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EXISTENCE TO SINGULAR BOUNDARY VALUE PROBLEMS WITH SIGN CHANGING NONLINEARITIES USING AN APPROXIMATION METHOD APPROACH*

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Abstract. This paper studies the existence of solutions to the singular boundary value problem

$$\begin{cases} -u'' = g(t,u) + h(t,u), & t \in (0,1), \\ u(0) = 0 = u(1), \end{cases}$$

where $g\colon (0,1) \times (0,\infty) \to \mathbb{R}$ and $h\colon (0,1) \times [0,\infty) \to [0,\infty)$ are continuous. So our nonlinearity may be singular at $t = 0,1$ and $u = 0$ and, moreover, may change sign. The approach is based on an approximation method together with the theory of upper and lower solutions.

Keywords: singular boundary value problem, positive solution, upper and lower solution

MSC 2000: 34B15, 34B16

1. Introduction

The singular boundary value problems of the form

$$\text{(1.1)} \qquad \begin{cases} -u'' = f(t,u), & t \in (0,1), \\ u(0) = 0 = u(1) \end{cases}$$

occurs in several problems in applied mathematics [1]–[3]. In many papers, the critical condition is that either

$$f(t,r) \geqslant 0 \quad \text{for } (t,r) \in (0,1) \times (0,\infty)$$

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or there exists a constant $L > 0$ such that for any compact set $K \subset (0,1)$ there is $\varepsilon = \varepsilon_K > 0$ such that

$$f(t,r) \geqslant L \quad \text{for all } t \in K, \ r \in (0,\varepsilon].$$

We refer the reader to [1]–[3]. In the case when $f(t,r)$ may change sign in a neighborhood of $r = 0$, very few existence results are available in literature.

In this paper we study positive solutions of the boundary value problem

$$\begin{cases} -u'' = g(t,u) + h(t,u), & t \in (0,1), \\ u(0) = 0 = u(1); \end{cases} \tag{1.2}$$

here $g\colon (0,1) \times (0,\infty) \to \mathbb{R}$ and $h\colon (0,1) \times [0,\infty) \to [0,\infty)$ are continuous so that our nonlinearity may be singular at $t = 0, 1$ and $u = 0$. Also our nonlinearity may change sign. Our main existence results (Theorem 1.1 and Theorem 1.2) are new (see Remark 1.2, Example 3.1 and Example 3.2).

A function u is a solution of the boundary value problem (1.2) if $u\colon [0,1] \to \mathbb{R}$, u satisfies the differential equation (1.2) on $(0,1)$ and the stated boundary data.

Let $C[0,1]$ denote the class of maps u continuous on $[0,1]$, with the norm $|u|_\infty = \max\limits_{t\in[0,1]} |u(t)|$. Let

$$M = \Big\{ h \in C(0,1)\colon \int_0^1 |h(s)| \,\mathrm{d}s < \infty \text{ with } \lim_{t\to 0^+} t|h(t)| < \infty \text{ and } \lim_{t\to 1^-} (1-t)|h(t)| < \infty. \Big\}$$

Theorem 1.1. *Suppose the following conditions hold:*

(G1) *there exist continuous functions $g_i\colon (0,1)\times(0,\infty) \to (0,\infty)$ $(i = 1,2)$ such that*

$$\begin{cases} g_i(t,\cdot) \quad \text{is strictly decreasing for } t \in (0,1), \\ -g_1(t,r) \leqslant g(t,r) \leqslant g_2(t,r) \quad \text{for } (t,r) \in (0,1) \times (0,\infty), \\ g_1(\cdot, r\varphi_1(\cdot)), g_2(\cdot, r) \in M \quad \text{for all } r > 0; \end{cases} \tag{1.3}$$

(G2) *for all $r_2 > r_1 > 0$ there exists $\gamma(\cdot) \in M$ such that $g(t,r) + \gamma(t)r$ is increasing in (r_1, r_2);*

(G3) *there exist $c_1 > c_2 > 0$, $0 < \beta < 1$ such that*

$$0 \leqslant g(t,r), \ \ t \in (0,1), \ \ 0 < r < c_1 \tag{1.4}$$

and

$$\int_0^1 t(1-t)\overline{g}(t, c_2 l(t))\,\mathrm{d}t \geqslant c_2\pi \tag{1.5}$$

where $\overline{g}(t,r)=\min\{g(t,r), 1/r^{\beta}\}$ *and* $l(t)=\min\{t,1-t\}$ *for* $t\in(0,1)$*;*

(H1) *there exist continuous functions* $h_i\colon\ (0,1)\times[0,\infty)\to[0,\infty)$ $(i=1,2)$ *such that*

$$\begin{cases} h_i(t,\cdot) \quad \text{is increasing for } t\in(0,1),\\ h_1(\cdot,r), h_2(\cdot,r)\in M \quad \text{for } r>0,\\ h_1(t,r)\leqslant h(t,r)\leqslant h_2(t,r) \quad \text{for } (t,r)\in(0,1)\times(0,\infty),\\ \text{there exists } \overline{r}>0 \text{ such that } h_1(t,\overline{r})>0 \quad \text{for } t\in(0,1); \end{cases} \tag{1.6}$$

(H2) $\lim\limits_{r\to\infty} h_2(t,r)/r=0$ *for* $t\in(0,1)$.

Then problem (1.2) *has at least one solution* $u\in C[0,1]\cap C^1(0,1)$ *and* $u(t)>0$ *for* $t\in(0,1)$. *Moreover, there exist* $c_i=c_i(g,h,\varphi_1)>0$, $i=3,4$ *such that*

$$c_3\varphi_1(t)\leqslant u(t)\leqslant c_4(\varphi_1(t)+1) \ \text{ for } t\in[0,1],$$

where φ_1 *is defined in Lemma* 2.1.

Remark 1.1. Notice that $\overline{g}(t,r)$ satisfies (G1). Moreover, for all $r_2>r_1>0$ there exists $\overline{\gamma}(\cdot)\in M$ such that $\overline{g}(t,r)+\overline{\gamma}(t)r$ is increasing in (r_1,r_2),

$$\overline{g}(\cdot, cl(\cdot))\in M \quad \text{for } c>0$$

and

$$g(t,r)\geqslant\overline{g}(t,r). \tag{1.7}$$

Theorem 1.2. *Suppose* (G1), (G2), (H1), (H2) *and the following conditions hold:*

(G3′) *there exists* $\lambda\geqslant\lambda_1$ *such that*

$$\lim_{r\to0^+}\frac{\lambda r+g^-(t,r)}{h(t,r)}=0 \tag{1.8}$$

where λ_1 *is defined in Lemma* 2.1 *and* $g^+(t,r)=\max\{0,g(t,r)\}$, $g^-(t,r)=\max\{0,-g(t,r)\}$*;*

(H3) *for all* $r_2>r_1>0$ *there exists* $\tau(\cdot)\in M$ *such that* $h(t,r)+\tau(t)r$ *is increasing in* (r_1,r_2).

Then problem (1.2) *has at least one solution* $u \in C[0,1] \cap C^1(0,1)$ *and* $u > 0$ *for* $t \in (0,1)$. *Moreover, there exist* $c_i = c_i(g,h) > 0$ $(i = 5,6)$ *such that*

$$c_5\varphi_1(t) \leqslant u(t) \leqslant c_6(\varphi_1(t) + 1) \quad \textit{for } t \in [0,1].$$

R e m a r k 1.2. In [2], the authors consider the boundary value problem (1.1) under the conditions

(i_1) there exists a constant $L > 0$ such that for any compact set $K \subset (0,1)$ there is $\varepsilon = \varepsilon_K > 0$ such that

$$f(t,r) \geqslant L \quad \text{for all } t \in K, \ \ r \in (0,\varepsilon];$$

(i_2) for any $\delta > 0$ there is $h_\delta \in C((0,1),\mathbb{R}^+)$ with

$$|f(t,r)| \leqslant h_\delta(t) \quad \text{for all } t \in (0,1), \ \ r \geqslant \delta,$$

and

$$\int_0^1 t(1-t)h_\delta(t)\,\mathrm{d}t < \infty.$$

Then problem (1.1) has at least one solution $u \in C[0,1] \cap C^1(0,1)$.

In Section 3 we give two examples (see Example 3.1 and Example 3.2) which satisfy the conditions of Theorem 1.1 or Theorem 1.2 but do not satisfy the conditions from Remark 1.2.

2. Proof of main results

We first give some lemmas which will help us to prove Theorem 1.1 and Theorem 1.2. We assume throughout this section that (G1), (G2), (H1) and (H2) hold.

Lemma 2.1. *Consider the eigenvalue problem*

$$\begin{cases} -u'' = \lambda u(t), & t \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

Then the eigenvalues are

$$\lambda_m = (m\pi)^2 \quad \textit{for } m = 1,2,\ldots$$

and the corresponding eigenfunctions are

$$\varphi_m(t) = \sin m\pi t \quad \textit{for } m = 1,2,\ldots.$$

Let $G(t,s)$ be the Green function for the BVP

$$\begin{cases} -u'' = 0 & \text{for } t \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

Then

$$G(t,s) = \begin{cases} s(1-t), & 0 \leqslant s < t \leqslant 1, \\ t(1-s), & 0 \leqslant t < s \leqslant 1. \end{cases}$$

For all $(t,s) \in [0,1] \times [0,1]$ define

$$N(t,s) = \begin{cases} \dfrac{G(t,s)}{\varphi_1(t)} & \text{if } t \neq 0, 1, \\ \dfrac{1-s}{\pi} & \text{if } t = 0 \\ \dfrac{s}{\pi} & \text{if } t = 1. \end{cases}$$

It follows easily that

$$0 < G(t,s) \leqslant t(1-t) \quad \text{for } (t,s) \in (0,1) \times (0,1) \tag{2.1}$$

and

$$\frac{s(1-s)}{2\pi} \leqslant N(t,s) \leqslant \frac{1}{2} \quad \text{for } (t,s) \in (0,1) \times (0,1). \tag{2.2}$$

Define operators $A, B\colon\ M \to C[0,1]$ by

$$Ax(t) = \int_0^1 G(t,s)x(s)\,\mathrm{d}s \tag{2.3}$$

and

$$Bx(t) = \int_0^1 N(t,s)x(s)\,\mathrm{d}s. \tag{2.4}$$

The next six results can be found in [4].

Lemma 2.2. *Let $n_0 \in \mathbb{N}$. Assume that for every $n > n_0$ there exist $a_n, \delta_n, \delta \in M$ such that*

$$0 \leqslant a_n(t), \quad |\delta_n(t)| \leqslant \delta(t) \quad \text{and} \quad \lim_{n\to\infty} \delta_n(t) = 0 \ \text{ for } \ t \in (0,1)$$

and there exist $\overline{u}, \overline{u}_n, \hat{u}_n, \hat{u} \in C[0,1]$ such that

$$0 < \overline{u}(t) \leqslant \overline{u}_n(t) \leqslant \hat{u}_n(t) \leqslant \hat{u}(t) \quad \text{for } \ t \in (0,1)$$

and $\hat{u}(0) = \hat{u}(1) = 0$. If

$$\overline{u}_n + A(a_n\overline{u}_n) \leqslant A\Big(g\Big(\cdot, \frac{1}{n} + v\Big) + h(\cdot, v) + a_n v + \delta_n\Big) \leqslant \hat{u}_n + A(a_n\hat{u}_n) \quad \text{in } \ (0,1)$$

and $v \in [\overline{u}_n, \hat{u}_n] = \{u \in C[0,1],\ \overline{u}_n(t) \leqslant u(t) \leqslant \hat{u}_n(t)$ for $t \in [0,1]\}$, then problem (1.2) has a solution $u \in C[0,1] \cap C^1(0,1)$ such that $\overline{u}(t) \leqslant u(t) \leqslant \hat{u}(t)$ for $t \in [0,1]$.

Lemma 2.3. *Let $\psi\colon (0,1) \times (0,\infty) \to (0,\infty)$ be a continuous function such that*

$$\begin{cases} \psi(t,\cdot) \ \text{ is strictly decreasing,} \\ \psi(\cdot, r) \in M \ \text{ for all } \ r > 0. \end{cases}$$

Then the problem

$$\begin{cases} -\omega''(t) = \psi\Big(t, \omega(t) + \dfrac{1}{n}\Big) \quad \text{for } \ t \in (0,1), \\ \omega(0) = \omega(1) = 0 \end{cases}$$

has a solution $\omega_n \in C[0,1]$ such that

$$\omega_n(t) \leqslant \omega_{n+1}(t) \leqslant 1 + \omega_1(t) \quad \text{for } \ t \in [0,1] \ \text{ and } \ n \in \mathbb{N}.$$

If we let $\omega(t) = \lim_{n\to\infty} \omega_n(t)$ for $t \in [0,1]$, then

$$\omega \in C[0,1], \quad \omega(t) > 0 \quad \text{for } \ t \in (0,1)$$

and

$$\begin{cases} -\omega''(t) = \psi(t, \omega(t)) \quad \text{for } \ t \in (0,1), \\ \omega(0) = \omega(1) = 0. \end{cases}$$

Lemma 2.4. *Suppose $m\colon (0,1)\times[0,\infty)\to[0,\infty)$ is a continuous function such that*

$$\begin{cases} m(\cdot,r)\in M & \text{for all } r\geqslant 0,\\ m(t,\cdot) & \text{is increasing}\end{cases}$$

and there exists $b\in M$, $b(t)>0$ for $t\in(0,1)$ with

$$\lim_{r\to+\infty}\frac{m(t,r)}{b(t)r}=0 \quad \text{uniformly with respect to } t\in(0,1).$$

Then there exist $R_0>0$ and $\widetilde{v}\in C[0,1]$, $0\leqslant\widetilde{v}\leqslant R_0\varphi_1$ such that

$$\begin{cases} -\widetilde{v}''(t)=m(t,\widetilde{v}) & \text{for } t\in(0,1),\\ \widetilde{v}(0)=\widetilde{v}(1)=0.\end{cases}$$

Corollary 2.1. *Let $\psi(t,r)$, $m(t,r)$, $(\omega_n)_{n\in\mathbb{N}}$ and $R_0>0$ be as in Lemmas 2.3 and 2.4. There exists $\{\widetilde{v}_n\}_{n\in\mathbb{N}}\subset C[0,1]$ with $0\leqslant\widetilde{v}_n\leqslant R_0\varphi_1$ such that*

$$\begin{cases} -\widetilde{v}_n''(t)=m(t,\omega_n+\widetilde{v}_n) & \text{for } t\in(0,1),\\ \widetilde{v}_n(0)=\widetilde{v}_n(1)=0\end{cases}$$

and

$$-\hat{u}_n''(t)\geqslant\psi\Big(t,\frac{1}{n}+\hat{u}_n\Big)+m(t,\hat{u}_n) \quad \text{for } t\in(0,1)$$

where $\hat{u}_n(t)=\omega_n(t)+\widetilde{v}_n(t)$.

Next we consider the boundary value problem

$$\begin{cases} -u''+a(t)u(t)=f(t), & t\in(0,1),\\ u(0)=0=u(1)\end{cases} \tag{2.5}$$

where $a,f\in M$, $a(t)\geqslant 0$ for $t\in(0,1)$.

Lemma 2.5. *The following statements hold:*

i) *for any $f\in M$, (2.5) is uniquely solvable and*

$$u+A(au)=A(f);$$

ii) *if $f(t)\geqslant 0$ for $t\in(0,1)$, then the solution of (2.5) is nonnegative.*

Corollary 2.2. *Let $\Phi\colon M \to C[0,1]\cap C^1(0,1)$ be the operator such that $\Phi(f)$ is the solution of* (2.5).

i) *If $f_1(t) \leqslant f_2(t)$ for $t \in (0,1)$, then $\Phi(f_1)(t) \leqslant \Phi(f_2)(t)$ for $t \in [0,1]$.*

ii) *Let $E \subset M$ and $\beta \in M$. If $|f(t)| \leqslant \beta(t)$, $t \in (0,1)$ for all $f \in E$, then $\Phi(E)$ is relatively compact with respect to the topology of $C[0,1]$.*

Lemma 2.6. *Let $\underline{\underline{u}}(t) = c_2 l(t)$. Then*

$$\underline{\underline{u}}(t) \leqslant A\Big(\overline{g}\Big(\cdot, \frac{1}{n} + \underline{\underline{u}}\Big) + \delta_n\Big)(t) \quad \text{for } t \in [0,1],\ \ n \geqslant 1,$$

where

$$\delta_n(t) = \overline{g}(t, \underline{\underline{u}}(t)) - \overline{g}\Big(t, \frac{1}{n} + \underline{\underline{u}}(t)\Big).$$

P r o o f. From (1.5) we have

$$\begin{aligned} A(\overline{g}(\cdot, \underline{\underline{u}}))(t) &= \int_0^1 G(t,s)\overline{g}(s, \underline{\underline{u}}(s))\,\mathrm{d}s \\ &= \varphi_1(t)\int_0^1 N(t,s)\overline{g}(s,\underline{\underline{u}}(s))\,\mathrm{d}s \\ &\geqslant \frac{\varphi_1(t)}{2\pi}\int_0^1 s(1-s)\overline{g}(s, c_2 l(s))\,\mathrm{d}s \\ &\geqslant \frac{c_2}{2}\varphi_1(t) \geqslant c_2 l(t) = \underline{\underline{u}}(t) \quad \text{for } t \in [0,1], \end{aligned}$$

so

$$\begin{aligned} &A\Big(\overline{g}\Big(\cdot, \frac{1}{n} + \underline{\underline{u}}\Big) + \delta_n\Big)(t) \\ &\quad = \int_0^1 G(t,s)\Big[\overline{g}\Big(s, \frac{1}{n} + \underline{\underline{u}}(s)\Big) - \overline{g}\Big(s, \frac{1}{n} + \underline{\underline{u}}(s)\Big) + \overline{g}(s, \underline{\underline{u}}(s))\Big]\,\mathrm{d}s \\ &\quad = \int_0^1 G(t,s)\overline{g}(s, \underline{\underline{u}}(s))\,\mathrm{d}s \geqslant \underline{\underline{u}}(t) \quad \text{for } t \in [0,1]. \end{aligned}$$

□

Lemma 2.7. *Let*

$$\overline{c} = \begin{cases} \max\Big\{c_1, \pi \sup\limits_{t\in(0,1)}\Big[2B\Big(\dfrac{1}{(\underline{\underline{u}}(\cdot))^\beta}\Big)(t) + B\Big(\sup\limits_{r\in[c_1, 1+\frac{1}{2}c_2]} g^-(\cdot, r)\Big)(t)\Big]\Big\} \\ \quad \text{if } c_1 < 1 + \dfrac{1}{2}c_2, \\ \max\Big\{c_1, \pi \sup\limits_{t\in(0,1)}\Big[2B\Big(\dfrac{1}{(\underline{\underline{u}}(\cdot))^\beta}\Big)(t)\Big]\Big\} \\ \quad \text{if } c_1 \geqslant 1 + \dfrac{1}{2}c_2 \end{cases}$$

be independent of n*, and*

$$\overline{\overline{u}}(t)=\overline{c}l(t).$$

Then

$$A\Big(\overline{g}\Big(\cdot,\frac{1}{n}+v\Big)+\delta_n\Big)(t)\leqslant\overline{\overline{u}}(t)\quad\textit{for }t\in[0,1],\ \ v\in[\underline{\underline{u}},\overline{\overline{u}}],\ \ n\geqslant 1.$$

P r o o f. Without loss of generality we suppose $c_1<1+\frac{1}{2}c_2$. Note that $\underline{\underline{u}}\leqslant\overline{\overline{u}}$ since $\overline{c}\geqslant c_1>c_2$. Let $v\geqslant\underline{\underline{u}}$. Then (note that $g^-(\cdot,r)=0$ if $0<r<c_1$ by virtue of (1.4))

$$\begin{aligned}
&A\Big(\overline{g}\Big(\cdot,\frac{1}{n}+v\Big)+\delta_n\Big)(t)\\
&\quad=\int_0^1 G(t,s)\Big[\overline{g}\Big(s,\frac{1}{n}+v(s)\Big)-\overline{g}\Big(s,\frac{1}{n}+\underline{\underline{u}}(s)\Big)+\overline{g}(s,\underline{\underline{u}}(s))\Big]\,\mathrm{d}s\\
&\quad\leqslant\int_0^1 G(t,s)\Big[\frac{1}{(\frac{1}{n}+v(s))^\beta}+g^-\Big(s,\frac{1}{n}+\underline{\underline{u}}(s)\Big)+\overline{g}(s,\underline{\underline{u}}(s))\Big]\,\mathrm{d}s\\
&\quad\leqslant\int_0^1 G(t,s)\Big[\frac{1}{(v(s))^\beta}+\sup_{r\in[c_1,1+\frac{1}{2}c_2]}g^-(s,r)+\frac{1}{(\underline{\underline{u}}(s))^\beta}\Big]\,\mathrm{d}s\\
&\quad\leqslant\int_0^1 G(t,s)\Big[2\frac{1}{(\underline{\underline{u}}(s))^\beta}+\sup_{r\in[c_1,1+\frac{1}{2}c_2]}g^-(s,r)\Big]\,\mathrm{d}s\\
&\quad=\varphi_1(t)\Big[2B\Big(\frac{1}{(\underline{\underline{u}}(\cdot))^\beta}\Big)+B\Big(\sup_{r\in[c_1,1+\frac{1}{2}c_2]}g^-(\cdot,r)\Big)\Big](t)\\
&\quad\leqslant\pi l(t)\cdot\Big[2B\Big(\frac{1}{(\underline{\underline{u}}(\cdot))^\beta}\Big)+B\Big(\sup_{r\in[c_1,1+\frac{1}{2}c_2]}g^-(\cdot,r)\Big)\Big](t)\\
&\quad\leqslant\overline{c}l(t)\quad\text{for }t\in[0,1].
\end{aligned}$$

□

Lemma 2.8. *For all* $n>1$ *there exists* $\alpha_n\in C[0,1]$ *such that*

$$\underline{\underline{u}}(t)\leqslant\alpha_n(t)\leqslant\overline{\overline{u}}(t)\tag{2.6}$$

and

$$\begin{cases}-\alpha_n''(t)=\overline{g}\Big(t,\dfrac{1}{n}+\alpha_n(t)\Big)+\delta_n(t)\quad\textit{for }t\in(0,1),\\ \alpha_n(0)=\alpha_n(1)=0.\end{cases}\tag{2.7}$$

P r o o f. Fix $n > 1$. According to Remark 1.1 there exist $\gamma_n \in M$, $\gamma_n \geqslant 0$ such that $\overline{g}(t,r) + \gamma_n(t)r$ is increasing in $(\frac{1}{n}, \frac{1}{n} + \frac{1}{2}\overline{c})$. Let $\overline{\gamma}(t) = \gamma_n$. We have that $\overline{g}\big(t, \frac{1}{n}+r\big)+\overline{\gamma}(t)r$ is increasing in $(0, \frac{1}{2}\overline{c}]$. From Lemma 2.6 and Lemma 2.7 we obtain for fixed $v \in C[0,1]$, $\underline{\underline{u}}(t) \leqslant v(t) \leqslant \overline{\overline{u}}(t)$ that

$$\begin{aligned}\underline{\underline{u}}(t) + A(\overline{\gamma}\underline{\underline{u}})(t) &\leqslant A\Big(\overline{g}\Big(\cdot, \frac{1}{n} + \underline{\underline{u}}\Big) + \delta_n\Big)(t) + A(\overline{\gamma}\underline{\underline{u}})(t) \\ &= A\Big(\overline{g}\Big(\cdot, \frac{1}{n} + \underline{\underline{u}}\Big) + \overline{\gamma}\underline{\underline{u}} + \delta_n\Big)(t) \\ &\leqslant A\Big(\overline{g}\Big(\cdot, \frac{1}{n} + v\Big) + \overline{\gamma}v + \delta_n\Big)(t) \\ &\leqslant A\Big(\overline{g}\Big(\cdot, \frac{1}{n} + \overline{\overline{u}}\Big) + \overline{\gamma}\overline{\overline{u}} + \delta_n\Big)(t) \\ &\leqslant \overline{\overline{u}}(t) + A(\overline{\gamma}\overline{\overline{u}})(t).\end{aligned}$$

Fix $v \in C[0,1]$ with $\underline{\underline{u}}(t) \leqslant v(t) \leqslant \overline{\overline{u}}(t)$. By Lemma 2.5 there exists $\Psi(v) \in C[0,1]$ such that

$$\begin{cases} -\Psi''(v)(t) + \overline{\gamma}(t)\Psi(v)(t) = \overline{g}\Big(t, \dfrac{1}{n} + v\Big) + \overline{\gamma}(t)v(t) + \delta_n(t) \quad \text{for } t \in (0,1), \\ \Psi(v)(0) = \Psi(v)(1) = 0. \end{cases}$$

Then

$$\Psi(v)(t) + A(\overline{\gamma}\Psi(v))(t) = A\Big(\overline{g}\Big(\cdot, \frac{1}{n} + v\Big) + \overline{\gamma}v + \delta_n\Big)(t) \quad \text{for } t \in (0,1),$$

so the above inequality yields

$$\underline{\underline{u}}(t) + A(\overline{\gamma}\underline{\underline{u}})(t) \leqslant \Psi(v)(t) + A(\overline{\gamma}\Psi(v))(t) \leqslant \overline{\overline{u}}(t) + A(\overline{\gamma}\overline{\overline{u}})(t) \quad \text{for } t \in (0,1).$$

From Corollary 2.2 we have

$$\underline{\underline{u}}(t) \leqslant \Psi(v)(t) \leqslant \overline{\overline{u}}(t) \quad \text{for } t \in [0,1].$$

Also,

$$\begin{aligned}\Big|\overline{g}\Big(t, \frac{1}{n} + v\Big) + \overline{\gamma}v + \delta_n\Big| &\leqslant g_1\Big(t, \frac{\varphi_1(t)}{n}\Big) + g_2\Big(t, \frac{1}{n}\Big) + \overline{\gamma}|\overline{\overline{u}}|_\infty + |\delta_n(t)| \\ &\equiv \beta(t) \in M \quad \text{for } t \in (0,1).\end{aligned}$$

Now $\Psi\colon [\underline{\underline{u}}, \overline{\overline{u}}] \to [\underline{\underline{u}}, \overline{\overline{u}}]$ is relatively compact, so Schauder's fixed point theorem implies that there exists $\alpha_n \in C[0,1]$ such that $\underline{\underline{u}}(t) \leqslant \alpha_n(t) \leqslant \overline{\overline{u}}(t)$ and $\Psi(\alpha_n)(t) = \alpha_n(t)$ for $t \in (0,1)$. We conclude that

$$\begin{cases} -\alpha_n''(t) = \overline{g}\Big(t, \dfrac{1}{n} + \alpha_n\Big) + \delta_n(t) \quad \text{for } t \in (0,1), \\ \alpha_n(0) = \alpha_n(1) = 0. \end{cases}$$

$\square$

P r o o f of Theorem 1.1. Without loss of generality we suppose $c_1 < 1 + \frac{1}{2}c_2$. Let

$$\psi(t,r) = g_2(t,r) + \frac{1}{\underline{\underline{u}}(t)^\beta} + \sup_{r\in[c_1,1+\frac{1}{2}c_2]} g^-(t,r)$$

and

$$m(t,r) = h_2(t,r).$$

From (1.3) we see that ψ satisfies the assumptions of Lemma 2.3, so there exist $\omega, \omega_n \in C[0,1]$ such that

$$\begin{cases} -\omega_n''(t) = g_2\Big(t, \dfrac{1}{n} + \omega_n\Big) + \dfrac{1}{\underline{\underline{u}}(t)^\beta} + \sup\limits_{r\in[c_1,1+\frac{1}{2}c_2]} g^-(t,r) \quad \text{for } t\in(0,1), \\ \omega_n(0) = \omega_n(1) = 0 \end{cases}$$

and

$$\omega(t) = \lim_{n\to\infty} \omega_n(t) \quad \text{for } t\in[0,1].$$

From (1.6) it follows that m satisfies the assumption of Lemma 2.4, so by Corollary 2.1 there exist $R_0 > 0$ and $\widetilde{v}_n \in C[0,1]$, $0 \leqslant \widetilde{v}_n(t) \leqslant R_0\varphi_1(t)$ for $t\in[0,1]$ such that

$$\begin{cases} -\widetilde{v}_n''(t) = h_2(t, \omega_n + \widetilde{v}_n) \quad \text{for } t\in(0,1), \\ \widetilde{v}_n(0) = \widetilde{v}_n(1) = 0 \end{cases}$$

and

$$-\hat{u}_n''(t) \geqslant g_2\Big(t, \frac{1}{n} + \hat{u}_n\Big) + \frac{1}{\underline{\underline{u}}(t)^\beta} + \sup_{r\in[c_1,1+\frac{1}{2}c_2]} g^-(t,r) + h_2(t,\hat{u}_n) \quad \text{for } t\in(0,1)$$

where $\hat{u}_n(t) = \omega_n(t) + \widetilde{v}_n(t)$ for $t\in[0,1]$. Then $\hat{u}_n \in C[0,1]$ and $\hat{u}_n(0) = \hat{u}_n(1) = 0$.

Let

$$\hat{u}(t) = \omega(t) + R_0\varphi_1(t) \quad \text{for } t\in[0,1],$$

so

$$0 \leqslant \hat{u}_n(t) \leqslant \hat{u}(t) \quad \text{for } t\in[0,1]. \tag{2.8}$$

Now let us consider the problem

$$\begin{cases} -u''(t) = g\Big(t, \dfrac{1}{n} + u\Big) + h(t,u) + \delta_n(t) \quad \text{for } t\in(0,1), \\ u(0) = u(1) = 0 \end{cases} \tag{2.9}$$

where δ_n is defined in Lemma 2.6.

We will prove that α_n is a lower solution of problem (2.9) and $\hat{u}_n$ is an upper solution of problem (2.9).

Now (1.7), Lemma 2.8 and the positivity of $h(t,s)$ imply that

$$-\alpha_n''(t) = \overline{g}\Big(t, \frac{1}{n} + \alpha_n(t)\Big) + \delta_n(t) \leqslant g\Big(t, \frac{1}{n} + \alpha_n(t)\Big) + h(t, \alpha_n(t)) + \delta_n(t),$$

so α_n is a lower solution of (2.9). On the other hand, from the definition of $\overline{g}$ and $\underline{\underline{u}}$ we have

$$\overline{g}(t, \underline{\underline{u}}) = \min\Big\{g(t, \underline{\underline{u}}), \frac{1}{\underline{\underline{u}}(t)^\beta}\Big\} \leqslant \frac{1}{\underline{\underline{u}}(t)^\beta} \quad \text{for } t \in (0,1)$$

and

$$\begin{aligned} -\overline{g}\Big(t, \frac{1}{n} + \underline{\underline{u}}\Big) &= -\min\Big\{g^+\Big(t, \frac{1}{n} + \underline{\underline{u}}\Big), \frac{1}{(\frac{1}{n} + \underline{\underline{u}})^\beta}\Big\} + g^-\Big(t, \frac{1}{n} + \underline{\underline{u}}\Big) \\ &\leqslant g^-\Big(t, \frac{1}{n} + \underline{\underline{u}}\Big) \leqslant \sup_{r \in [c_1, 1 + \frac{1}{2}c_2]} g^-(t, r) \quad \text{for } t \in (0,1), \end{aligned}$$

so

$$\delta_n(t) \leqslant \frac{1}{\underline{\underline{u}}(t)^\beta} + \sup_{r \in [c_1, 1 + \frac{1}{2}c_2]} g^-(t, r) \quad \text{for } t \in (0,1). \tag{2.10}$$

Consequently, we have

$$\begin{aligned} -\hat{u}_n''(t) &\geqslant g_2\Big(t, \frac{1}{n} + \hat{u}_n\Big) + \frac{1}{\underline{\underline{u}}(t)^\beta} + \sup_{r \in [c_1, 1 + \frac{1}{2}c_2]} g^-(t, r) + h_2(t, \hat{u}_n) \\ &\geqslant g\Big(t, \frac{1}{n} + \hat{u}_n\Big) + \frac{1}{\underline{\underline{u}}(t)^\beta} + \sup_{r \in [c_1, 1 + \frac{1}{2}c_2]} g^-(t, r) + h(t, \hat{u}_n) \\ &\geqslant g\Big(t, \frac{1}{n} + \hat{u}_n\Big) + h(t, \hat{u}_n) + \delta_n(t), \end{aligned}$$

so $\hat{u}_n$ is an upper solution of (2.9). Next we prove

$$\alpha_n(t) \leqslant \hat{u}_n(t) \quad \text{for } t \in [0,1]. \tag{2.11}$$

Suppose (2.11) is not true. Let $y(t) = \alpha_n(t) - \hat{u}_n(t)$ and let $\sigma \in (0,1)$ be the point where $y(t)$ attains its maximum over $(0,1)$. We have

$$y(\sigma) > 0 \quad \text{and} \quad y''(\sigma) \leqslant 0.$$

On the other hand, since $\alpha_n(\sigma) > \hat{u}_n(\sigma)$ we have

$$\begin{aligned}
-\alpha_n''(\sigma) &= \overline{g}\Big(\sigma, \frac{1}{n} + \alpha_n(\sigma)\Big) + \delta_n(\sigma)\\
&\leqslant g\Big(\sigma, \frac{1}{n} + \alpha_n(\sigma)\Big) + \delta_n(\sigma)\\
&\leqslant g\Big(\sigma, \frac{1}{n} + \alpha_n(\sigma)\Big) + \frac{1}{\underline{\underline{u}}(\sigma)^\beta} + \sup_{r\in[c_1, 1+\frac{1}{2}c_2]} g^-(\sigma, r)\\
&\leqslant g_2(\sigma, \frac{1}{n} + \alpha_n(\sigma)) + \frac{1}{\underline{\underline{u}}(\sigma)^\beta} + \sup_{r\in[c_1, 1+\frac{1}{2}c_2]} g^-(\sigma, r)\\
&< g_2\Big(\sigma, \frac{1}{n} + \hat{u}_n(\sigma)\Big) + \frac{1}{\underline{\underline{u}}(\sigma)^\beta} + \sup_{r\in[c_1, 1+\frac{1}{2}c_2]} g^-(\sigma, r) + h_2(\sigma, \hat{u}_n(\sigma))\\
&\leqslant -\hat{u}_n''(\sigma),
\end{aligned}$$

so

$$y''(\sigma) = \alpha_n''(\sigma) - \hat{u}_n''(\sigma) > 0,$$

and this is a contradiction.

From (G2) we see that there exists $\gamma \in M$ such that $r \to g\big(t, \frac{1}{n} + r\big) + \gamma(t)r$ is increasing in $(0, |\hat{u}|_\infty)$. Let $\overline{u}(t) \equiv \underline{\underline{u}}(t)$, $\overline{u}_n(t) = \alpha_n(t)$. From (2.6), (2.8) and (2.11) we have

$$0 < \overline{u}(t) \leqslant \overline{u}_n(t) \leqslant \hat{u}_n(t) \leqslant \hat{u}(t) \quad \text{for } t \in (0,1).$$

Further, for $v \in C[0,1]$ with $\overline{u}_n(t) \leqslant v(t) \leqslant \hat{u}_n(t)$, $t \in [0,1]$, we have

$$\begin{aligned}
-\overline{u}_n''(t) + \gamma(t)\overline{u}_n(t) &\leqslant g\Big(t, \frac{1}{n} + \overline{u}_n\Big) + \gamma(t)\overline{u}_n + \delta_n(t)\\
&\leqslant g\Big(t, \frac{1}{n} + v\Big) + \gamma(t)v + \delta_n(t) + h(t, v)\\
&\leqslant g\Big(t, \frac{1}{n} + \hat{u}_n\Big) + \gamma(t)\hat{u}_n + \delta_n(t) + h(t, \hat{u}_n)\\
&\leqslant -\hat{u}_n''(t) + \gamma(t)\hat{u}_n(t).
\end{aligned}$$

On the other hand, by (2.10),

$$|\delta_n(t)| \leqslant \frac{1}{\overline{u}(t)^\beta} + \sup_{r\in[c_1, 1+\frac{1}{2}c_2]} g^-(t, r) \equiv \delta(t)$$

and

$$\lim_{n\to\infty} \delta_n(t) = 0 \quad \text{for } t \in (0,1).$$

Now Lemma 2.2 guarantees that there exists a solution $u \in C[0,1]$ to (1.2) with

$$\overline{u}(t) \leqslant u(t) \leqslant \hat{u}(t) \quad \text{for } t \in [0,1].$$

Moreover, because $\hat{u}(t) \leqslant |\omega|_\infty + R_0\varphi_1(t) \leqslant (|\omega|_\infty + R_0)(1+\varphi_1(t)) = c_4(1+\varphi_1(t))$ (here $c_4 = |\omega|_\infty + R_0$) and $c_3\varphi_1(t) \leqslant \overline{u}(t)$ (here $c_3 = \frac{1}{2}c_2/\pi$, see Lemma 2.6), the estimates asserted in the theorem follow. □

P r o o f of Theorem 1.2. From assumption (1.8) it follows that there are $c_7 > 0$, $\lambda \geqslant \lambda_1$ such that, if $n > 2/c_7$, $0 < k < \frac{1}{2}c_7/|\varphi_1|_\infty$, then

$$0 < k|\varphi_1|_\infty < \frac{c_7}{2}, \quad 0 < \frac{1}{n} + k\varphi_1(t) < c_7$$

and

$$\frac{\lambda(\frac{1}{n} + k\varphi_1(t)) + g^-(t, \frac{1}{n} + k\varphi_1(t))}{h(t, \frac{1}{n} + k\varphi_1(t))} \leqslant 1.$$

Thus

$$\frac{\lambda k\varphi_1(t) + g^-(t, \frac{1}{n} + k\varphi_1(t))}{h(t, \frac{1}{n} + k\varphi_1(t))} \leqslant 1.$$

Then for fixed $n > \frac{2}{c_7}$,

$$\lambda k\varphi_1(t) + g^-\Big(t, \frac{1}{n} + k\varphi_1(t)\Big) \leqslant h\Big(t, \frac{1}{n} + k\varphi_1(t)\Big),$$

and we have

$$\begin{aligned}
\lambda k\varphi_1(t) &\leqslant h\Big(t, \frac{1}{n} + k\varphi_1(t)\Big) - g^-\Big(t, \frac{1}{n} + k\varphi_1(t)\Big) \\
&\leqslant g^+\Big(t, \frac{1}{n} + k\varphi_1(t)\Big) - g^-\Big(t, \frac{1}{n} + k\varphi_1(t)\Big) + h\Big(t, \frac{1}{n} + k\varphi_1(t)\Big) \\
&= g\Big(t, \frac{1}{n} + k\varphi_1(t)\Big) + h\Big(t, \frac{1}{n} + k\varphi_1(t)\Big) - h(t, k\varphi_1(t)) + h(t, k\varphi_1(t)) \\
&= g\Big(t, \frac{1}{n} + k\varphi_1(t)\Big) + h\Big(t, k\varphi_1(t)) + \delta_n(t)
\end{aligned}$$

where

$$\delta_n(t) = h\Big(t, \frac{1}{n} + k\varphi_1(t)\Big) - h(t, k\varphi_1(t)). \tag{2.12}$$

Let $\overline{u}(t) = k\varphi_1(t)$, then we have

$$-\overline{u}''(t) = \lambda_1 k\varphi_1(t) \leqslant \lambda k\varphi_1(t) \leqslant g\Big(t, \frac{1}{n} + \overline{u}(t)\Big) + h(t, \overline{u}(t)) + \delta_n(t) \quad \text{for } t \in (0,1).$$

Let

$$\psi(t,s) = g_2(t,s) + h_2\Big(t, \frac{1}{2}c_7 + k\varphi_1(t)\Big) + \frac{1}{2}\frac{c_7\varphi_1(t)}{|\varphi_1|}$$

and

$$m(t,s)=h_2(t,s).$$

From (1.3) we see that ψ satisfies the assumptions of Lemma 2.3, so there exist $\omega,\omega_n\in C[0,1]$ such that

$$\begin{cases} -\omega_n''(t)=g_2\Big(t,\dfrac{1}{n}+\omega_n\Big)+h_2\Big(t,\dfrac{1}{2}c_7+k\varphi_1(t)\Big)+\dfrac{1}{2}\dfrac{c_7\varphi_1(t)}{|\varphi_1|} & \text{for } t\in(0,1),\\ \omega_n(0)=\omega_n(1)=0 \end{cases}$$

and

$$\omega(t)=\lim_{n\to\infty}\omega_n(t)\quad\text{for } t\in[0,1].$$

It follows from (1.6) that m satisfies the assumption of Lemma 2.4, so by Corollary 2.1 there exist $R_0>0$ and $\widetilde{v}_n\in C([0,1])$, $0\leqslant\widetilde{v}_n(t)\leqslant R_0\varphi_1(t)$ for $t\in[0,1]$, such that

$$\begin{cases} -\widetilde{v}_n''(t)=h_2(t,\omega_n+\widetilde{v}_n) & \text{for } t\in(0,1),\\ \widetilde{v}_n(0)=\widetilde{v}_n(1)=0 \end{cases}$$

and

$$-\hat{u}_n''(t)\geqslant g_2\Big(t,\frac{1}{n}+\hat{u}_n(t)\Big)+h_2\Big(t,\frac{1}{2}c_7+k\varphi_1(t)\Big)+\frac{1}{2}\frac{c_7\varphi_1(t)}{|\varphi_1|}+h_2(t,\hat{u}_n)\quad\text{for } t\in(0,1)$$

where $\hat{u}_n(t)=\omega_n(t)+\widetilde{v}_n(t)$ for $t\in[0,1]$. Then $\hat{u}_n\in C[0,1]$ and $\hat{u}_n(1)=\hat{u}_n(1)=0$.

Let

$$\hat{u}(t)=\omega(t)+R_0\varphi_1(t)\quad\text{for } t\in[0,1],$$

so

$$0\leqslant\hat{u}_n(t)\leqslant\hat{u}(t)\quad\text{for } t\in[0,1].$$

Next we prove that

$$\overline{u}(t)\leqslant\hat{u}_n(t)\quad\text{for } t\in[0,1]. \tag{2.13}$$

Suppose (2.13) is not true. Let $y(t)=\overline{u}(t)-\hat{u}_n(t)$ and let $\sigma\in(0,1)$ be the point where $y(t)$ attains its maximum over $(0,1)$. We have

$$y(\sigma)>0\quad\text{and}\quad y''(\sigma)\leqslant 0.$$

On the other hand, since $\overline{u}(\sigma) > \hat{u}_n(\sigma)$ we have

$$\begin{aligned}
-\overline{u}''(\sigma) &\leqslant g\Big(\sigma, \frac{1}{n} + \overline{u}(\sigma)\Big) + h(\sigma, \overline{u}(\sigma)) + \delta_n(\sigma) \\
&\leqslant g_2\Big(\sigma, \frac{1}{n} + \overline{u}(\sigma)\Big) + h(\sigma, \overline{u}(\sigma)) + \delta_n(\sigma) \\
&= g_2\Big(\sigma, \frac{1}{n} + \overline{u}(\sigma)\Big) + h\Big(\sigma, \frac{1}{n} + \overline{u}(\sigma)\Big) \\
&< g_2\Big(\sigma, \frac{1}{n} + \hat{u}_n(\sigma)\Big) + h_2\Big(\sigma, \frac{1}{2}c_7 + k\varphi_1(\sigma)\Big) \\
&\quad + \frac{1}{2}\frac{c_7\varphi_1(\sigma)}{|\varphi_1|} + h_2(\sigma, \hat{u}_n(\sigma)) \leqslant -\hat{u}_n''(\sigma),
\end{aligned}$$

so we have

$$y''(\sigma) = \overline{u}_n''(\sigma) - \hat{u}_n''(\sigma) > 0,$$

and this is a contradiction.

We have

$$|\delta_n(t)| = \Big|h\Big(t, \frac{1}{n} + k\varphi_1(t)\Big) - h(t, k\varphi_1(t))\Big| \leqslant 2h_2\Big(t, \frac{1}{2}c_7 + k|\varphi_1|\Big)$$

for $n > 2/c_7$. Consequently $\delta_n \to 0$ for $t \in (0,1)$ and $n \to \infty$.

By virtue of assumptions (G2) and (H3) there exist $\gamma, \tau \in M$, $n > 2/c_7$ such that $g(t, \frac{1}{n} + r) + h(t, r) + a(t)r$ is increasing in $(0, |\hat{u}|_\infty)$, where $a(t) = \gamma(t) + \tau(t)$. Taking $\overline{u}_n = \overline{u}(t)$ and observing that

$$g\Big(\cdot, \frac{1}{n} + v\Big) + h(\cdot, v) + a_n v + \delta_n \in M, \quad v \in [\overline{u}_n, \hat{u}_n]$$

we obtain

$$\begin{aligned}
\overline{u} + A(a\overline{u}) &\leqslant A\Big(g\Big(\cdot, \frac{1}{n} + \overline{u}\Big) + h(\cdot, \overline{u}) + \delta_n + a\overline{u}\Big) \\
&\leqslant A\Big(g\Big(\cdot, \frac{1}{n} + v\Big) + h(\cdot, v) + a_n v + \delta_n\Big) \\
&\leqslant A\Big(g\Big(\cdot, \frac{1}{n} + \hat{u}_n\Big) + h(\cdot, \hat{u}_n) + a_n\hat{u}_n + \delta_n\Big) \\
&\leqslant A\Big(g_2\Big(\cdot, \frac{1}{n} + \hat{u}_n\Big) + h_2(\cdot, \hat{u}_n) + a_n\hat{u}_n + h_2\Big(\cdot, \frac{1}{2}c_7 + k\varphi_1\Big) + \frac{1}{2}\frac{c_7\varphi_1}{|\varphi_1|}\Big) \\
&\leqslant \hat{u}_n + A(a_n\hat{u}_n).
\end{aligned}$$

Lemma 2.2 implies that problem (1.2) has a solution $u \in C[0,1] \cap C^1(0,1)$. Reasoning similar to that in the proof of Theorem 1.1 implies that there exist $c_i = c_i(g,h) > 0$ $(i = 5, 6)$ such that

$$c_5\varphi_1(t) \leqslant u(t) \leqslant c_6(\varphi_1(t) + 1) \quad \text{for } t \in [0,1].$$

□

3. Example

Example 3.1. Consider the boundary value problem

$$\begin{cases} -u'' = g(t,u) + h(t,u), & t \in (0,1), \\ u(0) = 0 = u(1) \end{cases} \tag{3.1}$$

where

$$g(t,r) = \begin{cases} \dfrac{1}{r^\alpha}\Big|\sin\dfrac{1}{r}\Big|, & 0 < r \leqslant \dfrac{1}{\pi}, \\ -\dfrac{1}{r^\alpha}\sin\dfrac{1}{r}, & \dfrac{1}{\pi} < r \end{cases}$$

and

$$h(t,r) = \begin{cases} r^2 & \text{for } 0 \leqslant r < 1, \\ r^\tau & \text{for } r \geqslant 1 \end{cases}$$

with $\alpha > 0$, $0 < \tau < 1$. Then Theorem 1.1 guarantees that (3.1) has at least a solution $u \in C[0,1]$ with $u(t) > 0$ for $t \in (0,1)$.

To see this let $\beta = \min\{\frac{1}{2}, \frac{1}{2}\alpha\}$, $g_1(t,r) = 1/r^\beta + \pi^\alpha$ and $g_2(t,r) = 1/r^\alpha$. Notice that (1.3) is satisfied. For all $r_2 > r_1 > 0$, let

$$\gamma(t) \equiv \sup_{r\in\Lambda}\Big|\frac{\partial g}{\partial r}\Big| + 1 < \infty$$

where $\Lambda = (r_1, r_2) \setminus \{n\pi\colon\ n \in \mathbb{N}\}$, so $g(t,r) + \gamma(t)r$ is increasing in (r_1, r_2).

Let n_0 be fixed such that

$$2^{1/(\alpha-\beta)} < n_0\pi + \frac{1}{6}\pi.$$

Let $c_1 = 1/\pi$ and $c_2 \in (0, c_1)$ be such that

$$c_2^3 < \frac{1}{2(2-\beta)\pi^{1-\beta}} \sum_{n=n_0}^{\infty} \Big[\Big(\frac{6}{6n+1}\Big)^{2-\beta} - \Big(\frac{6}{6n+5}\Big)^{2-\beta}\Big].$$

Now for $n \geqslant n_0$,

$$\frac{1}{(c_2 t)^\alpha}\Big|\sin\frac{1}{c_2 t}\Big| \geqslant \frac{1}{(c_2 t)^\beta} \quad \text{for } t \in \Big[\frac{1}{c_2(n\pi + \frac{5}{6}\pi)}, \frac{1}{c_2(n\pi + \frac{1}{6}\pi)}\Big],$$

so we have

$$\begin{aligned}\int_0^1 t(1-t)\overline{g}(t,c_2 l(t))\,\mathrm{d}t &\geqslant \int_0^{1/2} t(1-t)\overline{g}(t,c_2 l(t))\,\mathrm{d}t\\ &\geqslant \frac{1}{2}\int_0^{1/2} t\overline{g}(t,c_2 l(t))\,\mathrm{d}t\\ &\geqslant \frac{1}{2}\sum_{n=n_0}^{\infty}\int_{1/(c_2(n\pi+\frac{5}{6}\pi))}^{1/(c_2(n\pi+\frac{1}{6}\pi))} t\frac{1}{(c_2 t)^{\beta}}\,\mathrm{d}t\\ &= \frac{1}{2}\frac{1}{c_2^{\beta}}\sum_{n=n_0}^{\infty}\int_{1/(c_2(n\pi+\frac{5}{6}\pi))}^{1/(c_2(n\pi+\frac{1}{6}\pi))} t^{1-\beta}\,\mathrm{d}t\\ &= \frac{1}{2}\frac{1}{c_2^2(2-\beta)\pi^{2-\beta}}\sum_{n=n_0}^{\infty}\Big[\Big(\frac{6}{6n+1}\Big)^{2-\beta}-\Big(\frac{6}{6n+5}\Big)^{2-\beta}\Big]\\ &\geqslant c_2\pi.\end{aligned}$$

Thus (1.5) is satisfied.

Let $h_1(t,r)=h_2(t,r)=h(t,r)$ for $(t,r)\in(0,1)\times(0,\infty)$ and note that (H1) and (H2) are satisfied.

Example 3.2. Consider the boundary value problem

$$\begin{cases}-u''=g(t,u)+h(t,u), & t\in(0,1),\\ u(0)=0=u(1)\end{cases}\tag{3.2}$$

where

$$g(t,r)=\begin{cases}-\dfrac{1}{r^{\alpha}} & \text{for } r\in[1,\infty),\\[2mm] -\dfrac{4}{3}r+\dfrac{1}{3} & \text{for } r\in\Big[\dfrac{1}{4},1\Big],\\[2mm] (4k+1)^{\alpha+1}(-4kr+1) & \text{for } r\in\Big[\dfrac{1}{4k+1},\dfrac{1}{4k}\Big],\\[2mm] (4k+1)^{\alpha+1}[(4k+2)r-1] & \text{for } r\in\Big[\dfrac{1}{4k+2},\dfrac{1}{4k+1}\Big],\\[2mm] (4k+2)r-1 & \text{for } r\in\Big[\dfrac{1}{4k+3},\dfrac{1}{4k+2}\Big],\\[2mm] 1-4(k+1)r & \text{for } r\in\Big[\dfrac{1}{4k+4},\dfrac{1}{4k+3}\Big]\end{cases}$$

for $k=1,2,\ldots$, $\alpha>0$ and

$$h(t,r)=\frac{1}{8}\sqrt{r}.$$

Then

$$g^-(t,r) = \begin{cases} \dfrac{1}{r^\alpha} & \text{for } r \in [1,\infty), \\ \dfrac{4}{3}r - \dfrac{1}{3} & \text{for } r \in \Big[\dfrac{1}{4}, 1\Big], \\ 0 & \text{for } r \in \Big[\dfrac{1}{4k+2}, \dfrac{1}{4k}\Big], \\ 1-(4k+2)r & \text{for } r \in \Big[\dfrac{1}{4k+3}, \dfrac{1}{4k+2}\Big], \\ 4(k+1)r - 1 & \text{for } r \in \Big[\dfrac{1}{4k+4}, \dfrac{1}{4k+3}\Big]. \end{cases}$$

Let $\lambda \equiv \lambda_1$. Then we have

$$\lim_{r\to 0^+} \frac{r\lambda + g^-(t,r)}{h(t,r)} = 0.$$

By Theorem 1.2, problem (3.2) has a solution $u \in C[0,1] \cap C^1(0,1)$.

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