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FLENSTED-JENSEN'S FUNCTIONS ATTACHED TO THE LANDAU PROBLEM ON THE HYPERBOLIC DISC

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Abstract. We give an explicit expression of a two-parameter family of Flensted-Jensen's functions $\Psi_{\mu,\alpha}$ on a concrete realization of the universal covering group of $U(1,1)$. We prove that these functions are, up to a phase factor, radial eigenfunctions of the Landau Hamiltonian on the hyperbolic disc with a magnetic field strength proportional to μ, and corresponding to the eigenvalue $4\alpha(\alpha-1)$.

Keywords: Flensted-Jensen's functions, universal covering group, Landau Hamiltonian, hyperbolic disc

MSC 2000: 33C05, 35J10, 35Q40, 43A90, 57M10, 58C40

1. Introduction

In quantum mechanics, considerable attention has been paid to the physics of a charged particle evolving in a plane under the influence of a perpendicular uniform magnetic field. This problem, called the planar Landau problem [1], has been generalized to two-dimensional curved surfaces with a normal stationary magnetic field [2].

For a charged particle evolving in the hyperbolic plane under the influence of a uniform magnetic field, eigenfunctions and eigenvalues of the corresponding Landau Hamiltonian have been discussed in the context of the spectral theory [3]. Eigenstates can also be obtained as representation coefficients of the Lie group describing the symmetry of the quantum system [4]. Spherical functions are special coefficients of group representations, which are usually indexed by a spectral parameter.

Here our main aim is to attach to the Landau problem on the hyperbolic disc a set of Flensted-Jensen's (FJ's) spherical functions [5] defined on the universal covering group of $U(1,1)$. We prove that these FJ's functions are eigenstates of the particle.

The advantage of these functions is that they are indexed by two parameters, one of them being proportional to the magnetic field strength while the other occurs in the parametrization of the eigenvalue of the Hamiltonian.

The paper is organized as follows. In Section 2 we discuss the hyperbolic Landau Hamiltonian. In Section 3, we give the general form of eigenfunctions of the Hamiltonian. Section 4 deals with a realization of the universal covering group of $U(1,1)$. An Iwasawa decomposition corresponding to this realization is given in Section 5. In Section 6, we give an explicit expression of a two-parameter family of FJ's spherical functions and we show that it constitutes a family of eigenstates of the particle. Section 7 is devoted to some concluding remarks.

2. The Landau Hamiltonian on the unit disc

Let $\mathbf{H}_a^2 := \{\xi \in \mathbb{C}, \Im\xi > 0\}$ be the upper halfplane endowed with the metric $\mathrm{d}s^2 = a^2y^{-2}(\mathrm{d}x^2 + \mathrm{d}y^2)$, where $x = \Re\xi$, $y = \Im\xi$, and $a > 0$ is the parameter related to the curvature κ by $\kappa = -2/a^2$. The area element $\mathrm{d}\mu_a(\xi)$ has the form $\mathrm{d}\mu_a(\xi) = a^2y^{-2}\,\mathrm{d}x \wedge \mathrm{d}y$. A constant uniform magnetic field on $\mathbf{H}_a^2$ is given by a 2-form $\mathbf{B}$ defined as

$$\mathbf{B} = \frac{Ba^2}{y^2}\,\mathrm{d}x \wedge \mathrm{d}y,$$

where $B > 0$ is the field intensity. The form $\mathbf{B}$ can be represented as $\mathbf{B} = \mathrm{d}\mathbf{A}$, where $\mathbf{A} = Ba^2y^{-1}\,\mathrm{d}x$ is the vector potential (Landau gauge) we have chosen.

The Schrödinger operator describing a particle of charge e and mass m_* which lives on $\mathbf{H}_a^2$ and interacts with the magnetic field $\mathbf{B}$ is given by

$$H_b := -\frac{\hbar^2}{2m_*a^2}\big(y^2(\partial_x^2 + \partial_y^2) - 2\mathrm{i}by\partial_x - b^2\big) \tag{2.1}$$

where b is the dimensionless quantity $b := eBa^2/\hbar c$. For the sake of simplicity, we put $a = e = c = \hbar = 2m_* = 1$. We denote $\mathbf{H}^2 := \mathbf{H}_1^2$ and will consider a slight modification of H_b in (2.1). Actually, we will deal with the operator ([6], p. 8073):

$$H_B := y^2(\partial_x^2 + \partial_y^2) - 2\mathrm{i}By\partial_x \tag{2.2}$$

with $C_0^\infty(\mathbf{H}^2)$ as its regular domain in the Hilbert space $\mathcal{H} := L^2(\mathbf{H}^2, y^{-2}\,\mathrm{d}x\,\mathrm{d}y)$. The spectrum of H_B in $\mathcal{H}$ consists of two parts: (i) an absolutely continuous spectrum in the interval $(-\infty, 0]$, (ii) a point spectrum consisting of a finite number of infinitely degenerate eigenvalues of the form ([7], p. 11)

$$E_m^B := (B-m)(B-m-1), \quad 0 \leqslant m < B - \frac{1}{2}$$

when $2B > 1$.

The Hamiltonian H_B can be transferred to the unit disc $\mathbb{D}=\{z\in\mathbb{C}\colon\ |z|<1\}$ by means of the Cayley transform $\mathcal{C}\colon\ \mathbf{H}^2\to\mathbb{D}$ defined by $w\longmapsto\mathcal{C}(w)=(w-\mathrm{i})(w+\mathrm{i})^{-1}$ as

$$\Delta_B f(z):=4\Big(\frac{\overline{w}-\mathrm{i}}{w+\mathrm{i}}\Big)^{-B}H_B\Big(\frac{\overline{w}-\mathrm{i}}{w+\mathrm{i}}\Big)^{B}f(\mathcal{C}(w))$$

where $f\in C^2(\mathbb{D})$ and $z=\mathcal{C}(w)\in\mathbb{D}$, $w\in\mathbf{H}^2$. Explicitly,

$$\Delta_B=4(1-|z|^2)\Big((1-|z|^2)\frac{\partial^2}{\partial z\partial\overline{z}}+Bz\frac{\partial}{\partial z}-B\overline{z}\frac{\partial}{\partial\overline{z}}+B^2\Big). \tag{2.3}$$

This describes the motion of a charged particle moving in the hyperbolic disc under the influence of a uniform magnetic field of strength proportional to B.

3. Eigenfunction of Δ_B

Now, let $\alpha\in\mathbb{C}$ be a fixed complex number and let $\psi\colon\ \mathbb{D}\to\mathbb{C}$ be an eigenfunction of the operator Δ_B given in (2.3) corresponding to the eigenvalue $E(\alpha):=4\alpha(\alpha-1)$, i.e.,

$$\Delta_B\psi=E(\alpha)\psi \tag{3.1}$$

since the differential operator Δ_B is elliptic on $\mathbb{D}$, the function ψ is C^∞ on $\mathbb{D}$ and can be expanded into its Fourier series as

$$\psi(\varrho\mathrm{e}^{\mathrm{i}\theta})=\sum_{k\in\mathbb{Z}}\gamma_k(\varrho)\mathrm{e}^{\mathrm{i}k\theta}, \tag{3.2}$$

$0\leqslant\varrho<1$, $0\leqslant\theta\leqslant 2\pi$, where $\varrho\to\gamma_k(\varrho)$ is C^∞ on $[0,1[$ for each $k\in\mathbb{Z}$. Writing Δ_B in polar coordinates (ϱ,θ)

$$\Delta_B=(1-\varrho^2)\frac{\partial^2}{\partial\varrho^2}+(1-\varrho^2)^2\frac{1}{\varrho}\frac{\partial}{\partial\varrho}+(1-\varrho^2)^2\frac{1}{\varrho^2}\frac{\partial^2}{\partial\theta^2}+4\mathrm{i}B(1-\varrho^2)\frac{\partial}{\partial\theta}+4B^2(1-\varrho^2)$$

and inserting the expansion (3.2) of $\psi(\varrho\mathrm{e}^{\mathrm{i}\theta})$ into Eq. (3.1), we obtain for every $k\in\mathbb{Z}$ the differential equation

$$\begin{gathered}\varrho^2(1-\varrho^2)^2\gamma_k''(\varrho)+(1-\varrho^2)^2\varrho\gamma_k'(\varrho)\\ +\big[4\alpha(1-\alpha)\varrho^2+4B^2\varrho^2(1-\varrho^2)-k^2(1-\varrho^2)^2-4kB\varrho^2(1-\varrho^2)\big]\gamma_k(\varrho)=0.\end{gathered} \tag{3.3}$$

Changing the function and reducing Eq. (3.3) to a standard hypergeometric equation, we find after some calculation that $\gamma_k(\varrho)$ is given, up to a multiplicative constant, by

$${}_2F_1\Big(\alpha+B+\frac{1}{2}(|k|+k),\alpha-B+\frac{1}{2}(|k|-k),1+|k|,\varrho^2\Big)(1-\varrho^2)^\alpha\varrho^{|k|}$$

where ${}_2F_1$ is the Gauss hypergeometric function ([8]). Consequently, there exists a family of complex numbers $(c_{B,\alpha,k})_{k\in\mathbb{Z}}$ such that

$$(3.4)\quad \psi(\varrho e^{i\theta}) = (1-\varrho^2)^{\alpha} \sum_{k\in\mathbb{Z}} c_{B,\alpha,k}\, {}_2F_1\Big(\alpha + B + \frac{|k|+k}{2}, \alpha - B + \frac{|k|-k}{2}, 1+|k|, \varrho^2\Big)\varrho^{|k|}e^{ik\theta}.$$

4. The universal covering group of $U(1,1)$

For $\xi, \zeta \in \mathbb{C}^2$, $\xi = (\xi_1, \xi_2)$ and $\zeta = (\zeta_1, \zeta_2)$ let $\langle \xi, \zeta\rangle = \xi_1\overline{\zeta_1} - \xi_2\overline{\zeta_2}$. The group $U(1,1)$ is defined by

$$U(1,1) := \{g \in GL(2,\mathbb{C}),\ \langle g\xi, g\zeta\rangle = \langle \xi, \zeta\rangle,\ \text{for all } \xi,\zeta \in \mathbb{C}^2\}.$$

The elements of this group are matrices

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in GL(2,\mathbb{C}), \quad |a|^2 - |c|^2 = 1, \quad |d|^2 - |b|^2 = 1 \quad \text{and} \quad \bar{a}b = \bar{c}d.$$

Note that every element g of $U(1,1)$ decomposes to

$$g = \begin{pmatrix} a & b \\ c & d\end{pmatrix} = |d| \begin{pmatrix} 1 & bd^{-1} \\ \bar{b}(\bar{d})^{-1} & 1\end{pmatrix} \begin{pmatrix} a|a|^{-1} & 0 \\ 0 & d|d|^{-1}\end{pmatrix}.$$

From this decomposition we see that the Lie group $U(1,1)$ is diffeomorphic to the manifold $\mathbb{T}\times\mathbb{T}\times\mathbb{D}$, where $\mathbb{T} = [0, 2\pi]$. Indeed, setting $bd^{-1} = z$, $d|d|^{-1} = e^{i\theta}$ and $a|a|^{-1} = e^{i(\theta+\varphi)}$, we can write the elements of $U(1,1)$ as

$$g = \frac{e^{i\theta}}{\sqrt{1-|z|^2}} \begin{pmatrix} 1 & z \\ \bar{z} & 1\end{pmatrix} \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & 1\end{pmatrix}$$

with $z \in \mathbb{D}$, $\varphi \in \mathbb{R}$ and $\theta \in \mathbb{R}$. As a realization of the universal cover group of $U(1,1)$ we consider the 4-dimensional real analytic manifold $\widetilde{U(1,1)} := \mathbb{R}\times\mathbb{R}\times\mathbb{D}$ endowed with the group law

$$(\varphi_1, \theta_1, z_1)\cdot(\varphi_2, \theta_2, z_2) = \Big(\varphi_1 + \varphi_2 - 2\operatorname{Arg}(1+\overline{z_1}z_2e^{i\varphi_1}), \theta_1 + \theta_2 + \operatorname{Arg}(1+\overline{z_1}z_2e^{i\varphi_1}), \frac{z_1 + z_2e^{i\varphi_1}}{1+\overline{z_1}z_2e^{i\varphi_1}}\Big).$$

One can verify that this product is well defined, associative and $(0,0,0)$ is the identity. The inverse of an element (φ,θ,z) is $(-\varphi,-\theta,-z\mathrm{e}^{-\mathrm{i}\varphi})$. Furthermore, the projection map P from $\widetilde{U(1,1)}$ onto $U(1,1)$ defined by

$$(\varphi,\theta,z)\to P((\varphi,\theta,z))=\frac{\mathrm{e}^{\mathrm{i}\theta}}{\sqrt{1-|z|^2}}\begin{pmatrix}1 & z\\ \overline{z} & 1\end{pmatrix}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\varphi} & 0\\ 0 & 1\end{pmatrix}$$

is a morphism of groups whose kernel coincides with the fundamental group of $U(1,1)$. Indeed, we have

$$P^{-1}\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)=2\pi\mathbb{Z}\times 2\pi\mathbb{Z}\times\{0\}.$$

R e m a r k 3.1. An explicit realization of the universal cover group of the subgroup $SU(1,1)=\{g\in U(1,1),\ \det g=1\}$ can be found in [9, p. 260–261).

5. An Iwasawa decomposition of $\widetilde{U(1,1)}$

Now, to give an Iwasawa decomposition we need first to define some subgroups of $U(1,1)$. Let

$$K=\left\{\begin{pmatrix}\mathrm{e}^{\mathrm{i}\varphi} & 0\\ 0 & 1\end{pmatrix},\ \varphi\in\mathbb{R}\right\},\quad L=\left\{\begin{pmatrix}\mathrm{e}^{\mathrm{i}\theta} & 0\\ 0 & \mathrm{e}^{\mathrm{i}\theta}\end{pmatrix},\ \theta\in\mathbb{R}\right\},$$
$$A=\left\{\begin{pmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{pmatrix},\ t\in\mathbb{R}\right\},\quad N=\left\{\begin{pmatrix}1+\mathrm{i}s & -\mathrm{i}s\\ \mathrm{i}s & 1-\mathrm{i}s\end{pmatrix},\ s\in\mathbb{R}\right\}.$$

Then the product map $K\times L\times A\times N\to U(1,1)$ is a diffeomorphism. We lift these subgroups to the universal cover group $\widetilde{U(1,1)}$ and denote by $\widetilde{K}$, $\widetilde{L}$, $\widetilde{A}$ and $\widetilde{N}$ the connected components of the identity $(0,0,0)$ in their lifting. Precisely, we have

$$P^{-1}(K)=\mathbb{R}\times 2\pi\mathbb{Z}\times\{0\},\quad \widetilde{K}=\{(\varphi,0,0),\ \ \varphi\in\mathbb{R}\},$$
$$P^{-1}(L)=2\pi\mathbb{Z}\times\mathbb{R}\times\{0\},\quad \widetilde{L}=\{(0,\theta,0),\ \ \varphi\in\mathbb{R}\},$$
$$P^{-1}(A)=2\pi\mathbb{Z}\times 2\pi\mathbb{Z}\times\,]{-1},1[,\quad \widetilde{A}=\{(0,0,r),\ \ r\in\,]{-1},1[\},$$
$$P^{-1}(N)=\left\{\left(-\arg(\mathrm{i}s-1),\frac{\arg(\mathrm{i}s-1)}{2},\frac{\mathrm{i}s}{\mathrm{i}s-1}\right),\ \ s\in\mathbb{R}\right\},\ \ \widetilde{N}=P^{-1}(N).$$

We can show that the product map $\widetilde{K}\times\widetilde{L}\times\widetilde{A}\times\widetilde{N}\to\widetilde{U(1,1)}$ is also a diffeomorphism.

Now, let us consider the subgroup $A^1:=\widetilde{L}\cdot\widetilde{A}$. Explicitly,

$$A^1=\{(0,\theta,z),\ \theta\in\mathbb{R},\ -1<z<1\}.$$

A^1 is an Abelian Lie group whose Lie algebra $\mathfrak{A}^1$ is isomorphic to $\mathbb{R}^2 = \{(x, y),\ x, y \in \mathbb{R}\}$. The exponential map exp from $\mathfrak{A}^1$ onto A^1 is given by $\exp\colon\ (x, y) \to (x, \tanh y)$.

Using direct computation, we obtain that any element (φ, θ, z) of $\widetilde{U(1,1)}$ admits an Iwasawa decomposition of type $\widetilde{K}A^1\widetilde{N}$ given by the product

$$\begin{aligned}(\varphi, \theta, z) &= \big(\varphi - 2\operatorname{Arg}(1 + \bar{z}\mathrm{e}^{\mathrm{i}\varphi}), 0, 0\big)\\ &\quad\times \Big(0, \theta + \operatorname{Arg}(1 + \bar{z}\mathrm{e}^{\mathrm{i}\varphi}), \frac{|1 + \bar{z}\mathrm{e}^{\mathrm{i}\varphi}|^2 + |z|^2 - 1}{|1 + \bar{z}\mathrm{e}^{\mathrm{i}\varphi}|^2 - |z|^2 + 1}\Big)\\ &\quad\times \Big(-2\arg\Big(\frac{3}{2} - \frac{1 + |z|^2 + 2\bar{z}\mathrm{e}^{\mathrm{i}\varphi}}{2|1 + \bar{z}\mathrm{e}^{\mathrm{i}\varphi}|^2}\Big), \arg\Big(\frac{3}{2} - \frac{1 + |z|^2 + 2\bar{z}\mathrm{e}^{\mathrm{i}\varphi}}{2|1 + \bar{z}\mathrm{e}^{\mathrm{i}\varphi}|^2}\Big),\\ &\qquad\qquad \frac{z\mathrm{e}^{-\mathrm{i}\varphi} - \bar{z}\mathrm{e}^{\mathrm{i}\varphi}}{2(1 + |z|^2) + \bar{z}\mathrm{e}^{i\varphi} + 3z\mathrm{e}^{-\mathrm{i}\varphi}}\Big).\end{aligned}$$

6. Flensted-Jensen's spherical functions

Let $(\mathfrak{A}^1)^*_{\mathbb{C}}$ denote the complex dual of $\mathfrak{A}$. The Flensted-Jensen's spherical functions are functions $\Phi_{\mu,\lambda}$ on $\widetilde{U(1,1)}$ parametrized by numbers $(\mu, \lambda) \in (\mathfrak{A}^1)^*_{\mathbb{C}}$ and defined by the integral ([5], Theorem 5.1)

$$\Phi_{\mu,\lambda}((\varphi, \theta, z)) = \int_{\widetilde{K}/\widetilde{M}} \exp\big(\langle \mathrm{i}(\mu, \lambda) - \varrho, H^1((\varphi, \theta, z)^{-1}\widetilde{k})\rangle\big)\,\mathrm{d}[\widetilde{k}]$$

where $\widetilde{M}$ denotes the centralizer of A^1 in $\widetilde{K}$, $[\widetilde{k}]$ is the class of $\widetilde{k}$ modulo $\widetilde{M}$ and $H^1(\psi, \omega, w) \in \mathfrak{A}^1$ is the unique element such that $(\psi, \omega, w) \in \widetilde{K}\exp(H^1(\psi, \omega, w))\widetilde{M}$. Explicitly, we have that

- $\widetilde{K} \,/\, \widetilde{M} = \mathbb{T} \equiv [0, 2\pi]$,
- $(\varphi, \theta, z)^{-1}\widetilde{k} = (\chi - \varphi, -\theta, -z\mathrm{e}^{-\mathrm{i}\varphi})$, $\widetilde{k} = (\chi, 0, 0) \in \widetilde{K}$,
- $H^1((\varphi, \theta, z)^{-1}\widetilde{k}) = \big(-\theta + \operatorname{Arg}(1 - \bar{z}\mathrm{e}^{\mathrm{i}\chi}), -\frac{1}{2}\operatorname{Log}\big((1 - |z|^2)/|1 - \bar{z}\mathrm{e}^{\mathrm{i}\chi}|^2\big)\big)$
- $\varrho = (0, 1)$.

Note that the function H^1 is a 2π-periodic function with respect to φ. Therefore, the FJ's functions read

$$\Phi_{\mu,\lambda}(\theta, z) = \int_0^{2\pi} \exp\Big(-\mathrm{i}\mu\theta + \mathrm{i}\mu\operatorname{Arg}(1 - \bar{z}\mathrm{e}^{\mathrm{i}\chi}) - \frac{(\mathrm{i}\lambda - 1)}{2}\operatorname{Log}\Big(\frac{1 - |z|^2}{|1 - \bar{z}\mathrm{e}^{\mathrm{i}\chi}|^2}\Big)\Big)\,\mathrm{d}\chi.$$

These functions can also be written as

$$\Phi_{\mu,\lambda}(\theta, z) = \mathrm{e}^{-\mathrm{i}\mu\theta}(1 - |z|^2)^{\frac{1}{2}(1-\mathrm{i}\lambda)} \int_0^{2\pi} (1 - \bar{z}\mathrm{e}^{\mathrm{i}\chi})^{\frac{1}{2}(\mathrm{i}\lambda-1)+\frac{1}{2}\mu}(1 - z\mathrm{e}^{-\mathrm{i}\chi})^{\frac{1}{2}(\mathrm{i}\lambda-1)-\frac{1}{2}\mu}\,\mathrm{d}\chi.$$

Setting $\mu = 2B = \mu(B)$ and $\lambda = \mathrm{i}(2\alpha - 1) = \lambda(\alpha)$ and making use of the binomial formula

$$(1-x)^{-\beta} = \sum_{j=0}^{+\infty} \frac{\Gamma(\beta+j)}{\Gamma(\beta)} \frac{x^j}{j!}$$

we obtain that

$$\begin{aligned}\Phi_{\mu(B),\lambda(\alpha)}(\theta, z) &= \mathrm{e}^{-2B\mathrm{i}\theta}(1-|z|^2)^{\alpha} \int_0^{2\pi} (1-\bar{z}\mathrm{e}^{\mathrm{i}\chi})^{-(\alpha-B)}(1-z\mathrm{e}^{-\mathrm{i}\chi})^{-(\alpha+B)}\,\mathrm{d}\chi \\ &= \mathrm{e}^{-2B\mathrm{i}\theta}(1-|z|^2)^{\alpha} \sum_{m,l\geqslant 0} \frac{\Gamma(\alpha-B+m)}{\Gamma(\alpha-B)} \frac{\Gamma(\alpha+B+l)}{\Gamma(\alpha+B)} \frac{\bar{z}^m z^l}{m!\,l!} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(m-l)\chi}\,\mathrm{d}\chi.\end{aligned}$$

By virtue of

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(m-l)\chi}\,\mathrm{d}\chi = 2\pi\delta_{m,l}$$

we obtain

$$\begin{aligned}&\Phi_{\mu(B),\lambda(\alpha)}(\theta, z) \\ &= 2\pi\mathrm{e}^{-2B\mathrm{i}\theta}(1-|z|^2)^{\alpha} \sum_{m\geqslant 0} \frac{\Gamma(\alpha-B+m)}{\Gamma(\alpha-B)} \frac{\Gamma(\alpha+B+m)}{\Gamma(\alpha+B)} \frac{\Gamma(1)}{\Gamma(1+m)} \frac{(|z|^2)^m}{m!}.\end{aligned}$$

Recalling the expression for the Gauss hypergeometric function ${}_2F_1$ ([8]), we get that

$$\Phi_{\mu(B),\lambda(\alpha)}(\theta, z) = 2\pi\mathrm{e}^{-2B\mathrm{i}\theta}(1-|z|^2)^{\alpha}\,{}_2F_1(\alpha-B, \alpha+B, 1; |z|^2).$$

Finally, taking into account Eq. (3.4), we see that the function $z \longmapsto \Phi_{\mu(B),\lambda(\alpha)}(\theta, z)$ is, up to a phase factor, a radial eigenfunction of the Hamiltonian with the eingenvalue $4\alpha(\alpha-1)$.

7. Concluding remarks

When dealing with a particle moving on the hyperbolic disc under the influence of a normal uniform magnetic field (hyperbolic Landau problem) we have attached to this problem a set of Flensted-Jensen's (FJ's) spherical functions defined on the universal covering group of $U(1,1)$. We have proved that the constructed FJ's functions are, up to a phase factor, radial eigenfunctions of the Hamiltonian. On the one hand these functions present an advantage since they are indexed by two parameters which contain two important pieces of information. Indeed, one of these parameters is proportional to the magnetic field strength while the other occurs in the parametrization of the eigenvalue of the Hamiltonian. On the other hand, the above established connection indicates the possibility to look at the FJ's functions in the context of quantum mechanics.

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