappa(k_Z^{-1}\exp Z)=h^{-1}$ and the result follows. □

3. Representations

In the rest of the paper, we assume that the orbit $\mathcal{O}(\varphi_0)$ is associated with a unitary irreducible representation (ρ,E) of H as in [29, Section 4]. This correspondence can be described as follows. Let λ be the highest weight of (ρ,E). Let $\varphi_0\in\mathfrak{t}$ such that $\lambda(A)=i\langle\varphi_0,A\rangle$ for each $A\in\mathfrak{t}$. Then orbit of φ_0 under the adjoint action of H is said to be associated with the representation (ρ,E).

Since $\mathcal{O}(\varphi_0)$ is integral, the orbit $\mathcal{O}(\xi_0)$ is also integral [23]. In fact, $\mathcal{O}(\xi_0)$ is associated with the unitarily induced representation

$$\tilde{\pi} = \mathrm{Ind}_{V\times H}^{G}\,(e^{ip_0}\otimes\rho).$$

By a result of G. Mackey, π is irreducible because ρ is irreducible [25]. We denote by π_0 the usual realization of $\tilde{\pi}$ defined on a Hermitian vector bundle as follows [21], [24]. We introduce the Hilbert G-bundle $L := G\times_{e^{ip_0}\otimes\rho} E$ over $Z(p_0)\simeq K/H$. Recall that an element of L is an equivalence class

$$[g,u] = \{(g.(v,h), e^{-i\langle p_0,v\rangle}\rho(h)^{-1}u)\;:\;v\in V,\;h\in H\}$$

where $g\in G$, $u\in E$ and that G acts on L by left translations: $g\,[g',u] := [g.g',u]$. The action of G on $Z(p_0)\simeq K/H$ being given by $(v,k).p = k.p$, the projection map $[(v,k),u]\to k.p_0$ is G-equivariant. The G-invariant Hermitian structure on L is given by

$$\langle [g,u],[g,u']\rangle = \langle u,u'\rangle_E$$

where $g\in G$ and u, $u'\in E$. Let $\mathcal{H}_0$ be the space of sections s of L which are square-integrable with respect to the measure $d\mu(p)$, that is,

$$\|s\|^2_{\mathcal{H}_0} = \int_{Z(p_0)}\langle s(p),\,s(p)\rangle\,d\mu(p) < +\infty.$$

Then π_0 is the action of G on $\mathcal{H}_0$ defined by

$$(\pi_0(g)\,s)(p) = g\,s(g^{-1}.p).$$

Now, following [24], we introduce an alternative realization of $\tilde{\pi}$ on a space of functions. We associate with any $s\in\mathcal{H}_0$ the function $f_s:\mathfrak{n}^+\to E$ defined by $s(\gamma(Z)) = [(0,k_Z), f_s(Z)]$. For s and s' in $\mathcal{H}_0$, we have

$$\langle s(\gamma(Z)), s'(\gamma(Z))\rangle = \langle f_s(Z),\,f_{s'}(Z)\rangle_E.$$

This implies that

$$\langle s,\,s'\rangle_{\mathcal{H}_0} = \int_{\mathfrak{n}^+}\langle f_s(Z),\,f_{s'}(Z)\rangle_E\,\delta(Z)\,d\mu_L(Z).$$

where $\delta(Z) = \chi_\Lambda(\kappa(\exp Z^*\exp Z))$ (see Section 2). This leads us to introduce the Hilbert space $\mathcal{H}^0$ of functions $f:\mathfrak{n}^+\to E$ which are square-integrable with respect to the measure $\delta(Z)\,d\mu_L(Z)$. The norm on $\mathcal{H}^0$ is defined by

$$\|f\|^2_{\mathcal{H}^0} = \int_{\mathfrak{n}^+}\langle f(Z),\,f(Z)\rangle_E\,\delta(Z)\,d\mu_L(Z).$$

Moreover, for $s \in \mathcal{H}_0$, $g = (v, k) \in G$ and $Z \in \mathfrak{n}^+$, we have

$$\begin{aligned}(\pi_0(g)\, s)(\gamma(Z)) &= g\, s(g^{-1}.\gamma(Z)) = g\,[(0, k_{k^{-1}.Z}), f_s(k^{-1}\cdot Z)]\\ &= [(v, kk_{k^{-1}.Z}), f_s(k^{-1}\cdot Z)] = [(0, k_Z).(k_Z^{-1}.v, k_Z^{-1}kk_{k^{-1}.Z}), f_s(k^{-1}\cdot Z)]\\ &= [(0, k_Z), e^{i\langle p_0, k_Z^{-1}.v\rangle}\rho(k_Z^{-1}kk_{k^{-1}.Z})f_s(k^{-1}\cdot Z)].\end{aligned}$$

Hence we conclude that the equality

$$\pi^0(v,k)f\,(Z) = e^{i\langle k_Z.p_0, v\rangle}\,\rho(k_Z^{-1}kk_{k^{-1}.Z})f(k^{-1}\cdot Z) \tag{3.1}$$

defines a unitary representation π^0 on $\mathcal{H}^0$ which is unitarily equivalent to π_0.

Now we deduce from π^0 another realization of $\tilde{\pi}$ which is more convenient for explicit computations and for the Weyl calculus. First, we extend ρ to a representation $\tilde{\rho}$ of H^cN^- on E which is trivial on N^- and we note that

$$\begin{aligned}\rho(k_Z^{-1}kk_{k^{-1}.Z}) = \tilde{\rho}(k_Z^{-1}\exp Z)\tilde{\rho}(\exp(-Z)k\exp(k^{-1}\cdot Z))\\ \tilde{\rho}((\exp(k^{-1}\cdot Z))^{-1}k_{k^{-1}.Z}).\end{aligned} \tag{3.2}$$

On the other hand, by (2) of Lemma 2.2, we have

$$\begin{aligned}&\langle\tilde{\rho}(k_Z^{-1}\exp Z)u,\, \tilde{\rho}(k_Z^{-1}\exp Z)u'\rangle_E\\ &\quad= \langle\tilde{\rho}(\kappa(k_Z^{-1}\exp Z))u, \tilde{\rho}(\kappa(k_Z^{-1}\exp Z))u'\rangle_E\\ &\quad= \langle\tilde{\rho}(\kappa(k_Z^{-1}\exp Z))^*\tilde{\rho}(\kappa(k_Z^{-1}\exp Z))u,\, u'\rangle_E\\ &\quad= \langle\tilde{\rho}(\kappa(\exp Z^*\exp Z))u,\, u'\rangle_E.\end{aligned} \tag{3.3}$$

Let us denote by $R(v) = v^{1/2}$ the square root of a positive self-adjoint operator on E. In order to simplify the notation, we set $h(Z) := \kappa(\exp Z^*\exp Z)$ and $q(Z) = R(\tilde{\rho}(h(Z)))$. Then by (3.3) we have

$$\langle\tilde{\rho}(k_Z^{-1}\exp Z)u,\, \tilde{\rho}(k_Z^{-1}\exp Z)u'\rangle_E = \langle q(Z)u,\, q(Z)u'\rangle_E. \tag{3.4}$$

Let us introduce the Hilbert space $\mathcal{H}$ of functions $\phi : \mathfrak{n}^+ \to E$ which are square-integrable with respect to the measure $d\mu_L(Z)$. From equations (3.1), (3.2) and (3.4) we deduce immediately that π^0 is unitarily equivalent to the representation π of G on $\mathcal{H}$ defined by

$$\begin{aligned}\pi(v,k)\phi\,(Z) = e^{i\langle\exp Z\, p_0, v\rangle}\,\delta(Z)^{1/2}\delta(k^{-1}\cdot Z)^{-1/2}q(Z)\\ \tilde{\rho}(\exp(-Z)k\exp(k^{-1}\cdot Z))q(k^{-1}\cdot Z)^{-1}\phi(k^{-1}\cdot Z)\end{aligned} \tag{3.5}$$

the intertwining operator $f \in \mathcal{H}^0 \mapsto \phi \in \mathcal{H}$ being given by

$$\phi(Z) = \delta(Z)^{1/2}q(Z)\tilde{\rho}(k_Z^{-1}\exp Z)^{-1}f(Z).$$

4. Derived representation

In this section, we compute the differential $d\pi$ of the representation π of G. For $(w,A)\in\mathfrak{g}$, we can write

$$
\begin{aligned}
(d\pi(w,A)\,\phi)(Z) &= i\langle \exp Zp_0,\, w\rangle\,\phi(Z)\\
&+\delta(Z)^{1/2}\frac{d}{dt}\delta(k(t)^{-1}\cdot Z)^{-1/2}\big|_{t=0}\phi(Z)\\
&+q(Z)\,\frac{d}{dt}q(k(t)^{-1}\cdot Z)^{-1}\big|_{t=0}\phi(Z)\\
&+q(Z)d\tilde{\rho}\left(\frac{d}{dt}\exp(-Z)k(t)\exp(k(t)^{-1}\cdot Z)\big|_{t=0}\right)q(Z)^{-1}\phi(Z)\\
&+\frac{d}{dt}\phi(k(t)^{-1}\cdot Z)\big|_{t=0}
\end{aligned}
\tag{4.1}
$$

where $k(t):=\exp(tA)$. Recall that we have set $h(Z)=\kappa(\exp Z^*\exp Z)$ and $q(Z)=R(\tilde{\rho}(h(Z)))$ where R denotes square root. The following lemma can be easily deduced from results of [12]. We denote by $p_{\mathfrak{h}^c}$, $p_{\mathfrak{n}^+}$ and $p_{\mathfrak{n}^-}$ the projections of $\mathfrak{k}^c$ on $\mathfrak{h}^c$, $\mathfrak{n}^+$ and $\mathfrak{n}^-$ associated with the direct decomposition $\mathfrak{k}^c=\mathfrak{h}^c\oplus\mathfrak{n}^+\oplus\mathfrak{n}^-$.

Lemma 4.1. *Let $A\in\mathfrak{k}$ and $k(t)=\exp(tA)$. Then we have*

$$
\begin{aligned}
\frac{d}{dt}\tilde{\rho}\big(\exp(-Z)k(t)\exp(k(t)^{-1}\cdot Z)\big)\big|_{t=0} &= \frac{d}{dt}\tilde{\rho}\big(\kappa(k(t)^{-1}\exp Z)\big)^{-1}\big|_{t=0}\\
&= d\tilde{\rho}\big(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)\big)
\end{aligned}
\tag{4.2}
$$

and

$$
\frac{d}{dt}\,k(t)^{-1}\cdot Z\big|_{t=0} = -\frac{\mathrm{ad}\,Z}{1-e^{-\mathrm{ad}\,Z}}\,p_{\mathfrak{n}^+}(e^{-\mathrm{ad}\,Z}A).
\tag{4.3}
$$

Proof: Immediate consequence of [12], Proposition 4.1 and Proposition 5.1. □

Lemma 4.2. *Let $A\in\mathfrak{k}$ and $k(t)=\exp(tA)$. Then we have*

$$
\begin{aligned}
&\frac{d}{dt}q(Z)\,q(k(t)^{-1}\cdot Z)^{-1}\big|_{t=0} = -\,\mathrm{Ad}\,\tilde{\rho}(h(Z))^{1/2}(\mathrm{id}+\mathrm{Ad}\,\tilde{\rho}(h(Z))^{-1/2})^{-1}\\
&\big(d\tilde{\rho}\big(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)\big)+\mathrm{Ad}\,\tilde{\rho}(h(Z))^{-1}d\tilde{\rho}\big(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)^*\big)\big)
\end{aligned}
\tag{4.4}
$$

and similarly

$$
\begin{aligned}
&\frac{d}{dt}\delta(Z)^{1/2}\delta(k(t)^{-1}\cdot Z)^{-1/2}\big|_{t=0}\\
&= -\frac{1}{2}\big(\Lambda(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A))+\Lambda(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)^*)\big).
\end{aligned}
\tag{4.5}
$$

Proof: First, note that $\exp(k(t)^{-1}\cdot Z) = \zeta(k(t)^{-1}\exp Z)$. Then, applying Lemma 2.2, we have

$$h(k(t)^{-1}\cdot Z) = \kappa(k(t)^{-1}\exp Z)^{*-1}h(Z)\kappa(k(t)^{-1}\exp Z)^{-1}.$$

Hence

$$q(k(t)^{-1}\cdot Z)^{-1} = R\left(\tilde\rho\big(\kappa(k(t)^{-1}\exp Z)h(Z)^{-1}\kappa(k(t)^{-1}\exp Z)^*\big)\right)$$

and

$$\begin{aligned}&\frac{d}{dt}q(k(t)^{-1}\cdot Z)^{-1}\Big|_{t=0}\\&= dR(\tilde\rho(h(Z)^{-1}))d\tilde\rho(h(Z)^{-1})\left(\frac{d}{dt}\kappa(k(t)^{-1}\exp Z)h(Z)^{-1}\kappa(k(t)^{-1}\exp Z)^*\Big|_{t=0}\right)\\&= dR(\tilde\rho(h(Z)^{-1}))\left(d\tilde\rho(U)\tilde\rho(h(Z))^{-1}\right)\end{aligned}$$

where

$$U := \frac{d}{dt}\,\kappa(k(t)^{-1}\exp Z)h(Z)^{-1}\kappa(k(t)^{-1}\exp Z)^*h(Z)\Big|_{t=0}.$$

Applying Lemma 4.1, we find

$$\begin{aligned}U =&\frac{d}{dt}\kappa(k(t)^{-1}\exp Z)\Big|_{t=0} + \mathrm{Ad}(h(Z)^{-1})\frac{d}{dt}\kappa(k(t)^{-1}\exp Z)^*\Big|_{t=0}\\=&-p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A) - \mathrm{Ad}(h(Z)^{-1})p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)^*.\end{aligned}\tag{4.6}$$

On the other hand, using the equality

$$dR(u)v = (\mathrm{id} + \mathrm{Ad}\,u^{1/2})^{-1}(vu^{-1/2})$$

for any positive definite self-adjoint operator u on E, we get

$$\begin{aligned}&q(Z)\,\frac{d}{dt}q(k(t)^{-1}\cdot Z)^{-1}\Big|_{t=0}\\&\quad= \tilde\rho(h(Z))^{1/2}(\mathrm{id} + \mathrm{Ad}\,\tilde\rho(h(Z))^{-1/2})^{-1}\left(d\tilde\rho(U)\tilde\rho(h(Z))^{-1/2}\right)\\&\quad= \mathrm{Ad}\,\tilde\rho(h(Z))^{1/2}(\mathrm{id} + \mathrm{Ad}\,\tilde\rho(h(Z))^{-1/2})^{-1}(d\tilde\rho(U)).\end{aligned}$$

Taking equation (4.6) into account, we then obtain (4.4). Moreover, writing (4.6) for $\tilde\rho = \chi_\Lambda$, we also obtain (4.5). □

Proposition 4.1. *For* $(w, A) \in \mathfrak{g}$ *and* $\phi \in C_0^\infty(\mathfrak{n}^+, E)$ *we have*

$$\begin{aligned} d\pi(w,A)\,\phi(Z) &= \frac{d}{dt}\,(\pi(tw,\exp(tA))\phi)(Z)\big|_{t=0} \\ &= i\langle \exp Z p_0,\, w\rangle\,\phi(Z) \\ &\quad - \frac{1}{2}\big(\Lambda(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)) + \Lambda(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)^*)\big)\phi(Z) \\ &\quad + (\mathrm{id} + \mathrm{Ad}\,\tilde\rho(h(Z))^{-1/2})^{-1}\big(d\tilde\rho\big(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)\big) \\ &\quad - \mathrm{Ad}\,\tilde\rho(h(Z))^{-1/2} d\tilde\rho\big(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)^*\big)\big)\phi(Z) \\ &\quad - \partial_Z\phi(Z,Z^*)\left(\frac{\mathrm{ad}\,Z}{1-e^{-\mathrm{ad}\,Z}}\,p_{\mathfrak{n}^+}(e^{-\mathrm{ad}\,Z}\,A)\right) \\ &\quad - \partial_{Z^*}\phi(Z,Z^*)\left(\frac{\mathrm{ad}\,Z}{1-e^{-\mathrm{ad}\,Z}}\,p_{\mathfrak{n}^+}(e^{-\mathrm{ad}\,Z}\,A)\right)^*. \end{aligned}$$

PROOF: Using Lemma 4.1 and Lemma 4.2 and writing

$$\begin{aligned} &q(Z)d\tilde\rho\left(\frac{d}{dt}\exp(-Z)k(t)\exp(k(t)^{-1}\cdot Z)\big|_{t=0}\right)\,q(Z)^{-1} \\ &= \mathrm{Ad}\,\tilde\rho(h(Z))^{1/2} d\tilde\rho\big(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)\big) \\ &= \mathrm{Ad}\,\tilde\rho(h(Z))^{1/2}(\mathrm{id} + \mathrm{Ad}\,\tilde\rho(h(Z))^{-1/2})^{-1}(\mathrm{id} + \mathrm{Ad}\,\tilde\rho(h(Z))^{-1/2}) \\ &\quad d\tilde\rho\big(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)\big) \end{aligned}$$

we see that

$$\begin{aligned} &q(Z)d\tilde\rho\left(\frac{d}{dt}\exp(-Z)k(t)\exp(k(t)^{-1}\cdot Z)\big|_{t=0}\right)\,q(Z)^{-1} \\ &\quad + q(Z)\,\frac{d}{dt}q(k(t)^{-1}\cdot Z)^{-1}\big|_{t=0} \\ &= (\mathrm{id} + \mathrm{Ad}\,\tilde\rho(h(Z))^{-1/2})^{-1}\Big(d\tilde\rho\big(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)\big) \\ &\quad - \mathrm{Ad}\,\tilde\rho(h(Z))^{-1/2} d\tilde\rho\big(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)^*\big)\Big). \end{aligned}$$

The result then follows. □

5. Dequantization

We first introduce the Berezin calculus on the orbit $\mathcal{O}(\varphi_0)$. The Berezin calculus associates with each operator B on the finite-dimensional complex vector space E a complex-valued function $s(B)$ on the orbit $\mathcal{O}(\varphi_0)$ called the symbol of the operator B (see [4]). The following properties of the Berezin calculus can be found in [13], [5], [12].

Proposition 5.1. (1) *The map $B \to s(B)$ is injective.*

(2) *For each operator B on E, we have $s(B^*) = \overline{s(B)}$.*

(3) *For $\varphi \in \mathcal{O}(\varphi_0)$, $h \in H$ and for an operator B on E, we have*

$$s(B)(\mathrm{Ad}(h)\varphi) = s(\rho(h)^{-1}B\rho(h))(\varphi).$$

(4) *For $A \in \mathfrak{h}$ and $\varphi \in \mathcal{O}(\varphi_0)$, we have $s(d\rho(A))(\varphi) = i\langle\varphi, A\rangle$.*

Now we introduce the Berezin-Weyl calculus on $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$. We first recall the definition of the Berezin-Weyl calculus on $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathcal{O}(\varphi_0)$ (see [9]). We say that a smooth function $f : (T, S, \varphi) \to f(T, S, \varphi)$ is a symbol on $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathcal{O}(\varphi_0)$ if for each $(T, S) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ the function $\varphi \to f(T, S, \varphi)$ is the symbol in the Berezin calculus on $\mathcal{O}(\varphi_0)$ of an operator on E denoted by $\hat{f}(T, S)$. A symbol f on $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathcal{O}(\varphi_0)$ is called an $\mathcal{S}$-symbol if the function $\hat{f}$ belongs to the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ with values in $\mathrm{End}(E)$. Now we consider the Weyl calculus for $\mathrm{End}(E)$-valued functions [18]. For any $\mathcal{S}$-symbol f on $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathcal{O}(\varphi_0)$ we define an operator $\mathcal{W}(f)$ on the Hilbert space $L^2(\mathbb{R}^{2n}, E)$ by

$$(5.1)\quad (\mathcal{W}(f)\phi)(T) = (2\pi)^{-2n} \int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} e^{i\langle S,S'\rangle} \hat{f}\big(T + \frac{1}{2}S, S'\big)\, \phi(T+S)\, dS\, dS'$$

for $\phi \in C_0^\infty(\mathbb{R}^{2n}, E)$.

The Weyl-Berezin calculus can be extended to much larger classes of symbols (see for instance [18]). Here we are only concerned with a class of polynomial symbols. We say that a symbol f on $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathcal{O}(\varphi_0)$ is a P-symbol if the function $\hat{f}(T, S)$ is polynomial in S. Let f be the P-symbol defined by $f(T, S, \varphi) = u(T)S^\alpha$ where $u \in C^\infty(\mathbb{R}^{2n}, E)$ and $S^\alpha := s_1^{\alpha_1} s_2^{\alpha_2} \ldots s_{2n}^{\alpha_{2n}}$ for each multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2n})$. Then we have (see [26]):

$$(5.2)\qquad (\mathcal{W}(f)\phi)(T) = (i\partial_S)^\alpha \big(u(T + \frac{1}{2}S)\, \phi(T+S)\big)\Big|_{S=0}.$$

In particular, if $f(T, S, \varphi) = u(T)$ then

$$(5.3)\qquad (\mathcal{W}(f)\phi)(T) = u(T)\, \phi(T)$$

and if $f(T, S, \varphi) = u(T)s_k$ then

$$(5.4)\qquad (\mathcal{W}(f)\phi)(T) = i\left(\frac{1}{2}(\partial_{t_k}u)(T)\, \phi(T) + u(T)(\partial_{t_k}\phi)(T)\right).$$

The correspondence $f \mapsto \mathcal{W}(f)$ is called the Berezin-Weyl calculus on $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathcal{O}(\varphi_0)$. In order to obtain the Berezin-Weyl calculus on $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$, we just rewrite the Berezin-Weyl calculus on $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathcal{O}(\varphi_0)$ in complex coordinates.

Let $j:\mathbb{R}^{2n}\to\mathfrak{n}^+$ be the map defined by

$$j(t_1,t_2,\ldots,t_n,t'_1,t'_2,\ldots,t'_n)=\sum_{k=1}^{n}(t_k+it'_k)E_{\alpha_k}$$

and let $\tilde{j}:\mathbb{R}^{2n}\times\mathbb{R}^{2n}\times\mathcal{O}(\varphi_0)\to\mathfrak{n}^+\times\mathfrak{n}^+\times\mathcal{O}(\varphi_0)$ be the map given by

$$\tilde{j}(T,S,\varphi)=(j(T),j(S),\varphi).$$

We say that a function $f:\mathfrak{n}^+\times\mathfrak{n}^+\times\mathcal{O}(\varphi_0)\to\mathbb{C}$ is a symbol (resp. an S-symbol, a P-symbol) on $\mathfrak{n}^+\times\mathfrak{n}^+\times\mathcal{O}(\varphi_0)$ if $f\circ\tilde{j}$ is a symbol (resp. an S-symbol, a P-symbol) on $\mathbb{R}^{2n}\times\mathbb{R}^{2n}\times\mathcal{O}(\varphi_0)$ and we define the Berezin-Weyl calculus on $\mathfrak{n}^+\times\mathfrak{n}^+\times\mathcal{O}(\varphi_0)$ by

$$W(f)\phi\circ j=\mathcal{W}(f\circ\tilde{j})\phi$$

for each $\phi\in C_0^\infty(\mathfrak{n}^+,E)$. Let $Y=\sum_{k=1}^n y_kE_{\alpha_k}$ be the decomposition of $Y\in\mathfrak{n}^+$ in the basis (E_{α_k}). An easy computation shows that if $f(Z,Y,\varphi)=u(Z)$ then

$$(W(f)\phi)(Z)=u(Z)\phi(Z),\tag{5.5}$$

if $f(Z,Y,\varphi)=u(Z)y_k$ then

$$(W(f)\phi)(Z)=i(\partial_{\overline{z}_k}u)(Z)\phi(Z)+2iu(Z)(\partial_{\overline{z}_k}\phi)(Z)\tag{5.6}$$

and if $f(Z,Y,\varphi)=u(Z)\overline{y}_k$ then

$$(W(f)\phi)(Z)=i(\partial_{z_k}u)(Z)\phi(Z)+2iu(Z)(\partial_{z_k}\phi)(Z).\tag{5.7}$$

In order to dequantize the derived representation $d\pi$, that is, to calculate the Berezin-Weyl symbol of the operators $d\pi(X)$ $(X\in\mathfrak{g})$, we need the following lemma.

Lemma 5.1. *For $A\in\mathfrak{k}^c$ let $u_A:\mathfrak{n}^+\to\mathfrak{n}^+$ be the holomorphic map defined by*

$$u_A(Z)=\frac{\operatorname{ad}Z}{1-e^{-\operatorname{ad}Z}}\,p_{\mathfrak{n}^+}(e^{-\operatorname{ad}Z}A).$$

Then

$$\operatorname{Tr}_{\mathfrak{n}^+}\,du_A(Z)=\Lambda(p_{\mathfrak{h}^c}(e^{-\operatorname{ad}Z}A)).$$

Proof: Since $\mathfrak{n}^+$ is a nilpotent Lie algebra, we can write $u_A(Z)=s(\operatorname{ad}Z)p_{\mathfrak{n}^+}(e^{-\operatorname{ad}Z}A)$ where $s(z)=\sum_{k=0}^N a_kz^k$ is a polynomial. For $Y\in\mathfrak{n}^+$ and $Z\in\mathfrak{n}^+$, we have

$$\begin{aligned}du_A(Z)(Y)=&\frac{d}{dt}s(\operatorname{ad}(Z+tY))\Big|_{t=0}p_{\mathfrak{n}^+}(e^{-\operatorname{ad}Z}A)\\&+s(\operatorname{ad}Z)p_{\mathfrak{n}^+}\left(\frac{d}{dt}\operatorname{Ad}(\exp(-Z-tY))A\Big|_{t=0}\right).\end{aligned}$$

Now

$$\begin{aligned}\frac{d}{dt}s(\operatorname{ad}(Z+tY))\Big|_{t=0} &= \sum_{k=0}^{N} a_k \frac{d}{dt}(\operatorname{ad}Z + t\operatorname{ad}Y)^k\Big|_{t=0}\\ &= \sum_{k=0}^{N} a_k \Big(\sum_{r=0}^{k-1}(\operatorname{ad}Z)^r\operatorname{ad}Y(\operatorname{ad}Z)^{k-r-1}\Big).\end{aligned}$$

Then, since for each $r = 0, 1, \ldots, k-1$ the endomorphism of $\mathfrak{n}^+$ defined by

$$\begin{aligned}Y \to\ & (\operatorname{ad}Z)^r \operatorname{ad}Y(\operatorname{ad}Z)^{k-r-1}p_{\mathfrak{n}^+}(e^{-\operatorname{ad}Z}A)\\ &= -(\operatorname{ad}Z)^r\operatorname{ad}\big((\operatorname{ad}Z)^{k-r-1}p_{\mathfrak{n}^+}(e^{-\operatorname{ad}Z}A)\big)(Y)\end{aligned}$$

is nilpotent, the endomorphism of $\mathfrak{n}^+$ given by

$$Y \to \frac{d}{dt}s(\operatorname{ad}(Z+tY))\Big|_{t=0}p_{\mathfrak{n}^+}(e^{-\operatorname{ad}Z}A)$$

has trace zero. On the other hand we have

$$\begin{aligned}&\frac{d}{dt}\operatorname{Ad}(\exp(-Z-tY))\,A\Big|_{t=0}\\ &\qquad= \frac{d}{dt}\operatorname{Ad}\left(\exp(-Z)\exp(Z+tY)\right)^{-1}\operatorname{Ad}\exp(-Z)\,A\Big|_{t=0}\\ &\qquad= -\operatorname{ad}\left(\frac{1-e^{-\operatorname{ad}Z}}{\operatorname{ad}Z}Y\right)\operatorname{Ad}\exp(-Z)\,A\\ &\qquad= \operatorname{ad}\left(\operatorname{Ad}\exp(-Z)\,A\right)\left(\frac{1-e^{-\operatorname{ad}Z}}{\operatorname{ad}Z}\right)Y.\end{aligned}$$

The trace of the endomorphism of $\mathfrak{n}^+$ defined by

$$Y \to s(\operatorname{ad}Z)p_{\mathfrak{n}^+}\left(\frac{d}{dt}\operatorname{Ad}(\exp(-Z-tY))\,A\Big|_{t=0}\right)$$

is then

$$\begin{aligned}&\operatorname{Tr}_{\mathfrak{n}^+}\left(s(\operatorname{ad}Z)p_{\mathfrak{n}^+}\circ\operatorname{ad}(\operatorname{Ad}\exp(-Z)\,A)\,\frac{1-e^{-\operatorname{ad}Z}}{\operatorname{ad}Z}\right)\\ &\quad= \operatorname{Tr}_{\mathfrak{n}^+}\left(\frac{1-e^{-\operatorname{ad}Z}}{\operatorname{ad}Z}s(\operatorname{ad}Z)\,p_{\mathfrak{n}^+}\circ\operatorname{ad}(\operatorname{Ad}\exp(-Z)\,A)\right)\\ &\quad= \operatorname{Tr}_{\mathfrak{n}^+}\left(p_{\mathfrak{n}^+}\circ\operatorname{ad}(\operatorname{Ad}\exp(-Z)\,A)\right).\end{aligned}$$

Consequently, the lemma will be proved if we show that, for each A in $\mathfrak{k}^c$, we have

$$\operatorname{Tr}_{\mathfrak{n}^+}\left(p_{\mathfrak{n}^+}\circ\operatorname{ad}A\right) = \Lambda(p_{\mathfrak{h}^c}(A)).$$

If $A \in \mathfrak{n}^+$ then $p_{\mathfrak{n}^+}\circ\operatorname{ad}A = \operatorname{ad}A$ is a nilpotent endomorphism of $\mathfrak{n}^+$. Thus $\operatorname{Tr}_{\mathfrak{n}^+}(p_{\mathfrak{n}^+}\circ\operatorname{ad}A) = 0$. If $A \in \mathfrak{n}^-$ then for each $k = 1, 2, \ldots, n$ we have $\operatorname{ad}A(E_{\alpha_k}) \in$

$\mathfrak{h}^c + \sum_{\alpha<\alpha_k} \mathfrak{k}_{\alpha_k}$ and we also find that $\mathrm{Tr}_{\mathfrak{n}^+}(p_{\mathfrak{n}^+} \circ \mathrm{ad}\, A) = 0$. Finally, if $A \in \mathfrak{h}^c$ then

$$\mathrm{Tr}_{\mathfrak{n}^+}(p_{\mathfrak{n}^+} \circ \mathrm{ad}\, A) = \mathrm{Tr}_{\mathfrak{n}^+}(\mathrm{ad}\, A) = \sum_{k=1}^{n} \alpha_k(A) = \Lambda(A).$$

This ends the proof of the lemma. □

We consider the Cartan decomposition $K^c = K \exp(i\mathfrak{k})$ [17, Chapter VI]. For $k \in K^c$ we can write $k = up$ where $u \in K$ and $p \in \exp(i\mathfrak{k})$. Since $u^*u = e$ and $p^* = p$ we have $k^*k = p^*u^*up = p^2$ and we can introduce the notation $p =: (k^*k)^{1/2}$.

Proposition 5.2. *For $X = (w, A) \in \mathfrak{g}$, the Berezin-Weyl symbol of the operator $-id\pi(X)$ is the P-symbol f_X on $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$ given by*

$$\begin{aligned} f_X(Z, Y, \varphi) &= \langle \exp Z p_0, w\rangle \\ &+ \langle \varphi,\ (\mathrm{id} + \mathrm{Ad}(h(Z))^{-1/2})^{-1}\big(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\, A) \\ &- \mathrm{Ad}(h(Z))^{-1/2} p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\, A)^*)\big)\rangle \\ &+ \mathrm{Re}\langle u_A(Z),\ Y^*\rangle \end{aligned}$$

where

$$u_A(Z) = \frac{\mathrm{ad}\, Z}{1 - e^{-\mathrm{ad}\, Z}}\, p_{\mathfrak{n}^+}(e^{-\mathrm{ad}\, Z} A).$$

Proof: Write $u_A(Z) = \sum_{k=1}^{n} u_k(Z) E_{\alpha_k}$. Then, by using (5.5), (5.6) and (5.7), we see that the operator

$$\phi \mapsto i(\partial_Z \phi)(Z, Z^*)(u_A(Z)) = i \sum_{k=1}^{n} u_k(Z) \partial_{z_k} \phi$$

has symbol

$$\frac{1}{2}\sum_{k=1}^{n} u_k(Z)\overline{y}_k - \frac{1}{2} i \sum_{k=1}^{n} \partial_{z_k} u_k = \frac{1}{2}\langle u_A(Z), Y^*\rangle - \frac{1}{2} i \Lambda(p_{\mathfrak{h}^c}(e^{-\mathrm{ad}\, Z} A)).$$

Similarly, the operator

$$\phi \mapsto i(\partial_{Z^*} \phi)(Z, Z^*)(u_A(Z)^*) = i \sum_{k=1}^{n} \overline{u_k(Z)} \partial_{\overline{z}_k} \phi$$

has symbol

$$\frac{1}{2}\sum_{k=1}^{n} \overline{u_k(Z)} y_k - \frac{1}{2} i \sum_{k=1}^{n} \partial_{\overline{z}_k} \overline{u}_k = \frac{1}{2}\overline{\langle u_A(Z), Y^*\rangle} - \frac{1}{2} i \overline{\Lambda(p_{\mathfrak{h}^c}(e^{-\mathrm{ad}\, Z} A))}.$$

The result follows from Proposition 4.1 and Proposition 5.1(3). □

6. Adapted Weyl correspondence

In this section we show how the dequantization procedure used in Section 5 allows us to obtain an explicit symplectomorphism from $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$ onto a dense open subset of $\mathcal{O}(\xi_0)$. Using this symplectomorphism we then construct an adapted Weyl correspondence on $\mathcal{O}(\xi_0)$. We retain the notation from the previous sections. Moreover, for $A \in \mathfrak{k}^c$, we set $\mathrm{Re}(A) = \frac{1}{2}(A + \theta(A))$.

Recall that f_X denotes the Berezin-Weyl symbol of the operator $-id\pi(X)$ for $X \in \mathfrak{g}$. Since the map $X \to f_X(Z, Y, \varphi)$ is linear there exists a map Ψ from $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$ to $\mathfrak{g}^* \simeq V^* \oplus \mathfrak{k}$ such that

$$f_X(Z, Y, \varphi) = \langle \Psi(Z, Y, \varphi),\, X\rangle \tag{6.1}$$

for each $X \in \mathfrak{g}$ and each $(Z, Y, \varphi) \in \mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$. From Proposition 5.2 we deduce a precise expression for Ψ.

Proposition 6.1. *For $(Z, Y, \varphi) \in \mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$, we have*

$$\Psi(Z, Y, \varphi) = \Bigg(\exp Z p_0,\, \mathrm{Re}\,\mathrm{Ad}(\exp Z)\Bigg[p_{\mathfrak{n}^-}\bigg(\frac{\mathrm{ad}\, Z}{1 - e^{\mathrm{ad}\, Z}}\,\theta(Y)\bigg) \\ + 2(\mathrm{id} + \mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi\Bigg]\Bigg).$$

Proof: For $(w, A) \in \mathfrak{g}$, we transform the expression for $f_X(Z, Y, \varphi)$ given in Proposition 5.2 as follows. First we have

$$\begin{aligned}
&\langle \varphi, (\mathrm{id} + \mathrm{Ad}(h(Z))^{-1/2})^{-1} p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\, A)\rangle \\
&\quad = \langle (\mathrm{id} + \mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi,\, p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\, A)\rangle \\
&\quad = \langle (\mathrm{id} + \mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi,\, \mathrm{Ad}\exp(-Z)\, A\rangle \\
&\quad = \langle \mathrm{Ad}(\exp Z)(\mathrm{id} + \mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi,\, A\rangle.
\end{aligned}$$

On the other hand, by using the properties $(\mathrm{Ad}(k^{-1})B)^* = \mathrm{Ad}(k^*)B^*$ for $k \in \mathfrak{k}^c$ and $B \in \mathfrak{k}^c$ and $\langle B_1^*, B_2^*\rangle = \overline{\langle B_1, B_2\rangle}$ for B_1 and B_2 in $\mathfrak{k}^c$, we have

$$\begin{aligned}
&\langle \varphi, (\mathrm{id} + \mathrm{Ad}(h(Z))^{-1/2})^{-1}\, \mathrm{Ad}(h(Z))^{-1/2} p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\, A)^*\rangle \\
&\quad = \langle (\mathrm{id} + \mathrm{Ad}(h(Z))^{-1/2})^{-1}\varphi,\, p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\, A)^*\rangle \\
&\quad = -\overline{\langle (\mathrm{id} + \mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi,\, p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\, A)\rangle} \\
&\quad = -\overline{\langle (\mathrm{id} + \mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi,\, \mathrm{Ad}\exp(-Z)\, A\rangle} \\
&\quad = -\overline{\langle \mathrm{Ad}(\exp Z)\, (\mathrm{id} + \mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi,\, A\rangle}.
\end{aligned}$$

Then

$$\begin{aligned}&\Big\langle\varphi,(\mathrm{id}+\mathrm{Ad}(h(Z))^{-1/2})^{-1}\Big(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)\\&\qquad-\mathrm{Ad}(h(Z))^{-1/2}p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)^*\Big)\Big\rangle\\&=\Big\langle 2\,\mathrm{Re}\Big(\,\mathrm{Ad}(\exp Z)\,(\mathrm{id}+\mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi\Big),\,A\Big\rangle.\end{aligned}$$

Moreover we have

$$\begin{aligned}\langle u_A(Z),\,Y^*\rangle&=\Big\langle\frac{\mathrm{ad}\,Z}{1-e^{-\mathrm{ad}\,Z}}p_{\mathfrak{n}^+}(e^{-\mathrm{ad}\,Z}A),\,Y^*\Big\rangle\\&=\Big\langle p_{\mathfrak{n}^+}(e^{-\mathrm{ad}\,Z}A),\,-\frac{\mathrm{ad}\,Z}{1-e^{\mathrm{ad}\,Z}}Y^*\Big\rangle\\&=\Big\langle e^{-\mathrm{ad}\,Z}A,\,p_{\mathfrak{n}^-}\Big(\frac{\mathrm{ad}\,Z}{1-e^{\mathrm{ad}\,Z}}\theta(Y)\Big)\Big\rangle\\&=\Big\langle A,\,e^{\mathrm{ad}\,Z}p_{\mathfrak{n}^-}\Big(\frac{\mathrm{ad}\,Z}{1-e^{\mathrm{ad}\,Z}}\theta(Y)\Big)\Big\rangle.\end{aligned}$$

The result therefore follows. □

Let ω_0 and ω_1 be the Kirillov 2-forms on $\mathcal{O}(\xi_0)$ and $\mathcal{O}(\varphi_0)$, respectively. Denote by $\{\cdot,\cdot\}_0$ and $\{\cdot,\cdot\}_1$ the Poisson brackets associated with ω_0 and ω_1. We endow $\mathfrak{n}^+\times\mathfrak{n}^+$ with the symplectic form

$$\omega_2:=\frac{1}{2}\sum_{k=1}^{n}(dz_k\wedge d\overline{y}_k+d\overline{z}_k\wedge dy_k).$$

The corresponding Poisson bracket on $C^\infty(\mathfrak{n}^+\times\mathfrak{n}^+)$ is

$$\{f,\,g\}_2:=2\sum_{k=1}^{n}\left(\partial f_{z_k}\partial_{\overline{y}_k}g-\partial_{\overline{y}_k}f\partial_{z_k}g+\partial f_{\overline{z}_k}\partial_{y_k}g-\partial_{y_k}f\partial_{\overline{z}_k}g\right).$$

We endow the product $\mathfrak{n}^+\times\mathfrak{n}^+\times\mathcal{O}(\varphi_0)$ with the symplectic form $\omega:=\omega_2\otimes\omega_1$ and we denote by $\{\cdot,\cdot\}$ the corresponding Poisson bracket. Let $u,\,v\in C^\infty(\mathfrak{n}^+\times\mathfrak{n}^+)$ and $a,\,b\in C^\infty(\mathcal{O}(\varphi_0))$. Then, for $f(Z,Y,\varphi)=u(Z,Y)a(\varphi)$ and $g(Z,Y,\varphi)=v(Z,Y)b(\varphi)$ we have

$$\{f,\,g\}=u(Z,Y)v(Z,Y)\{a,\,b\}_1+a(\varphi)b(\varphi)\{u,\,v\}_2.$$

Lemma 6.1. *Suppose that f and g are two P-symbols on $\mathfrak{n}^+\times\mathfrak{n}^+\times\mathcal{O}(\varphi_0)$ of the form*

$$u(Z)+\langle v(Z),\varphi\rangle+\sum_{k=1}^{n}(w_k(Z)y_k+w'_k(Z)\overline{y}_k)$$

where $u \in C^\infty(\mathfrak{n}^+)$, $v \in C^\infty(\mathfrak{n}^+, \mathfrak{k}^c)$ *and* w_k, $w'_k \in C^\infty(\mathfrak{n}^+)$ *for* $k = 1, 2, \ldots, n$. *Then we have*

$$[W(f), W(g)] = -i\, W(\{f, g\}).$$

Proof: By using (5.5), (5.6) and (5.7), one can prove the result by a direct computation. One can also deduce it from Lemma 6.2 of [9] by using the fact that $\tilde{j}$ is a symplectomorphism from $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathcal{O}(\varphi_0)$ endowed with its natural symplectic structure onto $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$. □

Let $\tilde{\mathcal{O}}(\xi_0)$ be the dense open subset of $\mathcal{O}(\xi_0)$ defined by

$$\tilde{\mathcal{O}}(\xi_0) = \{(v, k).(p_0, \varphi_0) \,:\, v \in V,\, k \in K \cap N^+ H^c N^-\}.$$

Proposition 6.2. *The map* Ψ *is a symplectomorphism from* $(\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0), \omega)$ *onto* $(\tilde{\mathcal{O}}(\xi_0), \omega_0)$.

Proof: (1) First, we show that for any $\xi \in \tilde{\mathcal{O}}(\xi_0)$ there exists a unique element (Z, Y, φ) in $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$ such that $\Psi(Y, Z, \varphi) = \xi$. Let $\xi \in \tilde{\mathcal{O}}(\xi_0)$. Write $\xi = (v, k).(p_0, \varphi_0)$ where $v \in V$ and $k \in K \cap N^+ H^c N^-$. If $\Psi(Y, Z, \varphi) = \xi$ then

$$(0, k)^{-1}.\Psi(Z, Y, \varphi) = (p_0, \varphi_0 + (k^{-1}.v) \wedge p_0). \tag{6.2}$$

This gives $k^{-1} \exp Z p_0 = p_0$ or, equivalently, $k^{-1} \exp Z \in H^c N^-$ and we can write $k^{-1} \exp Z = yh$ where $y \in N^-$ and $h \in H^c$. Thus, equation (6.2) implies

$$\begin{aligned} &2\,\mathrm{Re}\,\mathrm{Ad}(yh)(\mathrm{id} + \mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi + \mathrm{Re}\,\mathrm{Ad}(yh)p_{\mathfrak{n}^-}\Big(\frac{\mathrm{ad}\,Z}{1 - e^{\mathrm{ad}\,Z}}\theta(Y)\Big) \\ &\qquad = \varphi_0 + (k^{-1}.v) \wedge p_0. \end{aligned} \tag{6.3}$$

Hence, noting that the element $Y_{Z,\varphi}$ defined by

$$Y_{Z,\varphi} := \mathrm{Ad}(y)\,\mathrm{Ad}(h)(\mathrm{id} + \mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi - \mathrm{Ad}(h)(\mathrm{id} + \mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi$$

belongs to $\mathfrak{n}^-$ and applying Lemma 2.1, we see that equation (6.3) is equivalent to

$$\begin{cases} \text{(E1)} \qquad \mathrm{Re}\Big(Y_{Z,\varphi} + \mathrm{Ad}(yh)p_{\mathfrak{n}^-}\Big(\frac{\mathrm{ad}\,Z}{1 - e^{\mathrm{ad}\,Z}}\theta(Y)\Big)\Big) = (k^{-1}.v) \wedge p_0 \\ \text{(E2)} \qquad 2\,\mathrm{Re}\big(\mathrm{Ad}(h)(\mathrm{id} + \mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi\big) = \varphi_0. \end{cases}$$

But we have

$$\begin{aligned} &2\,\mathrm{Re}\big(\mathrm{Ad}(h)(\mathrm{id} + \mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi\big) \\ &\quad = \mathrm{Ad}(h)(\mathrm{id} + \mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi + \mathrm{Ad}(\theta(h))(\mathrm{id} + \mathrm{Ad}(\theta(h(Z)))^{1/2})^{-1}\theta(\varphi) \\ &\quad = \mathrm{Ad}(h)(\mathrm{id} + \mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi + \mathrm{Ad}(h^*)^{-1}(\mathrm{id} + \mathrm{Ad}(h(Z))^{-1/2})^{-1}\varphi \\ &\quad = \mathrm{Ad}(h)(\mathrm{id} + \mathrm{Ad}(h^*h)^{-1}\,\mathrm{Ad}(h(Z))^{1/2})(\mathrm{id} + \mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi \end{aligned}$$

and, since $h^*h = h(Z)$, we can write

$$\begin{aligned} &2\operatorname{Re}\bigl(\operatorname{Ad}(h)(\mathrm{id}+\operatorname{Ad}(h(Z))^{1/2})^{-1}\varphi\bigr)\\ &\quad = \operatorname{Ad}(h)(\mathrm{id}+\operatorname{Ad}(h(Z))^{-1/2})(\mathrm{id}+\operatorname{Ad}(h(Z))^{1/2})^{-1}\varphi\\ &\quad = \operatorname{Ad}(h)\operatorname{Ad}(h(Z))^{-1/2}\varphi. \end{aligned}$$

Finally, writing $h = up$, $u \in K$, $p = (h^*h)^{1/2} \in \exp(i\mathfrak{k})$ for the Cartan decomposition of h, we obtain

$$2\operatorname{Re}\bigl(\operatorname{Ad}(h)(\mathrm{id}+\operatorname{Ad}(h(Z))^{1/2})^{-1}\varphi\bigr) = \operatorname{Ad}(u)\varphi$$

where $u \in H^c \cap K = H$. Consequently, equation (E2) gives $\varphi = \operatorname{Ad}(u^{-1})\varphi_0$. Since $Z = \log\zeta(k)$, we have shown that Z and φ are unique. In order to verify that Y is also unique, we have just to use equation (E1) and the following facts: (1) the map $Y \to \operatorname{Re}(Y)$ from $\mathfrak{n}^+$ to the ortho-complement of $\mathfrak{h}$ in $\mathfrak{k}$ is injective and (2) the map

$$Y \to p_{\mathfrak{n}^-}\Bigl(\frac{\operatorname{ad} Z}{1-e^{\operatorname{ad} Z}}\theta(Y)\Bigr)$$

is a bijection from $\mathfrak{n}^+$ onto $\mathfrak{n}^-$, the inverse bijection being

$$U \to \theta\left(p_{\mathfrak{n}^-}\Bigl(\frac{1-e^{\operatorname{ad} Z}}{\operatorname{ad} Z}U\Bigr)\right).$$

It is also clear that the element (Y, Z, φ) obtained below satisfies the equation $\Psi(Y, Z, \varphi) = \xi$. Moreover, by similar considerations, we show that Ψ takes values in $\tilde{\mathcal{O}}(\xi_0)$ and we can conclude that Ψ is a bijection from $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$ onto $\tilde{\mathcal{O}}(\xi_0)$.

(2) For $X \in \mathfrak{g}$, we denote by $\tilde{X}$ the function on $\tilde{\mathcal{O}}(\xi_0)$ defined by $\tilde{X}(\xi) = \langle \xi, X\rangle$. Observe that $f_X = \tilde{X} \circ \Psi$.

Let X and Y in $\mathfrak{g}$. Then by Proposition 5.2 and Lemma 6.1 we have

$$[W(f_X), W(f_Y)] = -iW(\{f_X, f_Y\}).$$

But we also have

$$[W(f_X), W(f_Y)] = [-id\pi(X), -id\pi(Y)] = -d\pi([X, Y]) = -iW(f_{[X,Y]}).$$

Hence $f_{[X,Y]} = \{f_X, f_Y\}$. Since $\widetilde{[X,Y]} = \{\tilde{X}, \tilde{Y}\}_0$, we obtain

$$\bigl\{\tilde{X}, \tilde{Y}\bigr\}_0 \circ \Psi = \bigl\{\tilde{X}\circ\Psi, \tilde{Y}\circ\Psi\bigr\}.$$

This implies that $\Psi^*(\omega_0) = \omega$. Since the 2-form ω is non-degenerate, we also have that the map Ψ is regular. Finally, Ψ is a symplectomorphism. □

Remark 6.1. The map Ψ might define a symplectomorphism from $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$ onto $\tilde{\mathcal{O}}(\xi_0)$ even when the orbit $\mathcal{O}(\varphi_0)$ is not assumed to be integral.

Now, we are in position to construct an adapted Weyl transform on $\mathcal{O}(\xi_0)$ by transferring to $\mathcal{O}(\xi_0)$ the Berezin-Weyl calculus on $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$. We say that a smooth function f on $\mathcal{O}(\xi_0)$ is a symbol (resp. a P-symbol, an S-symbol) on $\mathcal{O}(\xi_0)$ if $f \circ \Psi$ is a symbol (resp. a P-symbol, an S-symbol) for the Berezin-Weyl calculus on $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$.

Proposition 6.3. *Let $\mathcal{A}$ be the space of P-symbols on $\mathcal{O}(\xi_0)$ and let $\mathcal{B}$ be the space of differential operators on $\mathfrak{n}^+$ with coefficients in $C^\infty(\mathfrak{n}^+, E)$. Then the map $\tilde{W} : \mathcal{A} \to \mathcal{B}$ defined by the $\tilde{W}(f) = W(f \circ \Psi)$ is an adapted Weyl correspondence in the sense of Definition* 1.1.

Proof: The properties (1), (2) and (3) of Definition 1.1 are clearly satisfied with $D = C_0^\infty(\mathfrak{n}^+, E)$. The property (4) follows from the corresponding properties for the Berezin calculus (see Proposition 5.1) and for the usual Weyl calculus [18]. Finally, the property (5) is an immediate consequence of Proposition 5.2 and Proposition 6.1. □

7. Final remarks and examples

7.1. If ρ is a character of H then $\mathcal{O}(\varphi_0)$ reduces to the point φ_0 and Ψ is given by

$$\Psi(Z, Y, \varphi) = \Big(\exp Z p_0,\, \mathrm{Re}\, \mathrm{Ad}(\exp Z)\Big[\varphi_0 + p_{\mathfrak{n}^-}\Big(\frac{\mathrm{ad}\, Z}{1 - e^{\mathrm{ad}\, Z}}\, \theta(Y)\Big)\Big]\Big). \tag{7.1}$$

7.2. If $Z(p_0) \simeq G/H$ is a symmetric space then $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are abelian and $[\mathfrak{n}^+, \mathfrak{n}^-] \subset \mathfrak{h}^c$ (see [17, Lemma VII 2.16]). Thus, for each Y and Z in $\mathfrak{n}^+$, we have

$$p_{\mathfrak{n}^-}\Big(\frac{\mathrm{ad}\, Z}{1 - e^{\mathrm{ad}\, Z}}\, \theta(Y)\Big) = \theta(Y).$$

Hence the expression for Ψ is

$$\Psi(Z, Y, \varphi) = \Big(\exp Z p_0,\, \mathrm{Re}\, \mathrm{Ad}(\exp Z)\, \Big[2(\mathrm{id} + \mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi + \theta(Y)\Big]\Big)\,. \tag{7.2}$$

7.3. In this subsection, we consider the case when V is equal to the Lie algebra $\mathfrak{k}$ of K and σ is the adjoint action of K on $\mathfrak{k}$. We identify $V^* = \mathfrak{k}^*$ to $V = \mathfrak{k}$ by means of the Killing form. Then we have $v \wedge p = [v, p]$ for each $v \in V = \mathfrak{k}$ and each $p \in V^* \simeq \mathfrak{k}$. The coadjoint action of G on $\mathfrak{g}^*$ is thus given by

$$(v, k).(p, f) = (\mathrm{Ad}(k)p, \mathrm{Ad}(k)f + [v, \mathrm{Ad}(k)p]).$$

Moreover, if $\xi_0 = (p_0, \varphi_0)$ is an element of $\mathfrak{g}^*$ such that $p_0 \neq 0$ and $\mathcal{O}(\varphi_0)$ is integral then the stabilizer H of p_0 in K is the centralizer of the torus of K generated by $\exp p_0$ and one can apply to $\mathcal{O}(\xi_0)$ the results of the previous sections.

7.4. We illustrate here the situation described in the previous subsection by the following example. We take $K = SU(m+n)$ and p_0 to be the element of $\mathfrak{k}$ defined by

$$p_0 = i\begin{pmatrix} -nI_m & 0 \\ 0 & mI_n \end{pmatrix}.$$

The torus T_1 generated by $\exp p_0$ consists of the matrices

$$\begin{pmatrix} e^{ia}I_m & 0 \\ 0 & e^{ib}I_n \end{pmatrix} \qquad a,\, b \in \mathbb{R}, \qquad (e^{ia})^m(e^{ib})^n = 1.$$

The torus T_1 is contained in the maximal torus $T \subset K$ consisting of the matrices

$$\mathrm{Diag}(e^{ia_1}, e^{ia_2}, \ldots, e^{ia_{m+n}}), \qquad a_1, a_2, \ldots, a_{m+n} \in \mathbb{R}, \qquad \prod_{k=1}^{m+n} e^{ia_k} = 1.$$

Moreover, the subgroup $H = \{k \in K \, : \, k.p_0 = p_0\}$ is the centralizer of T_1 in K and consists of the matrices

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \qquad A \in U(m),\, D \in U(n), \qquad \mathrm{Det}\, A.\, \mathrm{Det}\, D = 1,$$

that is, we have $H = S(U(m) \times U(n))$. The complexification T^c of T has Lie algebra

$$\mathfrak{t}^c = \left\{ X = \mathrm{Diag}(x_1, x_2, \ldots, x_{m+n}) \ : \ x_k \in \mathbb{C}, \sum_{k=1}^{m+n} x_k = 0 \right\}.$$

The set of roots of $\mathfrak{t}^c$ on $\mathfrak{g}^c$ is $\lambda_i - \lambda_j$ for $1 \leq i \neq j \leq m+n$ where $\lambda_i(X) = x_i$ for $X \in \mathfrak{t}^c$ as above. The set of roots of $\mathfrak{t}^c$ on $\mathfrak{h}^c$ is $\lambda_i - \lambda_j$ for $1 \leq i \neq j \leq m$ and $m+1 \leq i \neq j \leq m+n$. We take the set of positive roots Δ^+ to be $\lambda_i - \lambda_j$ for $1 \leq i < j \leq m+n$ and the set of positive roots Δ_1^+ to be $\lambda_i - \lambda_j$ for $1 \leq i < j \leq m$ and $m+1 \leq i < j \leq m+n$. Then we have

$$N^+ = \left\{ \begin{pmatrix} I_m & Z \\ 0 & I_n \end{pmatrix} : Z \in M_{mn}(\mathbb{C}) \right\},\ N^- = \left\{ \begin{pmatrix} I_m & 0 \\ Y & I_n \end{pmatrix} : Y \in M_{nm}(\mathbb{C}) \right\}.$$

We identify $\mathfrak{n}^+$ to $M_{mn}(\mathbb{C})$ by means of the map

$$Z \mapsto \tilde{Z} = \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}.$$

We also have

$$H^c = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} A \in M_m(\mathbb{C}), D \in M_n(\mathbb{C}),\, \mathrm{Det}\, A.\, \mathrm{Det}\, D = 1 \right\}.$$

We easily see that the $N^+H^cN^-$-decomposition of a matrix $k \in K^c$ is given by

$$k = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_m & BD^{-1} \\ 0 & I_n \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I_m & 0 \\ D^{-1}C & I_n \end{pmatrix}.$$

Observe that a matrix $k \in K^c$ have such a decomposition if and only if $\operatorname{Det} D \neq 0$. In particular we have $K \subset N^+H^cN^-$. Moreover, we deduce from the preceding decomposition that the action of K^c on $\mathfrak{n}^+$ is given by

$$k \cdot Z = (AZ + B)(CZ + D)^{-1}, \qquad k = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Note that, for $k \in K^c$, we have $k^* = \overline{k}^t$ (conjugate transpose of k) and $\tilde{\theta}(k) = (\overline{k}^t)^{-1}$. For $X \in \mathfrak{k}^c$, we have $X^* = \overline{X}^t$ and $\theta(X) = -\overline{X}^t$.

We are here in the situation of the subsection 7.2 and Ψ is then given by equation (7.2) with

$$h(\tilde{Z}) = \begin{pmatrix} (I_m + ZZ^\star)^{-1} & 0 \\ 0 & I_n + Z^\star Z \end{pmatrix}$$

and

$$\exp \tilde{Z}\, p_0 = i \begin{pmatrix} (I_m + ZZ^\star)^{-1}(mZZ^\star - nI_m) & (m+n)Z(I_n + Z^\star Z)^{-1} \\ (m+n)(I_n + Z^\star Z)^{-1}Z^\star & (mI_n - nZ^\star Z)(I_n + Z^\star Z)^{-1} \end{pmatrix}.$$

In particular, in the case when $m = n = 1$, we can take

$$\varphi_0 = \begin{pmatrix} -ia & 0 \\ 0 & ia \end{pmatrix}$$

where $a \in \mathbb{N} \setminus (0)$. We get

$$\exp \tilde{Z} p_0 = \frac{1}{\sqrt{1 + |z|^2}} i \begin{pmatrix} |z|^2 - 1 & 2z \\ 2\overline{z} & 1 - |z|^2 \end{pmatrix}$$

and

$$\Psi(\tilde{Z}, \tilde{Y}) = \left(\exp \tilde{Z} p_0, \frac{1}{2} \begin{pmatrix} -2ai + y\overline{z} - \overline{y}z & 2aiz + y + \overline{y}z^2 \\ 2ai\overline{z} - y - y\overline{z}^2 & 2ai - y\overline{z} + \overline{y}z \end{pmatrix} \right).$$

References

[1] Ali S.T., Englis M., *Quantization methods: a guide for physicists and analysts*, Rev. Math. Phys. **17** (2005), no. 4, 391–490.

[2] Arnal D., Cortet J.-C., *Nilpotent Fourier Transform and Applications*, Lett. Math. Phys. **9** (1985), 25–34.

[3] Baguis P., *Semidirect products and the Pukansky condition*, J. Geom. Phys. **25** (1998), 245–270.

[4] Berezin F.A., *Quantization*, Math. USSR Izv. 8, 5 (1974), 1109–1165.
[5] Cahen B., *Deformation program for principal series representations*, Lett. Math. Phys. **36** (1996), 65–75.
[6] Cahen B., *Quantification d'une orbite massive d'un groupe de Poincaré généralisé*, C.R. Acad. Sci. Paris Sér. I Math. **325** (1997), 803–806.
[7] Cahen B., *Quantification d'orbites coadjointes et théorie des contractions*, J. Lie Theory **11** (2001), 257–272.
[8] Cahen B., *Contractions of $SU(1,n)$ and $SU(n+1)$ via Berezin quantization*, J. Anal. Math. **97** (2005), 83–102.
[9] Cahen B., *Weyl quantization for semidirect products*, Differential Geom. Appl. **25** (2007), 177–190.
[10] Cahen B., *Weyl quantization for principal series*, Beiträge Algebra Geom. **48** (2007), no. 1, 175–190.
[11] Cahen B., *Berezin quantization for discrete series*, preprint Univ. Metz (2008), to appear in Beiträge Algebra Geom.
[12] Cahen B., *Berezin quantization on generalized flag manifolds*, preprint Univ. Metz (2008), to appear in Math. Scand.
[13] Cahen M., Gutt S., Rawnsley J., *Quantization on Kähler manifolds I, Geometric interpretation of Berezin quantization*, J. Geom. Phys. **7** (1990), 45–62.
[14] Cotton P., Dooley A.H., *Contraction of an adapted functional calculus*, J. Lie Theory **7** (1997), 147–164.
[15] Folland B., *Harmonic Analysis in Phase Space*, Princeton Univ. Press, Princeton, 1989.
[16] Gotay M., *Obstructions to quantization*, in Mechanics: From Theory to Computation (Essays in Honor of Juan-Carlos Simo), J. Nonlinear Science Editors, Springer, NewYork, 2000, pp. 271–316.
[17] Helgason S., *Differential Geometry, Lie Groups and Symmetric Spaces*, Graduate Studies in Mathematics, Vol. 34, American Mathematical Society, Providence, Rhode Island, 2001.
[18] Hörmander L., *The Analysis of Linear Partial Differential Operators III. Pseudodifferential Operators*, Grundlehren der Mathematischen Wissenschaften, 274, Springer, Berlin-Heidelberg-NewYork, 1985.
[19] Kirillov A.A., *Elements of the Theory of Representations*, Grundlehren der Mathematischen Wissenschaften, 220, Springer, Berlin-Heidelberg-New York, 1976.
[20] Kirillov A.A., *Lectures on the Orbit Method*, Graduate Studies in Mathematics, Vol. 64, American Mathematical Society, Providence, Rhode Island, 2004.
[21] Kostant B., *Quantization and unitary representations*, in Modern Analysis and Applications, Lecture Notes in Mathematics 170, Springer, Berlin-Heidelberg-New York, 1970, pp. 87–207.
[22] Neeb K.-H., *Holomorphy and Convexity in Lie Theory*, de Gruyter Expositions in Mathematics, Vol. 28, Walter de Gruyter, Berlin, New York, 2000.
[23] Rawnsley J.H., *Representations of a semi direct product by quantization*, Math. Proc. Cambridge Philos. Soc. **78** (1975), 345–350.
[24] Simms D.J., *Lie Groups and Quantum Mechanics*, Lecture Notes in Mathematics, 52, Springer, Berlin-Heidelberg-New York, 1968.
[25] Taylor M.E., *Noncommutative Harmonis Analysis*, Mathematical Surveys and Monographs, Vol. 22, American Mathematical Society, Providence, Rhode Island, 1986.
[26] Voros A., *An algebra of pseudo differential operators and the asymptotics of quantum mechanics*, J. Funct. Anal. **29** (1978), 104–132.
[27] Wallach N.R., *Harmonic Analysis on Homogeneous Spaces*, Pure and Applied Mathematics, Vol. 19, Marcel Dekker, New York, 1973.

[28] Wildberger N.J., *Convexity and unitary representations of a nilpotent Lie group*, Invent. Math. **89** (1989), 281–292.
[29] Wildberger N.J., *On the Fourier transform of a compact semisimple Lie group*, J. Austral. Math. Soc. A **56** (1994), 64–116.

Université de Metz, UFR-MIM, Département de mathématiques, LMMAS, ISGMP-Bât. A, Ile du Saulcy 57045, Metz cedex 01, France
Email: cahen@univ-metz.fr

(*Received* January 2, 2009)