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A NOTION OF ORLICZ SPACES FOR VECTOR VALUED FUNCTIONS

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Abstract. The notion of the Orlicz space is generalized to spaces of Banach-space valued functions. A well-known generalization is based on N-functions of a real variable. We consider a more general setting based on spaces generated by convex functions defined on a Banach space. We investigate structural properties of these spaces, such as the role of the delta-growth conditions, separability, the closure of $\mathcal{L}^\infty$, and representations of the dual space.

Keywords: vector valued function, Orlicz space, Luxemburg norm, delta-growth condition, duality

MSC 2000: 46E30, 46E40, 46B10

1. Introduction

Let Ω be a measure space. An Orlicz space is defined by

$$\mathcal{L}^{\varphi}_{\Omega}(\mathbb{R}) = \left\{ u\colon\ \Omega \to \mathbb{R} \mid \exists\, \alpha > 0 : \int_{\Omega} \varphi(\alpha^{-1} u(\omega))\, \mathrm{d}\mu(\omega) \leqslant 1 \right\}.$$

Here $\varphi\colon\ \mathbb{R} \to [0,\infty]$ is convex, proper, lower semi-continuous and even, with $\varphi(0) = 0$. The norm $\|u\|$ is the infimum of all α such that the above estimate holds. Orlicz spaces are a straightforward generalization of Lebesgue $\mathcal{L}^p$ spaces. They have been thoroughly investigated, and two excellent monographs [4] and [5] are available on this subject. Also [10] provides a good overview on the subject. Moreover, there have been generalizations of Orlicz spaces in several directions [3], [6], [7], [8], [9]. One of them is to consider functions with values in finite- or infinite-dimensional vector spaces. The usual approach is to consider the integral

$$\mathcal{L}^{\varphi}_{\Omega}(X) = \Big\{u\colon\ \Omega \to X \mid \text{Bochner measurable}, \\ \exists\, \alpha > 0\colon\ \int_{\Omega} \varphi(\alpha^{-1}|u(\omega)|)\,\mathrm{d}\mu(\omega) \leqslant 1\Big\}$$

with a convex function $\varphi\colon\ [0,\infty) \to [0,\infty]$. Such spaces have a certain kind of isotropy with respect to the underlying vector space X, in the sense that only $|u(\omega)|$ enters the norm of u in the Orlicz space. In this paper we take a more general approach and consider a proper, convex, lower semi-continuous function $\varphi\colon\ X \to [0,\infty]$ on a real Banach space X, where $\varphi(x) = \varphi(-x)$. We define

$$\mathcal{L}^{\varphi}_{\Omega}(X) = \Big\{u\colon\ \Omega \to X \mid \text{Bochner measurable}, \\ \exists\, \alpha > 0\colon\ \int_{\Omega} \varphi(\alpha^{-1}u(\omega))\,\mathrm{d}\mu(\omega) \leqslant 1\Big\}. \tag{1}$$

The aim of this paper is to investigate the basic structure of such spaces and to carry over what remains valid from the classical theory of Orlicz spaces. To our knowledge, this type of generalized Orlicz spaces has been considered only in [7], [8] with additional growth conditions on φ. In these papers the author considers the integral

$$\mathcal{L}^{\varphi}_{\Omega}(X) = \Big\{u\colon\ \Omega \to X \mid \text{Bochner measurable}, \\ \exists\, \alpha > 0\colon\ \int_{\Omega} \varphi(\omega, \alpha^{-1}u(\omega))\,\mathrm{d}\mu(\omega) \leqslant 1\Big\}$$

with a finite-dimensional Euclidean space X and a generalized N-function $\varphi\colon\ \Omega \times X \to [0,\infty)$, i.e. φ is a continuous, convex and coercive function of x for each ω, a measurable function of ω for each x and there exists a $d \geqslant 0$ such that

$$\inf_{\omega} \inf_{c \geqslant d} \frac{\inf_{\|x\|=c} \varphi(\omega, x)}{\sup_{\|x\|=c} \varphi(\omega, x)} > 0.$$

Under these assumptions the author shows the completeness of the Orlicz space and studies the relation between the Orlicz space and its Orlicz class with regard to a generalized Δ-condition. Moreover, the concept of modular convergence is introduced and conditions under which the norm and modular convergence are equivalent, are given. The author also considers duality and characterizes all continuous linear functionals defined on the Orlicz space. Moreover, in [7], [8] φ is not necessarily assumed to be an even function. Omitting the growth conditions in [7], [8], this generalization may lead to the situation that $\mathcal{L}^\varphi$ is no longer a vector space but a cone. This case has been discussed in more detail in [9].

The consideration of such anisotropic Orlicz spaces has been motivated by research on visco-plastic constitutive laws involving convex dissipation potentials [3]. If the rheological behavior of a visco-plastic material is described by certain dissipation potentials, it was shown that there exists a unique solution for the strain and the inner state at a material point, given the stress history at this point. To prove this, restrictions had to be placed on the dissipation potential. In particular, the Δ_2 and ∇_2-growth conditions known from classical Orlicz spaces were needed. The proof relies on the duality between the Orlicz spaces generated by the dissipation potential and its Fenchel conjugate. The integral in (1) appears very naturally as an estimate for the energy dissipated due to the plasticity of the material. From the viewpoint of continuum mechanics, it is a serious restriction to request that the dissipation potential depend only on the norm of the strain and the inner states. This would require not only isotropy of the material, but in addition it means that the plastic behavior depends only on the norm of the stress, i.e., the yield surfaces are spheres. In practice, the stress enters plasticity criteria in a more complicated manner (see e.g. [1], Chapter 5.2.2). Therefore, in [3] the concept of a generalized Orlicz space for $\mathbb{R}^n$-valued functions in the sense of (1) was introduced.

In this paper we investigate the following structural properties:

- Definition, completeness and criteria for separability of the generalized Orlicz spaces.
- The subspaces $\mathbf{E}^\varphi_\Omega(X)$, which is the closure of $\mathcal{L}^\infty$, and $\mathbf{C}^\varphi_\Omega(X)$, which consists of the functions with absolutely continuous norm. The relations of these subspaces to the whole space and to each other are nontrivial even in the case of classical Orlicz spaces.
- The dual space and its relation to the Orlicz space generated by the Fenchel conjugate of φ.

These features depend heavily on the Δ_2 and ∇_2-growth conditions, also on the property whether φ is bounded on bounded sets. Although much of the theory of classical Orlicz spaces can be recovered, there are some surprising counterexamples. In a sense, our paper gives positive results (such as reflexivity) for smooth φ with restricted growth. In the pathological case of φ attaining infinity within bounded sets—which reflects the case of elasto-plasticity in continuum mechanics—the structure of the space is complicated. The counterexamples in this paper may help to find the right conjectures to shed light also on this delicate case.

The paper is structured as follows: Chapter 2 provides a short overview of the definitions and properties of convex functions used in the subsequent sections. Chapter 3 deals with the main definitions of a generalized Orlicz space $\mathbf{L}^{\varphi}_{\Omega}(X)$, the subset $\overline{\mathbf{L}}^{\varphi}_{\Omega}(X)$, which consists of all functions u for which the integral $\int_{\Omega} \varphi(u)\,\mathrm{d}\mu$ is finite, and the generalized Luxemburg norm N^{φ}_{Ω}. If μ is diffuse on a subset with positive measure, then $\mathbf{L}^{\varphi}_{\Omega}(X) = \overline{\mathbf{L}}^{\varphi}_{\Omega}(X)$ if and only if φ satisfies the Δ_2-growth condition. In Chapter 4 we study the properties of $\mathbf{C}^{\varphi}_{\Omega}(X)$, the subspace of functions with absolutely continuous norm. The main result in this chapter is the equivalence between the absolutely continuous norm and a certain notion of monotone convergence. Chapter 5 is dedicated to the studies of $\mathbf{E}^{\varphi}_{\Omega}(X)$, the closure of $\mathcal{L}^{\infty}$. For φ bounded on bounded subsets of X, we get the equivalence between the Δ_2-growth condition and $\mathbf{E}^{\varphi}_{\Omega}(X) = \mathbf{L}^{\varphi}_{\Omega}(X)$, similar to the classical case. A surprising example shows that the implication $\varphi \in \Delta_2$ implies $\mathbf{E}^{\varphi}_{\Omega}(X) = \mathbf{L}^{\varphi}_{\Omega}(X)$ need not be true for a φ which is not bounded on a bounded subset of X, which is a major difference between the classical and the generalized Orlicz space. In Chapter 6 we provide the main theorems about completeness and separability of $\mathbf{L}^{\varphi}_{\Omega}(X)$. The results are similar to those for classical Orlicz spaces. Only if φ is bounded on bounded subsets, satisfies the Δ_2-condition and Ω is a compact metric space, a general result for the separability of $\mathbf{L}^{\varphi}_{\Omega}(X)$ can be obtained. If φ does not satisfy the Δ_2-condition and μ is diffuse on a subset with positive measure, then $\mathbf{L}^{\varphi}_{\Omega}(X)$ cannot be separable. Duality and reflexivity of the Orlicz space is considered in Chapter 7. Again, reflexivity is connected to the Δ_2-growth condition. If Ω is a finite measure space and φ is bounded on bounded subsets, then $\mathbf{L}^{\varphi}_{\Omega}(X)$ is reflexive if and only if φ satisfies the Δ_2- and ∇_2-growth conditions.

2. On convex functions

Throughout this paper let $(\Omega, \mathcal{A}, \mu)$ be a measure space, i.e. let μ be σ-finite. X denotes a Banach space with the dual X^*. If $E \in \mathcal{A}$, then χ_E denotes the indicator function of E.

Definition 2.1. Let φ be a function from X to $\mathbb{R}$. φ is called *convex*, if for any x, $y \in X$ and $0 < \lambda < 1$ the following inequality holds:

$$\varphi(\lambda x + (1-\lambda)y) \leqslant \lambda\varphi(x) + (1-\lambda)\varphi(y).$$

If $\varphi(0) = 0$ we get immediately

$$\varphi(\lambda x) \leqslant \lambda\varphi(x), \quad 0 \leqslant \lambda \leqslant 1$$

and

$$\lambda\varphi(x) \leqslant \varphi(\lambda x), \quad \lambda \geqslant 1.$$

Definition 2.2. Let φ be a nontrivial function from X to $[0,\infty]$. φ is called *coercive* iff $\lim\limits_{\|x\|\to\infty} \varphi(x)/\|x\| = \infty$.

Definition 2.3. Let φ be a convex function from X to $[0,\infty]$. The *Fenchel conjugate* φ^* of φ is defined by

$$\begin{cases} \varphi^*\colon\ X^* \to [0,\infty], \\ \varphi^*(x^*) := \sup\limits_{x\in X}\{\langle x^*, x\rangle - \varphi(x)\}. \end{cases}$$

Obviously, the so called *Fenchel inequality* holds for every $x \in X$ and $x^* \in X^*$:

$$\langle x^*, x\rangle \leqslant \varphi(x) + \varphi^*(x^*).$$

By [2], Theorem 4.2, page 63, for a continuous φ and an $x \in X$ with $\varphi(x) < \infty$, we can find an $x^* \in X^*$ such that

$$\langle x^*, x\rangle = \varphi(x) + \varphi^*(x^*).$$

Definition 2.4. A nontrivial function φ from X to $\mathbb{R}$ is called *lower semi-continuous at* x_0 (LSC) if

$$\varphi(x_0) \leqslant \liminf_{x\to x_0} \varphi(x).$$

The function φ is called *lower semi-continuous*, if φ is lower semi-continuous at every $x \in X$.

By [2], Theorem 3.1, page 37, a nontrivial convex function φ is LSC if and only if $\varphi = \varphi^{**}$, $\varphi^{**} = (\varphi^*)^*$.

Theorem 2.1. *Let φ be a nontrivial, convex function from X to $[0,\infty]$. Then the following statements are equivalent:*

1) *φ is bounded on an open subset of X,*
2) *φ is locally Lipschitz in the interior of* $\mathrm{Dom}(\varphi) = \{x \in X \mid \varphi(x) < \infty\}$.

Proof. The proof can be found in [2], Theorem 2.1, page 25. □

Note that if φ is bounded on bounded subsets of X, then $\mathrm{Dom}(\varphi) = X$ and φ is continuous on X.

Theorem 2.2. *Let X be reflexive, φ a nontrivial, convex and LSC function from X to $[0,\infty]$ and φ^* the Fenchel conjugate of φ. The following statements are equivalent:*

1) *φ is bounded on bounded subsets of X,*
2) *φ^* is coercive.*

Proof. We give a proof only for one direction, the other direction is proved in the same way. Suppose that φ is bounded on bounded subsets of X and φ^* is not coercive. Then there exists a sequence $\{x_n^*\}_{n\in\mathbb{N}} \subseteq X^*$ and an $M>0$ such that

$$\varphi^*(x_n^*) \leqslant M\|x_n^*\|, \quad \|x_n^*\| \to \infty.$$

Since X is reflexive we can find a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\langle x_n^*, x_n\rangle = (M+1)\|x_n^*\|$ and $\|x_n\| \leqslant M+1$. But then

$$\begin{aligned}\varphi(x_n) = \sup_{x^*\in X^*}\{\langle x^*, x_n\rangle - \varphi^*(x^*)\} &\geqslant \langle x_n^*, x_n\rangle - \varphi^*(x_n^*)\\ &\geqslant (M+1)\|x_n^*\| - M\|x_n^*\| = \|x_n^*\| \to \infty,\end{aligned}$$

which is a contradiction to the boundedness of φ on $\{x\in X, \|x\| \leqslant M+1\}$. □

The growth properties of the convex function φ are dominating when studying the duality, reflexivity or separability of a vector valued Orlicz space. The most important growth conditions, the Δ_2 and the ∇_2 condition ensure that the convex function φ can be compared with functions φ_p, where $\varphi_p(x) = \|x\|^p$ and $p>1$. In classical theory, this result can be found in [4], Proposition 12. In the case of $X=\mathbb{R}^N$, the proof is given in [3], but remains the same for an arbitrary Banach space.

Definition 2.5. Let φ be a function from X to $[0,\infty]$.

The function φ is said to satisfy the Δ_2-*condition* if there exists an $L>1$ and an $M\geqslant 0$ such that $\varphi(2x) \leqslant L\varphi(x)$ for all $x\in X$ with $\|x\|\geqslant M$.

The function φ is said to satisfy the ∇_2-*condition* if there exists an $l>1$ and an $M\geqslant 0$ such that $\varphi(x) \leqslant \frac{1}{2}l^{-1}\varphi(x/l)$ for all $x\in X$ with $\|x\|\geqslant M$.

We write $\varphi\in\Delta_2$ (or $\varphi\in\nabla_2$) if φ satisfies the Δ_2- (or ∇_2)-condition.

There is a strong relation between the growth condition of φ and its Fenchel conjugate φ^*. For convex functions $\varphi\colon \mathbb{R}\to[0,\infty)$ the correlations are well studied in [4]. For $\varphi\colon \mathbb{R}^N\to[0,\infty)$ the proof of the next remark is given in [3], but remains the same for an arbitrary Banach space.

R e m a r k 2.1. Let $\varphi\colon X \to [0,\infty]$ have φ^* as its Fenchel conjugate. Let φ and φ^* be coercive. Then

$$\varphi \in \Delta_2 \quad \text{if and only if} \quad \varphi^* \in \nabla_2.$$

3. Main definitions

In the whole section we assume that

- μ has the finite subset property, i.e. for every $E \in \mathcal{A}$ with positive measure we can find an $F \in \mathcal{A}$, $F \subseteq E$, such that $0 < \mu(F) < \mu(E)$;
- $\varphi\colon X \to [0,\infty]$, convex, LSC, $\varphi(x) = 0$ iff $x = 0$, $\varphi(x) = \varphi(-x)$;
- $\lim\limits_{\|x\|\to\infty} \varphi(x) = +\infty$;
- $\varphi^*\colon X^* \to [0,\infty]$ is the Fenchel conjugate of φ.

Definition 3.1. Let $\mathcal{M} := \{u\colon \Omega \to X \mid \text{Bochner measurable}\}$. We define the following subset of $\mathcal{M}$:

$$\mathcal{L}^{\varphi}_{\Omega}(X) = \Big\{u \in \mathcal{M} \mid \exists\, \alpha > 0\colon \int_{\Omega} \varphi(\alpha^{-1}u)\,\mathrm{d}\mu < +\infty\Big\}.$$

$\mathbf{L}^{\varphi}_{\Omega}(X)$ denotes the set of equivalence classes in $\mathcal{L}^{\varphi}_{\Omega}(X)$ with respect to equality almost everywhere. By $\overline{\mathbf{L}}^{\varphi}_{\Omega}(X)$ we define the following convex subset of $\mathbf{L}^{\varphi}_{\Omega}(X)$:

$$\overline{\mathbf{L}}^{\varphi}_{\Omega}(X) = \Big\{u \in \mathbf{L}^{\varphi}_{\Omega}(X) \mid \int_{\Omega} \varphi(u)\,\mathrm{d}\mu < +\infty\Big\}.$$

Definition 3.2. For $u \in \mathbf{L}^{\varphi}_{\Omega}(X)$ we define

$$N^{\varphi}_{\Omega}(u) := \inf\Big\{\alpha > 0,\ \int_{\Omega} \varphi(\alpha^{-1}u)\,\mathrm{d}\mu \leqslant 1\Big\}.$$

In classical theory $N^{\varphi}_{\Omega}(u)$ is called the Luxemburg norm of u.

R e m a r k 3.1. $\overline{\mathbf{L}}^{\varphi}_{\Omega}(X)$ is a convex subset of $\mathcal{M}$, $\mathbf{L}^{\varphi}_{\Omega}(X)$ is a linear space. Moreover, N^{φ}_{Ω} possesses the following properties:

1) $N^{\varphi}_{\Omega}(u) = 0$ iff $u = 0$ almost everywhere, $u \in \mathbf{L}^{\varphi}_{\Omega}(X)$;
2) $N^{\varphi}_{\Omega}(u+v) \leqslant N^{\varphi}_{\Omega}(u) + N^{\varphi}_{\Omega}(v)$ for all $u, v \in \mathbf{L}^{\varphi}_{\Omega}(X)$;
3) $N^{\varphi}_{\Omega}(\lambda u) = |\lambda| N^{\varphi}_{\Omega}(u)$ for all $u \in \mathbf{L}^{\varphi}_{\Omega}(X)$ and for all $\lambda \in \mathbb{R}$.

Theorem 3.1. *$N_\Omega^\varphi(\cdot)$ is LSC, i.e. for every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathbf{L}_\Omega^\varphi(X)$ which converges almost everywhere to some $u \in \mathbf{L}_\Omega^\varphi(X)$ we have*

$$N_\Omega^\varphi(u) \leqslant \liminf_{n\to\infty} N_\Omega^\varphi(u_n).$$

P r o o f. The result for the classical Luxemburg norm can be found in [4], Proposition 4, pages 56–57. With minor modifications the proof carries over to our assumptions. □

Definition 3.3. A set $E \in \mathcal{A}$, $\mu(E) > 0$, is called a *μ-atom*, if for each subset $F \subseteq E$, $F \in \mathcal{A}$ we have either $\mu(F) = 0$ or $\mu(E \setminus F) = 0$. μ is called *diffuse* on an $E \in \mathcal{A}$ if E does not contain an μ-atom.

Note that if μ is diffuse then for each λ, $0 \leqslant \lambda \leqslant \mu(E)$, there is a set $D \in \mathcal{A}$, $D \subseteq E$ such that $\mu(D) = \lambda$. See for example [11], Proposition 7, page 26.

Theorem 3.2. *Let μ be diffuse on an element $E \in \mathcal{A}$, $\mu(E) > 0$. If $\overline{\mathbf{L}}_\Omega^\varphi(X)$ is a linear space, then $\varphi \in \Delta_2$. If φ is bounded on bounded subsets and $\mu(\Omega) < \infty$, the converse is true as well.*

P r o o f. The proof for the first implication follows the proof of [4], Theorem 2, pages 46–47.

Suppose now that φ is bounded on bounded subsets of X and $\mu(\Omega) < \infty$. Let $u \in \overline{\mathbf{L}}_\Omega^\varphi(X)$, $\varphi \in \Delta_2$ and $M \geqslant 0$ be such that

$$\varphi(2x) \leqslant K\varphi(x) \quad \text{for all } \|x\| \geqslant M,\ K > 1.$$

Then

$$\int_\Omega \varphi(2u)\,\mathrm{d}\mu \leqslant \int_{\{\omega \mid \|u(\omega)\| \leqslant M\}} \varphi(2u)\,\mathrm{d}\mu + K\int_\Omega \varphi(u)\,\mathrm{d}\mu < \infty.$$

□

R e m a r k 3.2. Let μ be diffuse on $E \in \mathcal{A}$ with $\mu(E) > 0$ and assume that φ is not bounded on a bounded subset of X. Then $\varphi \in \Delta_2$ does not imply in general that $\overline{\mathbf{L}}_\Omega^\varphi(X)$ is a linear space.

E x a m p l e 3.1. $X = \mathbb{R}$, $\Omega = [0,1]$, μ is the Lebesgue measure.

$$\varphi(x) = \begin{cases} \dfrac{1}{1-x} - 1, & |x| < 1, \\ \infty, & |x| \geqslant 1. \end{cases}$$

Obviously φ is convex, LSC, coercive and $\varphi \in \Delta_2$. Consider $u\colon [0,1] \to \mathbb{R}$, $u(\omega) = \omega/2$. Then $u \in \mathbf{L}_\Omega^\varphi(X)$ since $\int_\Omega \varphi(u)\,\mathrm{d}\mu = \int_0^1 ((1 - \frac{1}{2}\omega)^{-1} - 1) < \infty$, but $\int_\Omega \varphi(2u) = \int_0^1 ((1-\omega)^{-1} - 1)\,\mathrm{d}\omega = \infty$.

The proofs of the next theorem and corollary follow the considerations of [5], § 8, subchapter 4.

Theorem 3.3. *Let μ be diffuse on an $E \in \mathcal{A}$, $\mu(E) > 0$, $\varphi \notin \Delta_2$. Then there exists a $u \in \overline{\mathbf{L}}^{\varphi}_{\Omega}(X)$ such that $\beta u \notin \overline{\mathbf{L}}^{\varphi}_{\Omega}(X)$ for all $\beta > 1$.*

Corollary 3.1. *Assume that $\varphi \notin \Delta_2, \mu$ is diffuse on $E \in \mathcal{A}$, $\mu(E) > 0$ and φ is bounded on bounded subsets. Then there exists a $u \in \mathbf{L}^{\varphi}_{\Omega}(X)$ such that*

$$\begin{aligned} &\beta u \in \overline{\mathbf{L}}^{\varphi}_{\Omega}(X) \quad \textit{for all } 0 \leqslant \beta < 1,\\ &\beta u \notin \overline{\mathbf{L}}^{\varphi}_{\Omega}(X) \quad \textit{for all } \beta \geqslant 1. \end{aligned}$$

4. Absolutely continuous norm

Definition 4.1. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathbf{L}^{\varphi}_{\Omega}(X)$, $u \in \mathbf{L}^{\varphi}_{\Omega}(X)$. We say that u_n converges *monotonically* to u if there exists a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ in $\mathcal{L}^1(\Omega, \mathbb{R})$ with $0 \leqslant \alpha_n(\omega) \leqslant \alpha_{n+1}(\omega) \leqslant 1$, $\alpha_n \to 1$ almost everywhere and $u_n(\omega) = \alpha_n(\omega)u(\omega)$.

Lemma 4.1. *Let $\varphi \in \Delta_2$, let φ be bounded on bounded subsets of X, $\mu(\Omega) < \infty$ and $u_n \to u$ monotonically. Then $\lim\limits_{n\to\infty} N^{\varphi}_{\Omega}(u - u_n) = 0$.*

P r o o f. A detailed proof is given in [3] for $X = \mathbb{R}^N$, but the proof remains the same for an arbitrary Banach space. $\square$

R e m a r k 4.1. If φ is not bounded on bounded subsets, the statement is not true in general.

E x a m p l e 4.1. Take $X = \mathbb{R}$, $\Omega = [0, 1]$,

$$\varphi(x) = \begin{cases} \dfrac{1}{1-|x|} - 1, & |x| < 1,\\ \infty, & |x| \geqslant 1. \end{cases}$$

Obviously $\varphi \in \Delta_2$, φ is convex, LSC and coercive. Put $u =: [0,1] \to \mathbb{R}$, $u(\omega) = \omega$, which is in $\mathbf{L}^{\varphi}_{\Omega}(X)$. Consider $E_n := [0, 1 - 2^{-n})$, $\alpha_n(\omega) = \chi_{E_n}(\omega)$, $u_n(\omega) = \alpha_n(\omega)u(\omega) = \omega\chi_{E_n}(\omega)$. Then u_n converges monotonically to u and $(u - u_n)(\omega) = \omega\chi_{\Omega\setminus E_n}(\omega)$. Let $\varepsilon_0 = 1$. Then $N^{\varphi}_{\Omega}(u - u_n) > \varepsilon_0$, since

$$\int_{\Omega} \varphi\Big(\frac{1}{\varepsilon_0}(u - u_n)\Big)(\omega)\,\mathrm{d}\mu(\omega) = \int_{1-2^{-n}}^{1} \Big(\frac{1}{1-\omega} - 1\Big)\,\mathrm{d}\omega = \infty.$$

Proposition 4.1. *If $\varphi \notin \Delta_2$, μ is diffuse on an $E \in \mathcal{A}$ with positive measure then there exists a $u \in \mathbf{L}^{\varphi}_{\Omega}(X)$ and a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathbf{L}^{\varphi}_{\Omega}(X)$ such that $u_n \to u$ monotonically, but $N^{\varphi}_{\Omega}(u-u_n)$ does not converge to 0.*

P r o o f. Take an α, $0<\alpha<\mu(E)\leqslant\infty$. Since μ is diffuse on E there exists an $F \subseteq E$, $F\in\mathcal{A}$ and $\mu(F)=\alpha$. Take $n_0\in\mathbb{N}$ such that $\sum\limits_{n\geqslant n_0} 1/n^2<\alpha$. Then we can find an $F_0\subseteq F$, $\mu(F_0)=\sum\limits_{n\geqslant n_0} 1/n^2$. Take a $D_1\in\mathcal{A}$, $D_1\subseteq F_0$ and $\mu(D_1)=1/n_0^2$. Since $\mu(F_0\setminus D_1)>0$ there exists a $D_2\in F_0\setminus D_1$, $\mu(D_2)=(n_0+1)^{-2}$. Continuing the procedure we obtain disjoint subsets $D_n\in\mathcal{A}$ with $\mu(D_n)=(n_0+n-1)^{-2}$ for $n\geqslant 1$. Since $\varphi\notin\Delta_2$ we can find a sequence $\{x_n\}\in X$ such that $\varphi(2x_n)>n\varphi(x_n)$, $\|x_n\|\to\infty$ and $\varphi(x_n)\geqslant 1$. So we can find disjoint subsets $F_n\in\mathcal{A}$ with $\mu(F_n)=\mu(D_n)/\varphi(x_n)$. Put $u=\sum\limits_{m=1}^{\infty} x_n\chi_{F_n}(\omega)$. Then $u\in\mathbf{L}^{\varphi}_{\Omega}(X)$, since

$$\int_{\Omega}\varphi(u)\,\mathrm{d}\mu=\sum_{n=1}^{\infty}\varphi(x_n)\mu(F_n)=\sum_{n\geqslant n_0}\frac{1}{(n_0+n-1)^2}<\infty.$$

Let $\Omega_n=\{\omega\in\Omega \mid \|u(\omega)\|\leqslant n\}$, $\alpha_n(\omega)=\chi_{\Omega_n}(\omega)$, $u_n(\omega)=\alpha_n(\omega)u(\omega)$ for all $\omega\in\Omega$. Then $u_n\to u$ monotonically. For an arbitrary $n\in\mathbb{N}$ we can find a $k_0\in\mathbb{N}$ such that $\|x_k\|\geqslant n$ for all $k\geqslant k_0$. Thus

$$\int_{\Omega}\varphi(2(u-u_n))\,\mathrm{d}\mu\geqslant\sum_{k\geqslant k_0}\varphi(2x_k)\mu(F_k)\geqslant\sum_{k\geqslant k_0}\frac{k}{(n_0+k-1)^2}=\infty,$$

which implies that $N^{\varphi}_{\Omega}(u-u_n)>\frac{1}{2}$ for all $n\in\mathbb{N}$. □

Lemma 4.2. *Let φ be bounded on bounded subsets, $\mu(\Omega)<\infty$, $\{u_n\}_{n\in\mathbb{N}}\subseteq\mathcal{L}^{\infty}(\Omega,X)$. Suppose that $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded and u_n converges to 0 almost everywhere. Then $N^{\varphi}_{\Omega}(u_n)\to 0$ for $n\to\infty$.*

P r o o f. A detailed proof is given in [3] for $X=\mathbb{R}^N$, but the proof remains the same for an arbitrary Banach Space. □

R e m a r k 4.2. If φ is not bounded on bounded subsets, the statement is not generally true.

E x a m p l e 4.2. Let $X=\mathbb{R}$, $\Omega=[0,1]$, φ, u, u_n as in Example 4.1. Put $v_n := u-u_n$. Then $v_n\to 0$ almost everywhere, $\|v_n\|_{\mathcal{L}^{\infty}}\leqslant 1$ but $N^{\varphi}_{\Omega}(v_n)>1$ for all $n\in\mathbb{N}$.

Definition 4.2. Let $(\Omega,\mathcal{A},\mu)$ be a measure space and $(B,\|\cdot\|)$ a normed vector space of measurable functions $\Omega\to X$. Let $u\in B$.

(1) We say that u has *absolutely continuous norm in the strong sense* (cf. [10], Definition 3.1, page 14), iff $\|\chi_{E_n} u\| \to 0$ for each sequence $E_n \in \mathcal{A}$ with $\chi_{E_n}(\omega) \to 0$ for almost all $\omega \in \Omega$.
(2) We say that u has *absolutely continuous norm in the weak sense*, iff $\|\chi_{E_n} u\| \to 0$ for each sequence $E_n \in \mathcal{A}$ with $\mu(E_n) \to 0$.

We will give an example showing that absolute continuity of the norm in the weak sense does not imply absolute continuity of the norm in the strong sense.

Lemma 4.3. *Let $\varphi\colon [0,\infty) \to [0,\infty)$ be continuous, monotone increasing, with $\varphi(0) = 0$ and such that*

$$\lim_{x\to 0+} \frac{\varphi(2x)}{\varphi(x)} = \infty.$$

Then there exists a measurable function $u\colon [0,\infty) \to [0,1]$ such that

$$\int_0^\infty \varphi(u(t))\,\mathrm{d}t \leqslant 1, \quad \text{and} \quad \int_0^\infty \varphi(2u(t))\,\mathrm{d}t = \infty.$$

Proof. Choose a sequence $1 \geqslant v_1 > v_2 > v_3 > \ldots > 0$ such that

$$\frac{\varphi(2v_n)}{\varphi(v_n)} \geqslant 2^n.$$

Construct a sequence $0 = t_0 < t_1 < t_2 < \ldots$ such that $(t_n - t_{n-1})\varphi(v_n) = 2^{-n}$. Put $u(t) = v_n$ if $t \in [t_{n-1}, t_n)$. Then

$$\int_0^\infty \varphi(u(t))\,\mathrm{d}t = \sum_{n=1}^\infty (t_n - t_{n-1})\varphi(v_n) = 1,$$
$$\int_0^\infty \varphi(2u(t))\,\mathrm{d}t = \sum_{n=1}^\infty (t_n - t_{n-1})\varphi(2v_n) \geqslant \sum_{n=1}^\infty (t_n - t_{n-1})\varphi(v_n)2^n = \sum_{n=1}^\infty 1 = \infty.$$

□

Example 4.3. Let

$$\varphi(x) = \int_0^{|x|} (|x| - y)\mathrm{e}^{-1/y}\,\mathrm{d}y.$$

Then φ is convex, $\varphi(x) = \varphi(-x)$, φ is increasing on $[0,\infty)$, $\varphi(0) = \varphi'(0) = 0$, and

$$\lim_{x\to 0+} \frac{\varphi(2x)}{\varphi(x)} = \infty.$$

Let $\|\cdot\|$ denote the Luxemburg norm with respect to φ. Let u be constructed according to Lemma 4.3. Then u has absolutely continuous norm in the weak sense, but not in the strong sense.

P r o o f. Most of the properties of φ are obvious. To see the convexity, notice that for positive x

$$\varphi(x)=\int_0^x\int_0^z \mathrm{e}^{-1/y}\,\mathrm{d}y\,\mathrm{d}z.$$

Thus

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi(x)=\mathrm{e}^{-1/x}>0.$$

Applying de l'Hospital's rule twice we infer that

$$\lim_{x\to 0+}\frac{\varphi(2x)}{\varphi(x)}=\lim_{x\to 0+}\frac{4\varphi''(2x)}{\varphi''(x)}=\lim_{x\to 0+}4\mathrm{e}^{1/(2x)}=\infty.$$

We have then

$$\int_0^\infty \varphi(u(t))\,\mathrm{d}t\leqslant 1,$$

thus $\|u\|\leqslant 1$. Moreover, let $\varepsilon>0$ and let $E\subset[0,\infty)$ be a measurable set with Lebesgue measure

$$\mu(E)\leqslant\frac{1}{\varphi(\varepsilon^{-1})}.$$

Then

$$\int_0^\infty \varphi(\varepsilon^{-1}\chi_E(t)u(t))\,\mathrm{d}t\leqslant\int_E\varphi(\varepsilon^{-1})\,\mathrm{d}t\leqslant 1,$$

thus $\|\chi_E u\|\leqslant\varepsilon$. We have shown that u has absolutely continuous norm in the weak sense. However, let $E_n=[n,\infty)$, thus $\chi_{E_n}(t)\to 0$ for all $t\in[0,\infty)$. Suppose that $\|\chi_{E_n}u\|\leqslant 1/2$ for some n. Then

$$\int_0^\infty\varphi(2u(t))\,\mathrm{d}t=\int_0^n\varphi(2u(t))\,\mathrm{d}t+\int_{E_n}\varphi(2u(t))\,\mathrm{d}t\leqslant n\varphi(2)+1<\infty,$$

in contradiction to the construction of u. Therefore, u has not absolutely continuous norm in the strong sense. □

With $\mathbf{C}^\varphi_\Omega(X)$ we denote the linear space of all functions of $\mathbf{L}^\varphi_\Omega(X)$ which have absolutely continuous norm in the weak sense.

Theorem 4.1. *Let $\mu(\Omega)<\infty$ and let u be in $\mathbf{L}^\varphi_\Omega(X)$. Then the following assertions are equivalent:*

1) $u\in\mathbf{C}^\varphi_\Omega(X)$,
2) *every sequence $\{u_n\}$ converging monotonically to u converges also in norm, i.e. $N^\varphi_\Omega(u-u_n)\to 0$.*

P r o o f. 1) $\Rightarrow$ 2) Suppose that $u\in\mathbf{C}^\varphi_\Omega(X)$. Let u_n be an arbitrary sequence converging monotonically to u. Then there exists a sequence $\{\alpha_n\}_{n\in\mathbb{N}}\subseteq\mathcal{L}^1(\Omega,\mathbb{R})$,

$0 \leqslant \alpha_1(\omega) \leqslant \ldots \leqslant \alpha_n(\omega) \leqslant 1$, $\alpha_n \to 1$ and $u_n(\omega) = \alpha_n(\omega)u(\omega)$. We have to show that for every $\varepsilon > 0$ we can find an $n_0 \in \mathbb{N}$ such that

$$\int_\Omega \varphi\Big(\frac{1-\alpha_n}{\varepsilon}u\Big)\,\mathrm{d}\mu \leqslant 1 \quad \text{for all } n \geqslant n_0.$$

Take an $\varepsilon > 0$. Since $u \in \mathbf{L}^\varphi_\Omega(X)$ we can find a $\gamma > 0$ such that

$$\int_\Omega \varphi\Big(\frac{1}{\gamma}u\Big)\,\mathrm{d}\mu \leqslant 1.$$

Put $\beta = 2\gamma$. Then

$$2\int_\Omega \varphi\Big(\frac{1}{\beta}u\Big)\,\mathrm{d}\mu \leqslant \int_\Omega \varphi\Big(\frac{2}{\beta}u\Big)\,\mathrm{d}\mu = \int_\Omega \varphi\Big(\frac{1}{\gamma}u\Big)\,\mathrm{d}\mu \leqslant 1 \quad \text{(cf. Definition 2.1).}$$

Thus we can find a $\beta > 0$ such that $\int_\Omega \varphi(\frac{1}{\beta}u)\,\mathrm{d}\mu \leqslant \frac{1}{2}$.

Moreover, since $u \in \mathbf{C}^\varphi_\Omega(X)$ we can find a $\delta > 0$ with $N^\varphi_\Omega(\chi_F u) \leqslant \frac{\varepsilon}{2}$ whenever $\mu(F) < \delta$. In particular,

$$1 \geqslant \int_\Omega \varphi\Big(\frac{2}{\varepsilon}\chi_F u\Big)\,\mathrm{d}\mu \geqslant 2\int_\Omega \varphi\Big(\frac{1}{\varepsilon}\chi_F u\Big)\,\mathrm{d}\mu.$$

Put

$$\Omega_n = \Big\{\omega \in \Omega \mid \frac{1-\alpha_n(\omega)}{\varepsilon}\beta > 1\Big\}.$$

Since $\mu(\Omega) < \infty$ and $\alpha_n \to 1$ almost everywhere we get that $\mu(\Omega_n) \to 0$ for $n \to \infty$. Hence there exists an $n_0 \in \mathbb{N}$ such that $\mu(\Omega_n) < \delta$ for all $n \geqslant n_0$. Take an $n \geqslant n_0$, then

$$\begin{aligned}\int_\Omega \varphi\Big(\frac{1-\alpha_n}{\varepsilon}u\Big)\,\mathrm{d}\mu &= \int_{\Omega_n} \varphi\Big(\frac{1-\alpha_n}{\varepsilon}u\Big)\,\mathrm{d}\mu + \int_{\Omega\setminus\Omega_n} \varphi\Big(\frac{1-\alpha_n}{\varepsilon}u\Big)\,\mathrm{d}\mu \\ &\leqslant \int_\Omega \varphi\Big(\chi_{\Omega_n}\frac{1}{\varepsilon}u\Big)\,\mathrm{d}\mu + \int_\Omega \varphi\Big(\frac{1}{\beta}u\Big)\,\mathrm{d}\mu \\ &\leqslant \frac{1}{2} + \frac{1}{2} = 1.\end{aligned}$$

2) $\Rightarrow$ 1) Suppose that $u \notin \mathbf{C}^\varphi_\Omega(X)$. Then we can find an $\varepsilon > 0$ such that for every $\delta > 0$ there exists an $F_\delta \in \mathcal{A}$ with $N^\varphi_\Omega(\chi_{F_\delta}u) > \varepsilon$. Put $\delta_n = 2^{-n}$, $F_n = F_{\delta_n}$ and $\overline{F}_n = \bigcup\limits_{m\geqslant n} F_m$. Since $\mu(F_n) \leqslant 2^{-n}$ we get that $\mu(\overline{F}_n) \leqslant 2^{-n+1}$. Put $\alpha_n = (1 - \chi_{\overline{F}_n})$ and $u_n = \alpha_n u$. Then u_n converges monotonically to u, but

$$N^\varphi_\Omega(u - u_n) = N^\varphi_\Omega((1-\alpha_n)u) = N^\varphi_\Omega(\chi_{\overline{F}_n}u) > \varepsilon.$$

$\square$

Theorem 4.2. *Let Ω be a σ-finite measure space and $u \in \mathbf{L}_\Omega^\varphi(X)$. Then the following assertions are equivalent:*

a) *u has absolutely continuous norm in the strong sense.*

b) *u has absolutely continuous norm in the weak sense, and there exists a sequence $\Omega_1 \subset \Omega_2 \subset \ldots$ such that $\mu(\Omega_k) < \infty$ and $N_\Omega^\varphi((1-\chi_{\Omega_k})u) \to 0$.*

c) *If $u_n \to u$ monotonically, then $N_\Omega^\varphi(u-u_n) \to 0$.*

Proof. a) $\Rightarrow$ c) Let $u_n = \alpha_n u$ with $\alpha_n \to 1$ monotonically a.e. Choose $\gamma < 1/N_\Omega^\varphi(u)$ such that

$$\int_\Omega \varphi(\gamma u(\omega))\,\mathrm{d}\mu(\omega) \leqslant 1.$$

Let $\varepsilon > 0$ be arbitrary. We define

$$E_n = \Big\{\omega \in \Omega \mid 1 - \alpha_n(\omega) > \frac{\varepsilon\gamma}{2}\Big\}.$$

Then $\chi_{E_n}(\omega) \to 0$ if $\alpha_n(\omega) \to 1$, i.e. almost everywhere. By a) we infer that for sufficiently large n we have

$$N_\Omega^\varphi(\chi_{E_n}(u-u_n)) \leqslant N_\Omega^\varphi(\chi_{E_n}u) \leqslant \frac{\varepsilon}{2}.$$

On the other hand,

$$\int_\Omega \varphi\Big(\frac{2}{\varepsilon}(1-\chi_{E_n})(u-u_n)\Big)\,\mathrm{d}\mu = \int_{\Omega\setminus E_n} \varphi\Big(\frac{2(1-\alpha_n)}{\varepsilon}u\Big)\,\mathrm{d}\mu \leqslant \int_{\Omega\setminus E_n} \varphi(\gamma u)\,\mathrm{d}\mu \leqslant 1$$

so that

$$N_\Omega^\varphi((1-\chi_{E_n})(u-u_n)) \leqslant \frac{\varepsilon}{2}.$$

Finally,

$$N_\Omega^\varphi(u-u_n) \leqslant N_\Omega^\varphi(\chi_{E_n}(u-u_n)) + N_\Omega^\varphi((1-\chi_{E_n})(u-u_n)) \leqslant \varepsilon.$$

c) $\Rightarrow$ b) First assume that $E_n \in \mathcal{A}$ with $\mu(E_n) \to 0$, but $N_\Omega^\varphi(\chi_{E_n}u)$ does not converge to 0. Taking subsequences if necessary, we may assume that $\mu(E_n) \leqslant 2^{-n}$ and $N_\Omega^\varphi(\chi_{E_n}u) \geqslant \varepsilon$ for some fixed $\varepsilon > 0$. Put $F_n = \bigcup\limits_{k\geqslant n} E_k$ and $u_n = (1-\chi_{F_n})u$. Notice that $\mu(F_n) \leqslant 2^{1-n} \to 0$ and $F_n \subset F_{n-1}$, so that $\chi_{F_n} \to 0$ monotonically a.e., i.e., $u_n \to u$ monotonically. Thus $N_\Omega^\varphi(u-u_n) \to 0$ by c). However,

$$N_\Omega^\varphi(u-u_n) = N_\Omega^\varphi(\chi_{F_n}u) \geqslant N_\Omega^\varphi(\chi_{E_n}u) \geqslant \varepsilon.$$

To show the second part of b), let Ω_k be any increasing sequence of sets of finite measure such that $\Omega = \bigcup\limits_k \Omega_k$. Such a sequence exists since Ω is sigma finite. Let $E_k = \Omega \setminus \Omega_k$, then $\chi_{E_k} \to 0$ almost everywhere. Thus, by c), $N^{\varphi}_{\Omega}((1-\chi_{\Omega_k})u) \to 0$.

b) $\Rightarrow$ a) Let $E_n \in \mathcal{A}$ with $\chi_{E_n} \to 0$ almost everywhere. Fix $\varepsilon > 0$. First we pick Ω_k such that $N^{\varphi}_{\Omega}((1-\chi_{\Omega_k})u) \leqslant \frac{1}{2}\varepsilon$. Since $\chi_{E_n} \to 0$ a.e. and $\mu(\Omega_k) < \infty$, we infer that $\mu(E_n \cap \Omega_k) \to 0$ for $n \to \infty$. For sufficiently large n we infer by b) that $N^{\varphi}_{\Omega}(\chi_{E_n\cap\Omega_k} u) \leqslant \frac{1}{2}\varepsilon$. Finally,

$$N^{\varphi}_{\Omega}(\chi_{E_n} u) \leqslant N^{\varphi}_{\Omega}(\chi_{E_n\cap\Omega_k} u) + N^{\varphi}_{\Omega}((1-\chi_{\Omega_k})u) \leqslant \varepsilon.$$

$\square$

5. The closure of $\mathcal{L}^{\infty}_{\Omega}(\mu)$

Definition 5.1. By $\mathbf{E}^{\varphi}_{\Omega}(X)$ we denote the set of all $u \in \mathcal{M}$ such that there exists a sequence of bounded functions $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathbf{L}^{\varphi}_{\Omega}(X)$ with $N^{\varphi}_{\Omega}(u-u_n) \to 0$ for $n \to \infty$.

Remark 5.1. $\mathbf{E}^{\varphi}_{\Omega}(X)$ is a linear subspace of $\mathbf{L}^{\varphi}_{\Omega}(X)$.

Proof. It is easy to see that $\mathbf{E}^{\varphi}_{\Omega}(X)$ is a linear space. To show that it is a subset of $\mathbf{L}^{\varphi}_{\Omega}(X)$, take an arbitrary $u \in \mathbf{E}^{\varphi}_{\Omega}(X)$. By definition there exists a $u_1 \in \mathbf{L}^{\varphi}_{\Omega}(X)$, u_1 is bounded and $N^{\varphi}_{\Omega}(u-u_1) \leqslant \frac{1}{2}$. Thus

$$\int_{\Omega} \varphi(2(u-u_1))\,\mathrm{d}\mu \leqslant 1, \quad \text{hence} \quad 2(u-u_1) \in \mathbf{L}^{\varphi}_{\Omega}(X).$$

Since $\mathbf{L}^{\varphi}_{\Omega}(X)$ is a linear space, we conclude that

$$u = \frac{1}{2}(2(u-u_1)) + \frac{1}{2}(2u_1) \in \mathbf{L}^{\varphi}_{\Omega}(X).$$

$\square$

Theorem 5.1. *Let φ be bounded on bounded subsets and $\mu(\Omega) < \infty$, then $\mathbf{E}^{\varphi}_{\Omega}(X) \subseteq \overline{\mathbf{L}}^{\varphi}_{\Omega}(X)$.*

Proof. Suppose that $u_1 \in \mathbf{E}^{\varphi}_{\Omega}(X)$, $\|u_1(\omega)\| \leqslant a$ for all $\omega \in \Omega$. Let $\alpha > 0$ be an arbitrary number. Define $C_1 := \sup\limits_{\|x\|\leqslant a} \varphi(\alpha x)$. Then we get

$$\int_{\Omega} \varphi(\alpha u_1)\,\mathrm{d}\mu \leqslant \mu(\Omega) C_1 < \infty, \quad \text{thus } \alpha u_1 \in \overline{\mathbf{L}}^{\varphi}_{\Omega}(X).$$

Now take an arbitrary $u \in \mathbf{E}^{\varphi}_{\Omega}(X)$. Then there exists a $u_1 \in \mathbf{E}^{\varphi}_{\Omega}(X)$, u_1 bounded and $N^{\varphi}_{\Omega}(u - u_1) \leqslant \frac{1}{2}$. Thus $2(u - u_1) \in \overline{\mathbf{L}}^{\varphi}_{\Omega}(X)$. Since $\overline{\mathbf{L}}^{\varphi}_{\Omega}(X)$ is convex we conclude that

$$u = \frac{1}{2}(2(u - u_1)) + \frac{1}{2}(2u_1) \in \overline{\mathbf{L}}^{\varphi}_{\Omega}(X).$$

□

Remark 5.2. Again, the theorem is not generally true if φ is not bounded on bounded subsets.

Example 5.1. Let $X = \mathbb{R}$, $\Omega = [0, 1]$, φ as in Example 4.1. Put $u(\omega) = \omega$ for all $\omega \in \Omega$. Then $\|u(\omega)\| \leqslant 1$ for all $\omega \in \Omega$ and $u \in \mathbf{L}^{\varphi}_{\Omega}(X)$. Thus $u \in \mathbf{E}^{\varphi}_{\Omega}(X)$, but $u \notin \overline{\mathbf{L}}^{\varphi}_{\Omega}(X)$.

Theorem 5.2. *Let φ satisfy the Δ_2-condition. Then $\mathbf{E}^{\varphi}_{\Omega}(X) = \mathbf{L}^{\varphi}_{\Omega}(X)$.*

Proof. Take an arbitrary $u \in \mathbf{L}^{\varphi}_{\Omega}(X)$ and an $\alpha > 0$ such that $\int_{\Omega} \varphi(\alpha u) \leqslant 1$. Since φ satisfies the Δ_2-condition, we can find an $M > 0$ and $L \geqslant 1$ such that $\varphi(2x) \leqslant L\varphi(x)$ for all $\|x\| \geqslant M$.

Put

$$u_n(\omega) = \begin{cases} u(\omega), & \|u(\omega)\| \leqslant n, \\ \dfrac{n}{\|u(\omega)\|} u(\omega), & \|u(\omega)\| > n. \end{cases}$$

We want to show that $N^{\varphi}_{\Omega}(u - u_n) \to 0$ for $n \to \infty$. This means that

$$\int_{\Omega} \varphi(\zeta_n(u - u_n)(\omega)) \,\mathrm{d}\mu(\omega) = \int_{\Omega} \varphi(\zeta_n \delta_n(\omega) u(\omega)) \,\mathrm{d}\mu(\omega) \leqslant 1$$

for a suitable sequence $\{\zeta_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}^+$ with $\zeta_n \to \infty$ for $n \to \infty$ and

$$\delta_n(\omega) = \begin{cases} 0, & \|u(\omega)\| \leqslant n, \\ \dfrac{\|u(\omega)\| - n}{\|u(\omega)\|}, & \|u(\omega)\| > n. \end{cases}$$

Take an arbitrary $\gamma > 1$ and $x \in X$ with $\|x\| \geqslant M$. Then the Δ_2-condition yields

$$\varphi(\gamma x) \leqslant \varphi\big(2^{\ln[\gamma]/\ln 2 + 1} x\big) \leqslant L^{\ln[\gamma]/\ln 2 + 1} \varphi(x) \leqslant L^{\ln \gamma/\ln 2 + 1} \varphi(x) \leqslant \gamma^k L \varphi(x)$$

with $k = \ln L / \ln 2$. Without loss of generality we may assume that $k \geqslant 1$. Thus

$$\varphi(\beta x) \leqslant \begin{cases} \beta \varphi(x), & \beta \leqslant 1, \\ L\beta^k \varphi(x), & \beta > 1. \end{cases}$$

For $n \geqslant M$ we can infer that

$$\varphi\Big(\frac{\zeta_n\delta_n(\omega)}{\alpha}(\alpha u(\omega))\Big) \leqslant L\frac{\zeta_n^k\delta_n(\omega)}{\alpha_0}\varphi(\alpha u(\omega)), \quad \alpha_0 = \min\{\alpha, \alpha^k\}.$$

Since $\int_\Omega \varphi(\alpha u)\,\mathrm{d}\mu \leqslant 1$ we know that $\varphi(\alpha u) \in \mathcal{L}^1(\Omega, \mathbb{R})$. Moreover, $\delta_n(\omega) \leqslant 1$ for all $\omega \in \Omega$ and $\delta_n(\omega) \downarrow 0$. Thus we get that $\delta_n\varphi(\alpha u) \in \mathcal{L}^1(\Omega, \mathbb{R})$ and $\|\delta_n\varphi(\alpha u)\|_{\mathcal{L}^1} \to 0$ for $n \to \infty$. We assume that $\delta_n\varphi(\alpha u)$ is not equal to zero almost everywhere, otherwise it would be trivial.

Now put $\zeta_n = (\|\delta_n\varphi(\alpha u)\|_{\mathcal{L}^1})^{-1/k}(\alpha_0/L)^{1/k}$. Then $\zeta_n \to \infty$ and

$$\begin{aligned}\int_\Omega \varphi(\zeta_n\delta_n(\omega)u(\omega))\,\mathrm{d}\mu(\omega) &\leqslant \frac{\zeta_n^k}{\alpha_0}L\int_\Omega \delta_n(\omega)\varphi(\alpha u(\omega))\,\mathrm{d}\mu \\ &= \frac{1}{\|\delta_n\varphi(\alpha u)\|_{\mathcal{L}^1}}\int_\Omega \delta_n(\omega)\varphi(\alpha u(\omega))\,\mathrm{d}\mu(\omega) = 1.\end{aligned}$$

□

Corollary 5.1. *Let φ be bounded on bounded subsets, $\mu(\Omega) < \infty$ and let μ be diffuse on an $E \in \mathcal{A}$ with $\mu(E) > 0$. Then $\mathbf{E}^\varphi_\Omega(X) = \mathbf{L}^\varphi_\Omega(X)$ iff $\varphi \in \Delta_2$.*

Proof. It remains to show that $\mathbf{E}^\varphi_\Omega(X) = \mathbf{L}^\varphi_\Omega(X)$ implies $\varphi \in \Delta_2$. Suppose that φ does not satisfy the Δ_2-condition. By Theorem 5.1, $\mathbf{E}^\varphi_\Omega(X)$ is a subset of $\overline{\mathbf{L}}^\varphi_\Omega(X)$. Since φ does not satisfy the Δ_2-condition, $\overline{\mathbf{L}}^\varphi_\Omega(X)$ is a proper subset of $\mathbf{L}^\varphi_\Omega(X)$ and thus so is $\mathbf{E}^\varphi_\Omega(X)$. □

The next remark is surprising, since it does not follow the classical Orlicz theory, where $\varphi \in \Delta_2$ is always equivalent to $\mathbf{E}^\varphi_\Omega(X) = \mathbf{L}^\varphi_\Omega(X)$.

Remark 5.3. If φ is not bounded on a bounded subset of X and φ does not satisfy the Δ_2-condition then it is not true in general that $\mathbf{E}^\varphi_\Omega(X)$ is a proper subset of $\mathbf{L}^\varphi_\Omega(X)$.

Example 5.2. Put $\Omega = [0,1]$, $X = \mathbb{R}^2$, let μ be the Lebesgue measure and define φ by

$$\varphi((x,y)) = \begin{cases} |x| + |y| & \text{if } |x| \leqslant 1, \\ \infty & \text{if } |x| > 1. \end{cases}$$

Then φ is convex and lower semi-continuous. We claim that $\varphi \notin \Delta_2$. Put $x_n = 1$ and $y_n = n$. Then $\|(x_n, y_n)\| \to \infty$ for $n \to \infty$ and

$$\infty = \varphi(2(x_n, y_n)) > n\varphi((x_n, y_n)) \quad \text{for all } n \in \mathbb{N} \text{ since } |2x_n| = 2.$$

Considering $\mathbf{L}^{\varphi}_{\Omega}(X)$ we see that

$$\mathbf{L}^{\varphi}_{\Omega}(X) = \mathcal{L}^{\infty}([0,1],\mathbb{R}) \times \mathcal{L}^{1}([0,1],\mathbb{R}).$$

Let $(u_1, u_1) \in \mathbf{L}^{\varphi}_{\Omega}(X)$. Put

$$u_{2,n}(s) = \begin{cases} u_2(s) & \text{if } |u_2(s)| \leqslant n, \\ 0 & \text{else.} \end{cases}$$

Since $|(u_2 - u_{2,n})(s)| \leqslant |u_2(s)|$ and $|u_2 - u_{2,n}| \to 0$ almost everywhere we get by the Lebesgue Theorem and by the fact that $u_2 \in \mathcal{L}^1([0,1],\mathbb{R})$ that

$$\|u_2 - u_{2,n}\|_{\mathcal{L}^1([0,1],\mathbb{R})} = \int_0^1 |(u_2 - u_{2,n})(s)| \, \mathrm{d}s \to 0 \quad \text{for } n \to \infty.$$

Put $(u_{1,n}, u_{2,n}) = (u_1, u_{2,n})$ and $\varepsilon_n = \|u_2 - u_{2,n}\|_{\mathcal{L}^1([0,1],\mathbb{R})}$. Then $u_n = (u_{1,n}, u_{2,n}) \in \mathbf{E}^{\varphi}_{\Omega}(X)$ and

$$\int_0^1 \varphi\Big(\frac{1}{\varepsilon_n}((u_1 - u_{1,n})(s), (u_2 - u_{2,n})(s))\Big) \, \mathrm{d}s = \frac{1}{\varepsilon_n} \int_0^1 |(u_2 - u_{2,n})(s)| \, \mathrm{d}s = 1.$$

Hence every $u = (u_1, u_2) \in \mathbf{L}^{\varphi}_{\Omega}(X)$ is in $\mathbf{E}^{\varphi}_{\Omega}(X)$.

Theorem 5.3. *If u has absolutely continuous norm in the strong sense, then $u \in \mathbf{E}^{\varphi}_{\Omega}(X)$. If in addition Ω is a finite measure space the same is true for any u with absolutely continuous norm in the weak sense, i.e., $\mathbf{C}^{\varphi}_{\Omega}(X) \subseteq \mathbf{E}^{\varphi}_{\Omega}(X)$.*

Proof. Let u have absolutely continuous norm in the strong sense. Put $E_n = \{\omega \in \Omega \mid |u(\omega)| > n\}$. Then $\chi_{E_n}(\omega) \to 0$ almost everywhere. Put $u_n = u(1 - \chi_{E_n})$. Evidently u_n is bounded and $N^{\varphi}_{\Omega}(u - u_n) = N^{\varphi}_{\Omega}(\chi_{E_n} u) \to 0$ since the norm is absolutely continuous.

Let $u \in \mathbf{C}^{\varphi}_{\Omega}(X)$, put $\Omega_n = \{\omega \in \Omega \mid \|u(\omega)\| \leqslant n\}$. Then $\mu(\Omega \setminus \Omega_n) \to 0$ for $n \to \infty$. Let $u_n(\omega) = \chi_{\Omega_n}(\omega) u(\omega)$. Then $u_n \in \mathbf{L}^{\varphi}_{\Omega}(X)$, it is bounded and

$$N^{\varphi}_{\Omega}(u - u_n) = N^{\varphi}_{\Omega}(\chi_{\Omega \setminus \Omega_n} u) \to 0 \quad \text{for } n \to \infty.$$

$\square$

Theorem 5.4. *If φ is bounded on bounded subsets and $\mu(\Omega) < \infty$ then $\mathbf{C}^{\varphi}_{\Omega}(X) = \mathbf{E}^{\varphi}_{\Omega}(X)$.*

P r o o f. 1) Let $u \in \mathbf{E}^{\varphi}_{\Omega}(X)$ be bounded, $\|u(\omega)\| \leqslant a$ for all $\omega \in \Omega$. Take an arbitrary $\varepsilon > 0$, put $C := \sup\limits_{\|x\|\leqslant a} \varphi(x/\varepsilon)$. Suppose, that $C \neq 0$ (otherwise it would be trivial). For an arbitrary $E \in \mathcal{A}$ we get

$$\int_{\Omega} \varphi\Big(\frac{1}{\varepsilon} u\chi_E\Big)\,\mathrm{d}\mu = \int_E \varphi\Big(\frac{1}{\varepsilon} u\Big)\,\mathrm{d}\mu \leqslant \mu(E)C \leqslant 1$$

whenever $\mu(E) \leqslant \delta = 1/C$.

2) Take an arbitrary $u \in \mathbf{E}^{\varphi}_{\Omega}(X)$, let $\varepsilon > 0$. By definition we can find a bounded $u_1 \in \mathbf{L}^{\varphi}_{\Omega}(X)$ with $N^{\varphi}_{\Omega}(u - u_1) \leqslant \frac{1}{2}\varepsilon$. Part 1) implies the existence of a $\delta > 0$ such that $N^{\varphi}_{\Omega}(\chi_E u) \leqslant \frac{1}{2}\varepsilon$ for all $E \in \mathcal{A}$, $\mu(E) < \delta$. Thus

$$N^{\varphi}_{\Omega}(u\chi_E) \leqslant N^{\varphi}_{\Omega}(u - u_1) + N^{\varphi}_{\Omega}(\chi_E u_1) \leqslant \varepsilon \quad \text{for all } E \in \mathcal{A} \text{ with } \mu(E) \leqslant \delta.$$

$\square$

R e m a r k 5.4. If φ is not bounded on a bounded subset, then it is not true in general that $\mathbf{E}^{\varphi}_{\Omega}(X) = \mathbf{C}^{\varphi}_{\Omega}(X)$, even if φ in Δ_2.

E x a m p l e 5.3. Take $X = \mathbb{R}$, $\Omega = [0,1]$, μ the Lebesgue measure and φ as in Example 4.1. Put $u(\omega) = \omega$, $\omega \in \Omega$. Take $\varepsilon = 1$ and put $E_n := [1 - 2^n, 1)$. Then $E_n \in \mathcal{A}$ and for any positive number δ we can find an $n \in \mathbb{N}$ such that $\mu(E_n) = 2^n \leqslant \delta$ but $N^{\varphi}_{\Omega}(u\chi_{E_n}) > 1$.

Corollary 5.2. *If φ is bounded on bounded subsets of X, $\mu(\Omega) < \infty$ and μ is diffuse on a subset $E \in \mathcal{A}$ with positive measure, then every $u \in \mathbf{L}^{\varphi}_{\Omega}(X)$ has absolutely continuous norm in the weak sense if and only if $\varphi \in \Delta_2$.*

P r o o f. By Theorem 5.4, $\mathbf{E}^{\varphi}_{\Omega}(X) = \mathbf{C}^{\varphi}_{\Omega}(X)$. Thus, by Corollary 5.1, $\mathbf{C}^{\varphi}_{\Omega}(X) = \mathbf{L}^{\varphi}_{\Omega}(X)$ if and only if $\varphi \in \Delta_2$. $\square$

Corollary 5.3. *Assume that φ is bounded on bounded subsets and $\mu(\Omega) < \infty$. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathbf{E}^{\varphi}_{\Omega}(X)$ converge monotonically to some $u \in \mathbf{E}^{\varphi}_{\Omega}(X)$. Then $N^{\varphi}_{\Omega}(u - u_n) \to 0$.*

P r o o f. By Theorem 5.4, $\mathbf{E}^{\varphi}_{\Omega}(X) = \mathbf{C}^{\varphi}_{\Omega}(X)$. Theorem 4.1 ensures that every monotonically convergent sequence in $\mathbf{C}^{\varphi}_{\Omega}(X)$ converges also in N^{φ}_{Ω}. $\square$

We are going to investigate the relation between $\mathbf{E}^{\varphi}_{\Omega}(X)$ and $\overline{\mathbf{L}}^{\varphi}_{\Omega}(X)$ in order to establish some useful properties for $\overline{\mathbf{L}}^{\varphi}_{\Omega}(X)$, especially for a φ which is bounded on bounded subsets of X.

Definition 5.2. For any $u \in \mathbf{L}^{\varphi}_{\Omega}(X)$ we define

$$d_{\mathbf{E}^{\varphi}_{\Omega}(X)}(u) := \inf\{N^{\varphi}_{\Omega}(u-v),\ \ v \in \mathbf{E}^{\varphi}_{\Omega}(X)\}.$$

It has the following properties:

1) $d_{\mathbf{E}^{\varphi}_{\Omega}(X)}(u+v) \leqslant d_{\mathbf{E}^{\varphi}_{\Omega}(X)}(u) + d_{\mathbf{E}^{\varphi}_{\Omega}(X)}(v)$ for all $u, v \in \mathbf{L}^{\varphi}_{\Omega}(X)$,
2) $d_{\mathbf{E}^{\varphi}_{\Omega}(X)}(\beta u) = |\beta| d_{\mathbf{E}^{\varphi}_{\Omega}(X)}(u)$ for all $u \in \mathbf{L}^{\varphi}_{\Omega}(X)$ and for all $\beta \in \mathbb{R}$.

Theorem 5.5. *Assume that φ is bounded on bounded subsets, $\mu(\Omega) < \infty$ and $u \in \mathbf{L}^{\varphi}_{\Omega}(X)$. We set*

$$u_n(\omega) = \begin{cases} u(\omega), & \|u(\omega)\| \leqslant n, \\ 0 & \textit{else.} \end{cases}$$

Then $d_{\mathbf{E}^{\varphi}_{\Omega}(X)}(u) = \liminf\limits_{n\to\infty} N^{\varphi}_{\Omega}(u-u_n)$.

P r o o f. The proof follows the proof of [4], Proposition 3, pages 92–93. □

Definition 5.3. We define the following subsets of $\mathbf{L}^{\varphi}_{\Omega}(X)$:

$$\begin{aligned} \mathcal{S}^{\varphi} &:= \{u \in \mathbf{L}^{\varphi}_{\Omega}(X) \mid d_{\mathbf{E}^{\varphi}_{\Omega}(X)}(u) < 1\}, \\ \overline{\mathcal{S}}^{\varphi} &:= \{u \in \mathbf{L}^{\varphi}_{\Omega}(X) \mid d_{\mathbf{E}^{\varphi}_{\Omega}(X)}(u) \leqslant 1\}. \end{aligned}$$

Theorem 5.6. *If φ is bounded on bounded subsets, $\mu(\Omega) < \infty$, then*

$$\mathcal{S}^{\varphi} \subseteq \overline{\mathbf{L}}^{\varphi}_{\Omega}(X) \subseteq \overline{\mathcal{S}}^{\varphi}.$$

If in addition $\varphi \notin \Delta_2$ and μ is diffuse on a subset $E \in \mathcal{A}$ with positive measure, then the inclusions are proper.

P r o o f. 1) $\mathcal{S}^{\varphi} \subseteq \overline{\mathbf{L}}^{\varphi}_{\Omega}(X)$: Take an arbitrary $u \in \mathcal{S}^{\varphi}$. By definition we can find an $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < 1$ and a $u_1 \in \mathbf{E}^{\varphi}_{\Omega}(X)$ bounded and such that $N^{\varphi}_{\Omega}(u-u_1) \leqslant \varepsilon$, e.g. $\varepsilon^{-1}(u-u_1) \in \overline{\mathbf{L}}^{\varphi}_{\Omega}(X)$. Thus

$$u = \varepsilon\Big(\frac{1}{\varepsilon}(u-u_1)\Big) + (1-\varepsilon)\Big(\frac{1}{1-\varepsilon}\,u_1\Big) \in \overline{\mathbf{L}}^{\varphi}_{\Omega}(X).$$

2) $\overline{\mathbf{L}}^{\varphi}_{\Omega}(X) \subseteq \overline{\mathcal{S}}^{\varphi}$: Let $u \in \overline{\mathbf{L}}^{\varphi}_{\Omega}(X)$, e.g. $\int_{\Omega} \varphi(u)\,\mathrm{d}\mu < \infty$. Put

$$u_n(\omega) = \begin{cases} u(\omega), & \|u(\omega)\| \leqslant n, \\ 0 & \text{else.} \end{cases}$$

Obviously $u_n \in \mathbf{E}^{\varphi}_{\Omega}(X)$ and u_n converges to u almost everywhere. Thus $\lim\limits_{n\to\infty} \varphi(u - u_n) = 0$ almost everywhere. By the construction of u_n, we have that $\varphi(u-u_n) \leqslant$

$\varphi(u)$. By the Lebesgue Theorem we conclude that $\lim\limits_{n\to\infty} \int_\Omega \varphi(u-u_n)\,\mathrm{d}\mu = 0$. However, this means that we can find an $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ we have $\int_\Omega \varphi(u-u_n)\,\mathrm{d}\mu \leqslant 1$, which implies that $N^\varphi_\Omega(u-u_n) \leqslant 1$. By Theorem 5.5 we infer that

$$d_{\mathbf{E}^\varphi_\Omega(X)}(u) = \lim_{n\to\infty} N^\varphi_\Omega(u-u_n) \leqslant 1 \quad \text{for all } n \geqslant n_0.$$

It remains to show that the inclusions are proper if $\varphi \notin \Delta_2$.

a) $\mathcal{S}^\varphi$ is a proper subset of $\overline{\mathbf{L}}^\varphi_\Omega(X)$: By Corollary 3.1 we can construct a $u \in \overline{\mathbf{L}}^\varphi_\Omega(X)$ with $\beta u \notin \overline{\mathbf{L}}^\varphi_\Omega(X)$ for all $\beta > 1$. Suppose that u is in $\mathcal{S}^\varphi$. Then we can find a $\beta > 1$ such that

$$d_{\mathbf{E}^\varphi_\Omega(X)}(\beta u) = \beta \inf\{N^\varphi_\Omega(u-v),\ v \in \mathbf{E}^\varphi_\Omega(X)\} < 1.$$

This together with 1) implies that $\beta u \in \overline{\mathbf{L}}^\varphi_\Omega(X)$ in contradiction to the construction of u.

b) $\overline{\mathbf{L}}^\varphi_\Omega(X)$ is a proper subset of $\overline{\mathcal{S}}^\varphi$: By Corollary 3.1 we can find a $u \in \mathbf{L}^\varphi_\Omega(X)$ with $\beta u \notin \overline{\mathbf{L}}^\varphi_\Omega(X)$ for all $\beta \geqslant 1$ and $\beta u \in \overline{\mathbf{L}}^\varphi_\Omega(X)$ for all $\beta < 1$. Then $d_{\mathbf{E}^\varphi_\Omega(X)}(u) \geqslant 1$, otherwise we get a contradiction with $\mathcal{S}^\varphi \subseteq \mathbf{L}^\varphi_\Omega(X)$. Moreover, $d_{\mathbf{E}^\varphi_\Omega(X)}(u) \leqslant 1$ and so $u \in \overline{\mathcal{S}}^\varphi$, since otherwise there is a $\beta \in (0,1)$ such that $d_{\mathbf{E}^\varphi_\Omega(X)}(\beta u) > 1$ and we obtain a contradiction with $\overline{\mathbf{L}}^\varphi_\Omega(X) \subseteq \overline{\mathcal{S}}^\varphi$. $\square$

6. Completeness and separability of $\mathbf{L}^\varphi_\Omega(X)$

Theorem 6.1. *Let $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathbf{L}^\varphi_\Omega(X)$ be a Cauchy sequence, i.e. for every $\varepsilon > 0$ we can find an $n_0 \in \mathbb{N}$ such that $N^\varphi_\Omega(u_{n+m}-u_n) < \varepsilon$ for all $n \geqslant n_0$ and $m \geqslant 1$. Then there exists a $u \in \mathbf{L}^\varphi_\Omega(X)$ such that $N^\varphi_\Omega(u-u_n) \to 0$ for $n \to 0$.*

Proof. Fix a $\delta > 0$ and an $\varepsilon > 0$. Choose an $\alpha > 0$ such that $\varphi(\alpha x) > 2/\delta$ if $\|x\| \geqslant \varepsilon$, which is possible since $\varphi(x) \to \infty$ for $\|x\| \to \infty$. Let n_0 be large enough so that

$$N^\varphi_\Omega(u_n-u_m) < \alpha^{-1}$$

for $m, n \geqslant n_0$, i.e.,

$$\int_\Omega \varphi(\alpha(u_n-u_m))\,\mathrm{d}\mu \leqslant 1.$$

Put

$$E = \Big\{\omega \mid \varphi(\alpha(u_n(\omega)-u_m(\omega))) > \frac{2}{\delta}\Big\}.$$

Then

$$\mu(E)\frac{2}{\delta} < \int_\Omega \varphi(\alpha(u_n-u_m))\,\mathrm{d}\mu \leqslant 1,$$

i.e.,

$$\mu\Big(\Big\{\omega \mid \varphi(\alpha(u_n(\omega) - u_m(\omega))) > \frac{2}{\delta}\Big\}\Big) < \frac{\delta}{2}.$$

Consequently,

$$\mu\Big(\Big\{\omega \mid \|u_n(\omega) - u_m(\omega)\| \geqslant \varepsilon\Big\}\Big) < \frac{\delta}{2},$$

which shows that $\{u_n\}$ is a Cauchy sequence in measure.

The rest of the proof is standard. $\square$

Theorem 6.2. *Suppose that φ is bounded on bounded subsets, $\mu(\Omega) < \infty$, μ is diffuse on a subset with positive measure and $\varphi \notin \Delta_2$. Then $\mathbf{L}^\varphi_\Omega(X)$ is not separable.*

Proof. Suppose that $\{u_n\}_{n\in\mathbb{N}}$ is a dense sequence in $\mathbf{L}^\varphi_\Omega(X)$. Take $E \subset \Omega$ with positive measure such that μ is diffuse on E. We choose a monotone sequence $M_i > 0$ such that $\mu(G_i) < 2^{-i-1}\mu(E)$ where

$$G_i = \{\omega \mid \|u_i(\omega)\| > M_i\}.$$

If $G = \bigcup\limits_{i=1}^{\infty} G_i$, then $\mu(G) \leqslant \frac{1}{2}\mu(E)$. So μ is a diffuse measure on the set $E_1 = (\Omega \setminus G) \cap E$ while $\mu(E_1) > 0$. By Corollary 5.1 we infer that $\mathbf{L}^\varphi_{\Omega\setminus G}(X) \neq \mathbf{E}^\varphi_{\Omega\setminus G}(X)$, which contradicts the fact that the restrictions of u_i to $\Omega \setminus G$ are bounded. $\square$

Theorem 6.3. *If Ω is a compact metric space with finite measure and φ is bounded on bounded subsets of X, then $\mathbf{E}^\varphi_\Omega(X)$ is separable.*

Proof. Take a $u \in \mathbf{E}^\varphi_\Omega(X)$ bounded, $\|u(\omega)\| \leqslant a$ for all $\omega \in \Omega$. Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary small numbers. By the Lusin Theorem we can find a compact $\Omega_1 \subseteq \Omega$, $\mu(\Omega \setminus \Omega_1) \leqslant \delta$, and a continuous function $u_1\colon \Omega \to X$ with $u_1(\omega) = u(\omega)$ for all $\omega \in \Omega_1$, $\|u_1(\omega)\| \leqslant a$ for all $\omega \in \Omega$, such that

$$\int_\Omega \varphi\Big(\frac{1}{\varepsilon}(u - u_1)\Big)\,\mathrm{d}\mu = \int_{\Omega\setminus\Omega_1} \varphi\Big(\frac{1}{\varepsilon}(u - u_1)\Big)\,\mathrm{d}\mu \leqslant \mu(\Omega \setminus \Omega_1)C \leqslant 1$$

for $\delta \leqslant 1/C$ and $C = \sup\limits_{\|x\|\leqslant 2a} \varphi(\frac{1}{\varepsilon}x)$.

For an arbitrary $v \in \mathbf{E}^\varphi_\Omega(X)$ we can find a bounded $u \in \mathbf{E}^\varphi_\Omega(X)$ with $N^\varphi_\Omega(v - u) \leqslant \frac{1}{2}\varepsilon$. By the above arguments, we can find a continuous u_1 with $N^\varphi_\Omega(u - u_1) \leqslant \frac{1}{2}\varepsilon$. Thus

$$N^\varphi_\Omega(v - u_1) \leqslant N^\varphi_\Omega(v - u) + N^\varphi_\Omega(u - u_1) \leqslant \varepsilon.$$

Since Ω is a compact metric space, we can find a countable dense subset D of $\mathcal{C}(\Omega, X)$ such that every continuous function can be approximated uniformly by functions in D.

By the uniform convergence and the boundedness of continuous functions we get, using the same arguments as above, the approximation in N^φ_Ω. $\square$

Corollary 6.1. *If Ω is a compact metric space, φ is bounded on bounded subsets of X, $\mu(\Omega) < \infty$, then $\mathbf{L}^{\varphi}_{\Omega}(X)$ is separable iff $\varphi \in \Delta_2$.*

Proof. By Theorem 6.2, $\varphi \notin \Delta_2$ implies that $\mathbf{L}^{\varphi}_{\Omega}(X)$ is not separable. For the other implication suppose that φ does satisfy the Δ_2-condition. By Theorem 5.2 $\mathbf{L}^{\varphi}_{\Omega}(X) = \mathbf{E}^{\varphi}_{\Omega}(X)$ which is separable by Theorem 6.2. □

7. Duality properties of $\mathbf{L}^{\varphi}_{\Omega}(X)$

Definition 7.1. By $(\mathbf{L}^{\varphi}_{\Omega}(X))^*$ we denote the set of all functions $F\colon \mathbf{L}^{\varphi}_{\Omega}(X) \to \mathbb{R}$ with the following properties:

1) F is linear and
2) there exists an $M > 0$ such that $|F(u)| \leqslant M N^{\varphi}_{\Omega}(u)$ for all $u \in \mathbf{L}^{\varphi}_{\Omega}(X)$, $O_{\varphi}(F) := \inf\{M > 0, \forall u \in \mathbf{L}^{\varphi}_{\Omega}(X)\colon\ |F(u)| \leqslant M N^{\varphi}_{\Omega}(u)\}$.

Since monotone convergence plays an important role, we want to consider those functions which have the "monotone convergence" property separately.

Definition 7.2. An $F \in (\mathbf{L}^{\varphi}_{\Omega}(X))^*$ is said to have the *monotone convergence property*, iff for every $u \in \mathbf{L}^{\varphi}_{\Omega}(X)$ and for every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathbf{L}^{\varphi}_{\Omega}(X)$ which converges monotonically to u we have that $F(u_n) \to F(u)$.

We denote by $\mathbf{P}^{\varphi^*}_{\Omega}(X) \subseteq (\mathbf{L}^{\varphi}_{\Omega}(X))^*$ the subset of all functions with the monotone convergence property.

Theorem 7.1. *If φ is bounded on bounded subsets and satisfies the Δ_2-condition, $\mu(\Omega) < \infty$, then*

$$(\mathbf{L}^{\varphi}_{\Omega}(X))^* = \mathbf{P}^{\varphi^*}_{\Omega}(X).$$

Proof. Suppose that $F \in (\mathbf{L}^{\varphi}_{\Omega}(X))^*$. Take a $u \in \mathbf{L}^{\varphi}_{\Omega}(X)$ and a sequence $\{u_n\} \subseteq \mathbf{L}^{\varphi}_{\Omega}(X)$ which converges monotonically to u. By Theorem 5.2 and Corollary 5.3 every sequence which is monotonically convergent, is also norm convergent. Thus $N^{\varphi}_{\Omega}(u - u_n) \to 0$ for $n \to \infty$ and we get that $F(u) = \lim\limits_{n\to\infty} F(u_n)$ and hence every $F \in (\mathbf{L}^{\varphi}_{\Omega}(X))^*$ has the monotone convergence property. □

Theorem 7.2. *For an arbitrary $v^* \in \mathbf{L}^{\varphi}_{\Omega}(X^*)$ the function*

$$F(u) := \int_{\Omega} \langle v^*(\omega), u(\omega)\rangle \,\mathrm{d}\mu(\omega), \quad u \in \mathbf{L}^{\varphi}_{\Omega}(X)$$

is in $\mathbf{P}^{\varphi^*}_{\Omega}(X)$ and $O_{\varphi}(F) \leqslant 2N^{\varphi^*}_{\Omega}(v^*)$.

P r o o f. The proof for $X = \mathbb{R}^N$ is given in [3].

We want to prove the theorem for an arbitrary Banach space.

Take an arbitrary $u \in \mathbf{L}_\Omega^\varphi(X)$. Suppose that $N_\Omega^\varphi(u) \leqslant 1$ and $N_\Omega^{\varphi^*}(v^*) \leqslant 1$. By the Fenchel inequality we get

$$\langle v^*(\omega), u(\omega)\rangle \leqslant \varphi^*(v^*(\omega)) + \varphi(u(\omega)).$$

So we always have

$$\int_\Omega \langle v^*(\omega), u(\omega)\rangle \,\mathrm{d}\mu(\omega) \leqslant \int_\Omega (\varphi^*(v^*(\omega)) + \varphi(u(\omega)))\,\mathrm{d}\mu(\omega).$$

Thus

$$F(u) = \int_\Omega \langle v^*(\omega)\,\mathrm{d}\mu(\omega), u(\omega)\rangle \in [-\infty, \infty) \quad \text{and} \quad F(u) \leqslant 2.$$

F is positive homogeneous and additive, also the monotone convergence property is clear. □

R e m a r k 7.1. Assume that $\mu(\Omega) < \infty$, μ is diffuse on Ω and φ is coercive. Moreover, let $v^* \in \mathcal{L}^1(\Omega, X)$ and let each piecewise constant function $u \in \mathbf{L}_\Omega^\varphi(X)$ satisfy

$$\int_\Omega \langle v^*(\omega), u(\omega)\rangle \,\mathrm{d}\mu(\omega) \leqslant M N_\Omega^\varphi(u).$$

Then $v^* \in \mathbf{L}_\Omega^{\varphi^*}(X^*)$ with $N_\Omega^{\varphi^*}(v^*) \leqslant M$.

P r o o f. Again, the proof for $X = \mathbb{R}^N$ is given in [3].

Assume without loss of generality that $M = 1$. Regard the approximating step functions

$$v_{n,i}^* = \frac{n}{\mu(\Omega)} \int_{E_i} v^*(\omega)\,\mathrm{d}\mu(\omega), \ \ n \in \mathbb{N}, \ \ E_i \in \mathcal{A} \text{ disjoint}, \ \ \Omega = \bigcup_{i=1}^n E_i, \ \ \mu(E_i) = \frac{\mu(\Omega)}{n},$$

$$v_n^* = \sum_{i=1}^n v_{n,i}^* \chi_{E_i}.$$

Let $u \in \mathbf{L}_\Omega^\varphi(X)$ be a step function, $N_\Omega^\varphi(u) \leqslant 1$ and let u_n be the same approximation as above for u. Then

$$\int_\Omega \langle v_n^*(\omega), u(\omega)\rangle \,\mathrm{d}\mu(\omega) = \int_\Omega \langle v^*(\omega), u_n(\omega)\rangle \,\mathrm{d}\mu(\omega).$$

Since u is a step function, we can find $F_i \in \mathcal{A}$, $i = 1, \ldots, m$, F_i disjoint and $x_i \in X$, $i = 1, \ldots, m$ with

$$u = \sum_{i=1}^m x_i \chi_{F_i}.$$

For step functions we have a generalized Jensen inequality:

$$\varphi\Big(\frac{1}{\mu(\Omega)}\int_\Omega u(\omega)\,\mathrm{d}\mu(\omega)\Big) = \varphi\Big(\sum_{i=1}^m x_i\frac{\mu(F_i)}{\mu(\Omega)}\Big) \leqslant \frac{1}{\mu(\Omega)}\sum_{i=1}^m \varphi(x_i)\mu(F_i) = \frac{1}{\mu(\Omega)}\int_\Omega \varphi(u)\,\mathrm{d}\mu.$$

Thus we have

$$\int_\Omega \varphi(u_n)\,\mathrm{d}\mu = \sum_{i=1}^n \mu(E_i)\varphi\Big(\frac{1}{\mu(E_i)}\int_{E_i} u(\omega)\,\mathrm{d}\mu(\omega)\Big) \leqslant \sum_{i=1}^n \int_{E_i}\varphi(u) = \int_\Omega \varphi(u)\,\mathrm{d}\mu \leqslant 1.$$

Since φ is coercive we get that φ^* is continuous. Thus we can find for each $v^*_{n,i}$ a $z_{n,i}\in X$ such that

$$\langle v^*_{n,i}, z_{n,i}\rangle = \varphi^*(v^*_{n,i}) + \varphi(z_{n,i}).$$

1) $\mu(\Omega)n^{-1}\sum\limits_{i=1}^n \varphi(z_{n,i}) > 1$. Then we can find a $\beta < 1$ such that

$$\frac{\mu(\Omega)}{n}\sum_{i=1}^n \varphi(\beta z_{n,i}) = 1.$$

Put

$$u = \sum_{i=1}^n \beta z_{n,i}\chi_{E_i} \quad \text{then} \quad \int_\Omega \varphi(u)\,\mathrm{d}\mu \leqslant 1.$$

Then

$$\frac{\mu(\Omega)}{n}\sum_{i=1}^n \varphi^*(v^*_{n,i}) = \frac{\mu(\Omega)}{n}\Big[\sum_{i=1}^n \langle v^*_{n,i}, z_{n,i}\rangle - \sum_{i=1}^n \varphi(z_{n,i})\Big] \leqslant \frac{1}{\beta}\Big[\int_\Omega \langle v^*_n(\omega), u(\omega)\rangle\,\mathrm{d}\mu(\omega) - \frac{\mu(\Omega)}{n}\sum_{i=1}^n \varphi(\beta z_{n,i})\Big] \leqslant 0.$$

2) $\mu(\Omega)n^{-1}\sum\limits_{i=1}^n \varphi(z_{n,i}) \leqslant 1$. With $\beta = 1$ and by the same computation as in 1) we get that

$$\frac{\mu(\Omega)}{n}\sum_{i=1}^n \varphi^*(v^*_{n,i}) \leqslant 1 - 0 = 1.$$

In both cases we have $\int_\Omega \varphi^*(v_n)\,\mathrm{d}\mu \leqslant 1$.

Since φ^* is LSC and $v^*_n \to v$ almost everywhere we can conclude that $\varphi^*(v^*(\omega)) \leqslant \liminf\limits_{n\to\infty} \varphi^*(v^*_n(\omega))$. By Fatou's Lemma we get $\int_\Omega \varphi^*(v^*)\,\mathrm{d}\mu \leqslant 1$. □

Theorem 7.3. *Suppose that φ is coercive and bounded on bounded subsets of X, Ω is a finite measure space and μ is diffuse on Ω. Then we have with the identification of Theorem* 7.2

$$\mathbf{P}_\Omega^{\varphi^*}(X) = \mathbf{L}_\Omega^{\varphi^*}(X^*).$$

Proof. The proof for $X = \mathbb{R}^N$ is given in [3].

1) Take an arbitrary F in $\mathbf{P}_\Omega^{\varphi^*}(X)$. We want to prove the existence of a $v^* \in \mathbf{L}_\Omega^{\varphi^*}(X^*)$ such that

$$F(u) = \int_\Omega \langle v^*(\omega), u(\omega)\rangle \,\mathrm{d}\mu(\omega).$$

Without loss of generality we may assume that $O_\varphi(F) \leqslant 1$.

For an $E \in \mathcal{A}$ we define a functional x_E^* by $x_E^*(x) := F(x\chi_E)$. We want to show that $x_E^* \in X^*$. It is clear that $x_E^*\colon X \to \mathbb{R}$ and linear. Is still remains to show that x_E^* is continuous. Define $M := \sup\{\varphi(y),\ \|y\| \leqslant 1\}$. Without loss of generality, we may assume that $M\mu(\Omega) \geqslant 1$. Then

$$\begin{aligned}\int_\Omega \varphi\Big(\frac{\pm x\chi_E}{\|x\| M\mu(\Omega)}\Big)\,\mathrm{d}\mu &= \varphi\Big(\frac{\pm x}{\|x\| M\mu(\Omega)}\Big)\mu(E)\\ &\leqslant \frac{1}{M\mu(\Omega)}\varphi\Big(\frac{\pm x}{\|x\|}\Big)\mu(E) \leqslant \frac{1}{M\mu(\Omega)} M\mu(E) \leqslant 1.\end{aligned}$$

Thus $N_\Omega^\varphi(\pm x\chi_E) \leqslant M\mu(\Omega)$ and $|x_E^*(x)| \leqslant N_\Omega^\varphi(x\chi_E) \leqslant M\mu(\Omega)$ if $\|x\| \leqslant 1$.

Now define the function

$$\begin{cases}\tau\colon \mathcal{A} \to X^*,\\ \tau(E) = x_E^*,\end{cases}$$

which is a vector valued measure, i.e. τ is σ-additive, since F has the monotone convergence property.

We want to work with the Radon-Nikodym property of the reflexive Banach space X^*. To be able to do so, we have to ensure that τ is of bounded variation. We first define the variation of τ versus φ^*, i.e.

$$V^{\varphi^*}(\tau, E) := \sup\Big\{\sum_{i=1}^n \varphi^*\Big(\frac{\tau(E_i)}{\mu(E_i)}\Big)\mu(E_i),\ \mu(E_i) \neq 0,\ \{E_i\}_{i=1}^n \in P(E)\Big\},$$

when $P(E)$ denotes the family of all finite partitions of E. We claim that $V^{\varphi^*}(\tau, E)$ is always smaller than one, independently of the choice of $E \in \mathcal{A}$. To prove the assertion, take an arbitrary $\{E_i\}_{i=1}^n \in P(E)$ and $\mu(E_i) \neq 0$. Since φ is bounded on bounded subsets of X, we can find $x_i \in X$ such that

$$\Big\langle \frac{\tau(E_i)}{\mu(E_i)}, x_i\Big\rangle = \varphi^*\Big(\frac{\tau(E_i)}{\mu(E_i)}\Big) + \varphi(x_i).$$

Define a function $u \in \mathbf{L}_{\Omega}^{\varphi}(X)$ by $u = \sum\limits_{i=1}^{n} x_i \chi_{E_i}$. Since $\int_\Omega \varphi(u) < \infty$ we get the following estimate:

$$
\begin{aligned}
1 + \int_\Omega \varphi(u)\,\mathrm{d}\mu \geqslant N_\Omega^\varphi(u) \geqslant F(u) &= \sum_{i=1}^{n} F(x_i \chi_{E_i}) = \sum_{i=1}^{n} \langle \tau(E_i), x_i \rangle \\
&= \sum_{i=1}^{n} \Big\langle \frac{\tau(E_i)}{\mu(E_i)}, x_i \Big\rangle \mu(E_i) \\
&= \sum_{i=1}^{n} \varphi^* \Big(\frac{\tau(E_i)}{\mu(E_i)} \Big) \mu(E_i) + \int_\Omega \varphi(u)\,\mathrm{d}\mu.
\end{aligned}
$$

Thus

$$
\sum_{i=1}^{n} \varphi^* \Big(\frac{\tau(E_i)}{\mu(E_i)} \Big) \mu(E_i) \leqslant 1
$$

is independent of the choice of $\{E_i\}_{i=1}^n \in P(E)$, which implies that $V^{\varphi^*}(\tau, E) \leqslant 1$ for all $E \in \mathcal{A}$.

Take now an arbitrary $E \in \mathcal{A}$, $\{E_i\}_{i=1}^n \in P(E)$ and $\{x_i\}_{i=1}^n \in X$ such that $\|x_i\| \leqslant 1$. Let $s(E_i)$ be the sign of $\langle \tau(E_i), x_i \rangle$ and $v = \sum\limits_{i=1}^{n} s(E_i) x_i \chi_{E_i}$. Then

$$
\begin{aligned}
\sum_{i=1}^{n} |\langle \tau(E_i), x_i \rangle| &= \sum_{i=1}^{n} \langle \tau(E_i), s(E_i) x_i \rangle \\
&= (N_\Omega^\varphi(v) + 1) \Big[\sum_{i=1}^{n} \Big\langle \frac{\tau(E_i)}{\mu(E_i)}, \frac{s(E_i) x_i}{N_\Omega^\varphi(v) + 1} \Big\rangle \mu(E_i) \Big] \\
&\leqslant (N_\Omega^\varphi(v) + 1) \Big[\sum_{i=1}^{n} \varphi^* \Big(\frac{\tau(E_i)}{\mu(E_i)} \Big) \mu(E_i) + \int_\Omega \varphi\Big(\frac{v}{N_\Omega^\varphi(v) + 1} \Big)\,\mathrm{d}\mu \Big] \\
&\leqslant (N_\Omega^\varphi(v) + 1)(V^{\varphi^*}(\tau, E) + 1) \\
&\leqslant 2(N_\Omega^\varphi(v) + 1).
\end{aligned}
$$

To get an estimate for $N_\Omega^\varphi(v)$, define $M := \sup\{\varphi(y),\ \|y\| \leqslant 1\}$. Again, without loss of generality, assume that $M\mu(\Omega) \geqslant 1$. Then

$$
\begin{aligned}
\int_\Omega \varphi\Big(\frac{v}{M\mu(\Omega)} \Big)\,\mathrm{d}\mu &= \sum_{i=1}^{n} \varphi\Big(\frac{s(E_i) x_i}{M\mu(\Omega)} \Big) \mu(E_i) \\
&\leqslant \frac{1}{\mu(\Omega)} \sum_{i=1}^{n} \frac{1}{M} \varphi(s(E_i) x_i) \mu(E_i) \\
&\leqslant \frac{1}{\mu(\Omega)} \sum_{i=1}^{n} \mu(E_i) = \frac{\mu(E)}{\mu(\Omega)} \leqslant 1.
\end{aligned}
$$

Hence $N_\Omega^\varphi(v) \leqslant M\mu(\Omega)$ and $\sum_{i=1}^{n}\langle\tau(E_i), x_i\rangle \leqslant 2(M\mu(\Omega)+1)$ is independent of the choice of $\{E_i\}$ and $\{x_i\}$.

In order to show that τ is of bounded variation, we have to prove that

$$|\tau|(E) := \sup\left\{\sum_{i=1}^{n}\|\tau(E_i)\|_{X^*},\ \{E_i\}_{i=1}^n \in P(E)\right\} < +\infty \quad \text{for every } E \in \mathcal{A}.$$

For any $\{E_i\}_{i=1}^n \in P(E)$ choose $\{x_i\}_{i=1}^n \in X$ such that $\|x_i\| \leqslant 1$ and

$$\|\tau(E_i)\| \leqslant \langle\tau(E_i), x_i\rangle + \frac{1}{n}.$$

Then

$$\sum_{i=1}^{n}\|\tau(E_i)\| \leqslant \sum_{i=1}^{n}\langle\tau(E_i), x_i\rangle + 1 \leqslant 2(M\mu(\Omega)+1)+1,$$

independently of the choice of $\{E_i\} \in P(E)$. Thus $|\tau|(E) < \infty$ for every $E \in \mathcal{A}$.

We also know that τ is absolutely μ-continuous, since $\tau(E) = 0$ whenever $\mu(E) = 0$.

With the Radon-Nikodym Property of X^*, we can find a function $v^* \in \mathcal{L}^1(\mu, X^*)$ such that

$$\tau(E) = \int_E v^*(\omega)\,\mathrm{d}\mu(\omega) \quad \text{for every } E \in \mathcal{A}.$$

This implies that for any $x \in X$ and any $E \in \mathcal{A}$ we have

$$F(x\chi_E) = \langle\tau(E), x\rangle = \left\langle\int_E v^*(\omega)\,\mathrm{d}\mu(\omega), x\right\rangle = \int_E\langle v^*(\omega), x\chi_E(\omega)\rangle\,\mathrm{d}\mu(\omega).$$

Take now an arbitrary step function $u \in \mathbf{L}_\Omega^\varphi(X)$, $u = \sum_{k=1}^{m}\chi_{Q_k}u_k$, $Q_k \in \mathcal{A}$, Q_k disjoint, $u_k \in X$. Then

$$F(u) = \sum_{k=1}^{m}F(u_k\chi_{Q_k}) = \sum_{k=1}^{m}\int_{Q_k}\langle v^*(\omega), u_k\chi_{Q_k}(\omega)\rangle\,\mathrm{d}\mu(\omega) = \int_\Omega\langle v^*(\omega), u(\omega)\rangle\,\mathrm{d}\mu(\omega).$$

By Remark 7.1 we get that $v^* \in \mathbf{L}_\Omega^{\varphi^*}(X^*)$ and $N_\Omega^{\varphi^*}(v^*) \leqslant 1$.

Assume now that u is bounded. Without loss of generality assume that $N_\Omega^\varphi(u) \leqslant 1$. Choose an averaging approximation such that

$$u_n(\omega) = \sum_{i=1}^{n}\frac{n}{\mu(\Omega)}\int_{E_i}u(\omega)\,\mathrm{d}\mu(\omega),\ \ E_i \in \mathcal{A},\ \text{ disjoint},$$

$$\mu(E) = \frac{\mu(\Omega)}{n},\ \ \Omega = \bigcup_{i=1}^{n}E_i.$$

Then $u_n \to u$ almost everywhere. Since u is bounded, u_n is uniformly bounded, which implies that

$$\int_\Omega \langle v^*(\omega), u_n(\omega)\rangle \,\mathrm{d}\mu(\omega) \to \int_\Omega \langle v^*(\omega), u(\omega)\rangle \,\mathrm{d}\mu(\omega).$$

Since u is bounded, we get by the continuity of φ that $u_n \in \mathbf{L}_\Omega^\varphi(X)$. Hence

$$\pm F(u) = F(\pm u_n) + F(\pm(u - u_n)) \leqslant \pm(F(u_n)) + N_\Omega^\varphi(\pm(u - u_n)),$$

from which we can infer that

$$\pm F(u) \leqslant \limsup_{n\to\infty} \pm F(u_n).$$

Therefore

$$F(u) = \lim_{n\to\infty} F(u_n) = \lim_{n\to\infty} \int_\Omega \langle v^*(\omega), u_n(\omega)\rangle \,\mathrm{d}\mu(\omega) = \int_\Omega \langle v^*(\omega), u(\omega)\rangle \,\mathrm{d}\mu(\omega).$$

For an arbitrary $u \in \mathbf{L}_\Omega^\varphi(X)$, define

$$u_n(\omega) = \begin{cases} u(\omega), & \|u(\omega)\| \leqslant n, \\ 0 & \text{else.} \end{cases}$$

By the monotone convergence property and the above statements we get

$$F(u) = \lim_{n\to\infty} F(u_n) = \int_\Omega \langle v^*(\omega), u(\omega)\rangle \,\mathrm{d}\mu(\omega).$$

2) It still remains to show that v^* is unique. Suppose that v_1^*, v_2^* represent F. Then we have

$$\int_\Omega \langle v_1^*(\omega), u(\omega)\rangle \,\mathrm{d}\mu(\omega) = \int_\Omega \langle v_2^*(\omega), u(\omega)\rangle \,\mathrm{d}\mu(\omega) \quad \text{for all } u \in \mathcal{L}^\infty(\Omega, X) \subseteq \mathbf{L}_\Omega^\varphi(X).$$

Thus $v_1^* = v_2^*$ almost everywhere. $\square$

Corollary 7.1. *Suppose that Ω is σ-finite, φ is bounded on bounded subsets and coercive. Then we have with the identification of Theorem* 7.2

$$\mathbf{P}_\Omega^{\varphi^*}(X) = \mathbf{L}_\Omega^{\varphi^*}(X^*).$$

P r o o f. Without loss of generality we may assume that $O_\varphi(F) \leqslant 1$. Suppose that $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ and $\mu(\Omega_n) < \infty$. Define $\overline{\Omega}_n = \bigcup_{k=1}^{n} \Omega_k$ and $F_n(u) = F(u)$ for all $u \in \mathbf{L}_\Omega^\varphi(X)$ which vanish outside of $\overline{\Omega}_n$. Obviously, $F_n \in \mathbf{P}_\Omega^{\varphi^*}(X)$. By Theorem 7.3 we can find a unique v_n^* such that

$$F_n(u) = \int_{\overline{\Omega}_n} \langle v_n^*(\omega), u(\omega)\rangle \, \mathrm{d}\mu(\omega)$$

for all $u \in \mathbf{L}_\Omega^\varphi(X)$ which vanishes outside of $\overline{\Omega}_n$ and $N_{\overline{\Omega}_n}^{\varphi^*}(v_n^*) \leqslant 1$. Take an arbitrary $n \in \mathbb{N}$ and a $u \in \mathbf{L}_\Omega^\varphi(X)$ which vanishes outside of $\overline{\Omega}_n$. Then

$$F(u) = F_n(u) = F_{n+1}(u), \quad \text{thus } v_n^* = v_{n+1}^* \text{ on } \overline{\Omega}_n \setminus M_n,$$

where $M_n \in \mathcal{A}$ is a set with $\mu(M_n) = 0$. If we put $M = \bigcup_{n=1}^{\infty} M_n$, then $\mu(M)$ is still zero and the function

$$v^*(\omega) = \begin{cases} v_n^*(\omega) & \text{if } \omega \in \overline{\Omega}_n \setminus M \text{ for some } n \in \mathbb{N}, \\ 0 & \text{if } \omega \in M \end{cases}$$

is well defined. Take an arbitrary $u \in \mathbf{L}_\Omega^\varphi(X)$ and define $u_n = \chi_{\overline{\Omega}_n} u$, which converges monotonicaly to u. Since $F \in \mathbf{P}_\Omega^{\varphi^*}(X)$ we get

$$\begin{aligned} F(u) = \lim_{n\to\infty} F(u_n) &= \lim_{n\to\infty} \left[\int_{\overline{\Omega}_n} \langle v_n^*(\omega), u_n(\omega)\rangle \, \mathrm{d}\mu(\omega) \right] \\ &= \lim_{n\to\infty} \left[\int_{\overline{\Omega}_n} \langle v^*(\omega), u(\omega)\rangle \, \mathrm{d}\mu(\omega) \right] \\ &= \int_\Omega \langle v^*(\omega), u(\omega)\rangle \, \mathrm{d}\mu(\omega). \end{aligned}$$

It still remains to show that $v^* \in \mathbf{L}_\Omega^{\varphi^*}(X^*)$. Since $N_{\overline{\Omega}_n}^{\varphi^*}(v_n^*) \leqslant 1$, we know that

$$\int_{\overline{\Omega}_n} \varphi^*(v_n^*) \, \mathrm{d}\mu = \int_\Omega \chi_{\overline{\Omega}_n} \varphi^*(v^*) \, \mathrm{d}\mu \leqslant 1.$$

By the Lebesgue Theorem we can conclude that $v^* \in \mathbf{L}_\Omega^{\varphi^*}(X^*)$ and $N_\Omega^{\varphi^*}(v^*) \leqslant 1$. The uniqueness follows from the uniqueness of v_n^*. $\square$

R e m a r k 7.2. If φ is bounded on bounded subsets of X, Ω is σ-finite and $\varphi \notin \Delta_2$, then we can find an $F \in (\mathbf{L}^{\varphi}_{\Omega}(X))^*$ which cannot be represented by a $v^* \in \mathbf{L}^{\varphi^*}_{\Omega}(X^*)$.

P r o o f. Since $\varphi \notin \Delta_2$, $\mathbf{E}^{\varphi}_{\Omega}(X)$ is a proper subset of $\mathbf{L}^{\varphi}_{\Omega}(X)$. Choose a $v \in \mathbf{L}^{\varphi}_{\Omega}(X) \setminus \mathbf{E}^{\varphi}_{\Omega}(X)$. The Hahn-Banach theorem guarantees the existence of a functional $F \in (\mathbf{L}^{\varphi}_{\Omega}(X))^*$ such that $F(u) = 0$ for all $u \in \mathbf{E}^{\varphi}_{\Omega}(X)$ and $F(v) > 0$. Suppose that there exists a $v^* \in \mathbf{L}^{\varphi^*}_{\Omega}(X^*)$ such that

$$F(w) = \int_{\Omega} \langle v^*(\omega), w(\omega)\rangle \,\mathrm{d}\mu(\omega) \quad \text{for all } w \in \mathbf{L}^{\varphi}_{\Omega}(X).$$

Let $\Omega = \bigcup\limits_{n=1}^{\infty} \Omega_n$, where $\mu(\Omega_n) < \infty$. Since φ is bounded on bounded subsets, $\mathcal{L}^{\infty}_{\Omega_n}(X) \subseteq \mathbf{E}^{\varphi}_{\Omega_n}(X)$. Thus we get for an arbitrary $u \in \mathcal{L}^{\infty}_{\Omega_n}(X)$ that

$$\langle v^*(\omega), u(\omega)\rangle = 0 \quad \text{for almost every } \omega \in \Omega_n$$

and hence $v^*(\omega) = 0$ for almost every $\omega \in \Omega_n$. Thus $v^*(\omega) = \sum\limits_{k=1}^{\infty} v^*(\omega)\chi_{\Omega_n} = 0$ almost everywhere and therefore $F(v) = 0$, in contradiction to the construction of F. $\square$

Corollary 7.2. *Suppose that φ is bounded on bounded subsets of X and coercive. Moreover, let $\mu(\Omega)$ be finite. Then $\mathbf{L}^{\varphi}_{\Omega}(X)$ is reflexive if and only if φ satisfies the Δ_2- and the ∇_2-growth condition.*

P r o o f. By Theorem 7.1, $\mathbf{L}^{\varphi}_{\Omega}(X))^* = \mathbf{P}^{\varphi^*}_{\Omega}(X)$, which can be identified with $\mathbf{L}^{\varphi^*}_{\Omega}(X^*)$ by Theorem 7.3. The ∇_2-condition ensures that $\varphi^* \in \Delta_2$. The coercivity of φ implies the continuity of φ^*. Applying Theorem 7.1 and Theorem 7.3 again, we get that $(\mathbf{L}^{\varphi^*}_{\Omega}(X^*))^*$ can be identified with $\mathbf{L}^{\varphi^{**}}_{\Omega}(X)$. Since φ is LSC, $\varphi^{**} = \varphi$ and thus $(\mathbf{L}^{\varphi}_{\Omega}(X)^*)^*$ can be identified with $\mathbf{L}^{\varphi}_{\Omega}(X)$. $\square$

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