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SIGNORINI PROBLEM WITH A SOLUTION DEPENDENT COEFFICIENT OF FRICTION (MODEL WITH GIVEN FRICTION): APPROXIMATION AND NUMERICAL REALIZATION*

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Abstract. Contact problems with given friction and the coefficient of friction depending on their solutions are studied. We prove the existence of at least one solution; uniqueness is obtained under additional assumptions on the coefficient of friction. The method of successive approximations combined with the dual formulation of each iterative step is used for numerical realization. Numerical results of model examples are shown.

Keywords: contact problems with given friction, unilateral contact and friction, solution dependent coefficient of friction

MSC 2000: 65N30

0. Introduction

Contact problems with friction have received great importance during last years. The mechanical setting of these problems is simple: one looks for an equilibrium state of deformable bodies being in mutual contact which are subject to body forces and surface tractions. In addition, effects of friction on the contacting parts are taken into account. A contact problem with given friction in which the slip bound is given a-priori represents the simplest model involving friction (see [8], [5]). Due to its simplicity it is not surprising that it does not reflect the physical situation very well. Therefore more realistic models of friction have to be used. Among them the model of friction obeying the Coulomb law is the most classical one. In spite of its simple formulation the mathematical treatment remained an open problem

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for a long time. The first existence result for the static case of Signorini problems based on the fixed point approach appeared in [9]. It was shown that there exists at least one solution provided that the coefficient of friction is sufficiently small. Another approach was used in [3], namely a simultaneous penalization of unilateral constraints combined with a regularization of the frictional term. The former (fixed point) approach is nowadays considered and widely accepted as an efficient and reliable tool for numerical realization of contact problems with Coulomb friction. In most papers the coefficient of friction $\mathcal{F}$ is supposed to be a function of the space variable but not depending on the solution itself. Nevertheless, from experiments we know that $\mathcal{F}$ may depend on the tangential component of displacements (or on the tangential velocity in the case of quasistatic problems). The paper [3] covers also the case when $\mathcal{F}$ depends on the solution. The technique of a simultaneous penalization and regularization used there for theoretical analysis and in [6] for discretization does seem to be not convenient for computations. It is the aim of this paper to extend the fixed point approach to contact problems with a solution-dependent coefficient of friction. For the sake of simplicity of our presentation we confine ourselves to a simpler case when a *given* slip bound is multiplied by a coefficient of friction depending on the tangential component of the displacement field. Coulomb friction will be treated in the next paper.

The paper is organized as follows: in Section 1 solutions to the problem are defined as fixed points of a mapping acting on the trace space. In Section 2 we prove the existence of at least one fixed point of this mapping. The solution is unique provided that the coefficient of friction is Lipschitz with sufficiently small modulus. The method of successive approximations is proposed for finding fixed points. Each iterative step is defined by a contact problem with given friction in which the coefficient of friction is updated. We briefly mention the dual formulation of this auxiliary problem in terms of contact stresses which will be used for its numerical realization. Finally, in Section 3 two iterative strategies are described and numerical results of several model examples are presented.

1. Setting of the problem

Let Ω be a bounded domain representing an elastic body which is supported by a rigid foundation S along a part Γ_c of the boundary $\partial\Omega$. For the sake of simplicity of our presentation we will suppose that $S = \mathbb{R}^2_- = \{(x_1, x_2) \mid x_2 \leqslant 0\}$ is the halfplane and Γ_c is a straight line placed on the x_1-axis (see Fig. 1.1).

The body is subject to body forces of density $F \in (L^2(\Omega))^2$ and surface tractions of density $P \in (L^2(\Gamma_p))^2$ that act on a part $\Gamma_p \subseteq \partial\Omega$.

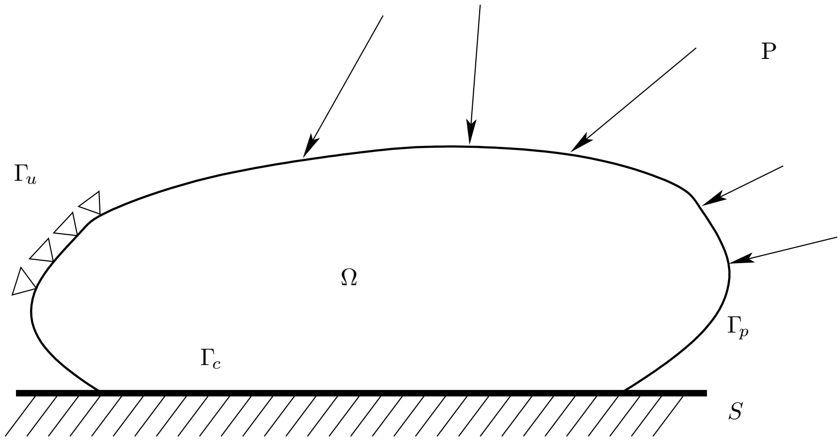

Figure 1.1.

The following *unilateral* and *friction conditions* are prescribed on the contact part:

$$u_2 \geqslant 0, \quad T_2(u) \geqslant 0, \quad u_2 T_2(u) = 0 \quad \text{on } \Gamma_c, \tag{1.1}$$

$$\begin{cases} |T_1(u)| \leqslant \mathcal{F}(|u_1|)g & \text{on } \Gamma_c, \\ |T_1(u)(x)| < \mathcal{F}(|u_1(x)|)g(x) \Rightarrow u_1(x) = 0 & x \in \Gamma_c \\ |T_1(u)(x)| = \mathcal{F}(|u_1(x)|)g(x) > 0 \Rightarrow \exists\, \lambda(x) \geqslant 0\colon & \\ \qquad\qquad u_1(x) = -\lambda(x) T_1(u(x)) & x \in \Gamma_c \end{cases} \tag{1.2}$$

where $\mathcal{F}$ is a *continuous, positive, bounded* function in $\mathbb{R}^1_+$ which defines the coefficient of friction depending on the magnitude of the tangential component u_1 of the displacement vector u, $g \in L^2(\Gamma_c)$, $g \geqslant 0$ is a given slip bound, $T_1(u)$, $T_2(u)$ are the tangential and normal components, respectively, of the stress vector $T(u)$ corresponding to u. Finally, the body is fixed on $\Gamma_u \subseteq \partial\Omega$, i.e. $u = 0$ on Γ_u.

Remark 1.1. In (1.1) and (1.2) the special geometry of Γ_c is reflected.

Our aim is to find an equilibrium state of Ω which is characterized by a displacement vector u satisfying the system of *equilibrium equations*

$$\frac{\partial \tau_{ij}}{\partial x_j} + F_i = 0 \quad \text{in } \Omega, \;\; i = 1, 2, [1] \tag{1.3}$$

[1] The summation convention is adopted.

the *classical boundary conditions*

$$u_i = 0 \quad \text{on } \Gamma_u, \ \ i = 1, 2; \tag{1.4}$$

$$T_i(u) = P_i \quad \text{on } \Gamma_p, \ \ i = 1, 2 \tag{1.5}$$

and the unilateral and friction conditions (1.1), (1.2). The symbol $\tau = (\tau_{ij})_{i,j=1}^2$ stands for the symmetric stress tensor which is related to the linearized strain tensor $\varepsilon = (\varepsilon_{ij})_{i,j=1}^2$ by means of a linear Hooke's law:

$$\tau_{ij} := \tau_{ij}(u) = c_{ijkl}\varepsilon_{kl}(u), \tag{1.6}$$

where $\varepsilon_{kl}(u) = \frac{1}{2}(\partial u_k/\partial x_l + \partial u_l/\partial x_k)$ and $c_{ijkl} \in L^\infty(\Omega)$, $i, j, k, l = 1, 2$, are linear elasticity coefficients which satisfy the usual symmetry and ellipticity conditions.

To give a weak formulation of our problem we introduce some notation. Let

$$V = \{v \in H^1(\Omega) \mid v = 0 \ \text{ on } \ \Gamma_u\},$$
$$\mathbb{V} = V \times V$$

and let $K \subseteq \mathbb{V}$ be a closed convex subset of kinematically admissible displacements:

$$K = \{v = (v_1, v_2) \in \mathbb{V} \mid v_2 \geqslant 0 \ \text{ a.e. on } \ \Gamma_c\}.$$

Further, let $H^{1/2}(\Gamma_c)$ denote the space of all traces on Γ_c of functions from V:

$$\varphi \in H^{1/2}(\Gamma_c) \quad \text{iff } \exists\, v \in V\colon \ v = \varphi \ \text{ on } \ \Gamma_c$$

and let $H_+^{1/2}(\Gamma_c)$ be the cone of all non-negative elements of $H^{1/2}(\Gamma_c)$. It is known that $H^{1/2}(\Gamma_c)$ is a Banach space if equipped with the norm

$$\|\varphi\|_{1/2,\,\Gamma_c} = \inf_{\substack{v \in V \\ v = \varphi \text{ on } \Gamma_c}} |v|_{1,\Omega}. \tag{1.7}$$

To simplify our presentation let the slip bound $g \equiv 1$ on Γ_c. For every $\varphi \in H_+^{1/2}(\Gamma_c)$ we shall define the following auxiliary problem:

$$(\mathcal{P}(\varphi)) \qquad \begin{cases} \text{Find } \ u := u(\varphi) \in K \ \text{ such that} \\ a(u, v - u) + \displaystyle\int_{\Gamma_c} \mathcal{F}(\varphi)(|v_1| - |u_1|)\,\mathrm{d}x_1 \geqslant L(v - u) \quad \forall\, v \in K, \end{cases}$$

where

$$a(u, v) = \int_\Omega c_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(v)\,\mathrm{d}x,$$
$$L(v) = \int_\Omega F_i v_i\,\mathrm{d}x + \int_{\Gamma_p} P_i v_i\,\mathrm{d}s.$$

It is well-known (see [5], [8]) that $(\mathcal{P}(\varphi))$ has a unique solution for every $\varphi \in H^{1/2}_+(\Gamma_c)$. This makes it possible to define a mapping $\Psi\colon H^{1/2}_+(\Gamma_c) \mapsto H^{1/2}_+(\Gamma_c)$ as follows:

$$\Psi\colon \varphi \mapsto \operatorname*{trace}_{\Gamma_c} |u_1(\varphi)| \tag{1.8}$$

where $u(\varphi) = (u_1(\varphi), u_2(\varphi)) \in K$ solves $(\mathcal{P}(\varphi))$.

Definition 1.1. By a weak solution to the Signorini problem with the solution-dependent coefficient of friction $\mathcal{F}$ we mean any function $u \in K$ solving $(\mathcal{P}(\varphi))$ where $\varphi \in H^{1/2}_+(\Gamma_c)$ is a fixed point of Ψ.

2. Existence result

Before we prove the existence of at least one fixed point, we shall examine basic properties of Ψ.

Lemma 2.1. *The mapping Ψ maps $H^{1/2}_+(\Gamma_c)$ into $H^{1/2}_+(\Gamma_c) \cap B_{1/2}$, where $B_{1/2} = \{\varphi \in H^{1/2}_+(\Gamma_c) \mid \|\varphi\|_{1/2,\Gamma_c} \leqslant \|L\|_\star/\alpha\}$, $\|L\|_\star$ is the dual norm of L and $\alpha > 0$ is the constant of Korn's inequality.*

Proof. Inserting $v = 0$ into $(\mathcal{P}(\varphi))$ we obtain

$$\alpha\|u\|_1^2 \leqslant a(u,u) + \int_{\Gamma_c} \mathcal{F}(\varphi)|u_1| \,\mathrm{d}x_1 \leqslant L(u) \leqslant \|L\|_\star \|u\|_1. \tag{2.1}$$

From this and the fact that

$$\|u\|_1 \geqslant \|u_1\|_{1/2,\Gamma_c} \geqslant \|\,|u_1|\,\|_{1/2,\Gamma_c} \tag{2.2}$$

the assertion of the lemma follows (observe that (2.2) is an easy consequence of (1.7)). $\square$

Lemma 2.2. *The mapping Ψ is weakly continuous:*

$$\varphi_n \rightharpoonup \varphi \text{ in } H^{1/2}(\Gamma_c), \quad \varphi_n, \varphi \in H^{1/2}_+(\Gamma_{c)} \Rightarrow \Psi(\varphi_n) \rightharpoonup \Psi(\varphi) \text{ in } H^{1/2}(\Gamma_c).$$

Proof. Let $u_n := u(\varphi_n) \in K$ be a solution to $(\mathcal{P}(\varphi_n))$, i.e.

$$a(u_n, v - u_n) + \int_{\Gamma_c} \mathcal{F}(\varphi_n)(|v_1| - |u_{n1}|) \,\mathrm{d}x_1 \geqslant L(v - u_n) \quad \forall\, v \in K\ . \tag{2.3}$$

From (2.1) it follows that $\{u_n\}$ is bounded:

$$\exists\, c > 0 \quad \text{such that } \|u_n\|_1 \leqslant c \;\; \forall\, n \in \mathbb{N}.$$

Thus there exist a subsequence $\{u_{n'}\} \subset \{u_n\}$ and an element $u \in K$ such that

$$u_{n'} \rightharpoonup u, \quad n' \to \infty \text{ in } (H^1(\Omega))^2. \tag{2.4}$$

Letting $n' \to \infty$ in (2.3) and using the relations

$$\limsup_{n' \to \infty} a(u_{n'}, v - u_{n'}) \leqslant a(u, v - u),$$
$$\lim_{n' \to \infty} L(v - u_{n'}) = L(v - u),$$
$$\lim_{n' \to \infty} \int_{\Gamma_c} \mathcal{F}(\varphi_{n'})(|v_1| - |u_{n'1}|)\,\mathrm{d}x_1 = \int_{\Gamma_c} \mathcal{F}(\varphi)(|v_1| - |u_1|)\,\mathrm{d}x_1$$

we see that u solves $(\mathcal{P}(\varphi))$ (the last limit passage follows from the assumption on $\mathcal{F}$, the Lebesgue dominated convergence theorem and the fact that $\mathrm{trace}_{\Gamma_c} |u_{n'1}| \to \mathrm{trace}_{\Gamma_c} |u_1|$ in $L^2(\Gamma_c)$ as $n' \to \infty$).

Since $(\mathcal{P}(\varphi))$ has a unique solution, the whole sequence $\{u_n\}$ tends weakly to u in $(H^1(\Omega))^2$. It is very easy to verify that

$$u_{n1} \rightharpoonup u_1 \text{ in } H^1(\Omega) \Rightarrow |u_{n1}| \rightharpoonup |u_1| \text{ in } H^1(\Omega)$$

and consequently

$$\operatorname*{trace}_{\Gamma_c} |u_{n1}| \rightharpoonup \operatorname*{trace}_{\Gamma_c} |u_1| \quad \text{in } H^{1/2}(\Gamma_c), \;\; n \to \infty.$$

$\square$

Since Ψ is weakly continuous and maps $H^{1/2}_+(\Gamma_c) \cap B_{1/2}$ into itself, the existence of at least one fixed point follows from the weak version of the Schauder fixed point theorem (see [5]). In what follows we give an alternative existence proof which is based on the existence of fixed points of the discretized problems. To this end we shall suppose that Ω is a *polygonal* domain.

Let $\{\mathcal{T}_h\}$, $h \to 0+$ be a family of *regular triangulations* of $\overline{\Omega}$ such that $\{\mathcal{T}_{h|\Gamma_c}\}$ is a *strongly regular* system of partitions of Γ_c. With any $\mathcal{T}_h$ the following sets will be associated:

$$\begin{aligned}
K_h = \{&v_h = (v_{h1}, v_{h2}) \in (C(\overline{\Omega}))^2 \mid v_{h|T} \in (P_1(T))^2,\\
&v_h = 0 \text{ on } \Gamma_u,\; v_{h2} \geqslant 0 \text{ on } \Gamma_c\},\\
\Lambda_h = \{&\varphi_h \in C(\overline{\Gamma}_c) \mid \varphi_h \text{ is piecewise linear on } \mathcal{T}_{h|\Gamma_c},\\
&\varphi_h \geqslant 0 \text{ on } \Gamma_c \text{ and } \varphi_h(x) = 0 \text{ if } x \in \overline{\Gamma}_c \cap \overline{\Gamma}_u \neq \emptyset\}.
\end{aligned}$$

That is, K_h is the convex set of all continuous, piecewise linear functions satisfying the unilateral constraints on Γ_c and Λ_h, contains all continuous, piecewise linear and non-negative functions over the partition of Γ_c generated by $\mathcal{T}_h$, vanishing at the common points of $\overline{\Gamma}_c$ and $\overline{\Gamma}_u$.

For every $\varphi_h \in \Lambda_h$ we define the following discrete problem:

$$(\mathcal{P}(\varphi_h))_h \qquad \begin{cases} \text{Find } u_h := u_h(\varphi_h) \in K_h \text{ such that} \\ a(u_h, v_h - u_h) + \displaystyle\int_{\Gamma_c} \mathcal{F}(\varphi_h)(|v_{h1}| - |u_{h1}|)\,\mathrm{d}x_1 \geqslant L(v_h - u_h) \\ \hfill \forall\, v_h \in K_h. \end{cases}$$

Let $\Psi_h \colon \Lambda_h \mapsto \Lambda_h$ be a mapping defined by

$$\Psi_h(\varphi_h) = r_h(|\operatorname*{trace}_{\Gamma_c} u_{h1}|),$$

where r_h is the piecewise linear Lagrange interpolation operator over $\mathcal{T}_{h|\Gamma_c}$ and $u_h = (u_{h1}, u_{h2}) \in K_h$ is the solution of $(\mathcal{P}(\varphi_h))_h$.

Lemma 2.3. *The mapping Ψ_h is continuous and maps $\Lambda_h \cap B_R$ into $\Lambda_h \cap B_R$, where $B_R = \{\varphi_h \in \Lambda_h \mid \|\varphi_h\|_{1/2,\Gamma_c} \leqslant R\}$ and $R \geqslant 0$ does not depend on h.*

Proof. Arguing exactly as in Lemma 2.1 one can show that

$$\| \, |u_{h1}| \, \|_{1/2,\Gamma_c} \leqslant \|u_{h1}\|_{1/2,\Gamma_c} \leqslant \frac{\|L\|_\star}{\alpha}. \tag{2.5}$$

Further,

$$\begin{aligned} \|r_h|u_{h1}|\,\|_{1/2,\Gamma_c} &\leqslant \|r_h|u_{h1}| - |u_{h1}|\,\|_{1/2,\Gamma_c} + \|\,|u_{h_1}|\,\|_{1/2,\Gamma_c} \\ &\leqslant ch^{1/2}\|u_{h1}\|_{1,\Gamma_c} + \|u_{h1}\|_{1/2,\Gamma_c} \\ &\leqslant c\|u_{h1}\|_{1/2,\Gamma_c} + \|u_{h1}\|_{1/2,\Gamma_c}, \end{aligned}$$

where $c > 0$ does not depend on h, which follows by the well-known approximation properties of r_h and the inverse inequality between $H^1(\Gamma_c)$ and $H^{1/2}(\Gamma_c)$. From this and (2.5) we see that Ψ_h maps $\Lambda_h \cap B_R$ into $\Lambda_h \cap B_R$ with $R := (c+1)\|L\|_\star/\alpha$. Continuity of Ψ_h can be easily verified. □

From the Brouwer fixed point theorem the existence of a fixed point of Ψ_h in $\Lambda_h \cap B_R$ follows. Let $\{\varphi_h^\star\}$, $h \to 0+$ be a sequence of fixed points of Ψ_h, i.e. $\varphi_h^\star = r_h(|u_{h1}^\star|) := r_h(|\operatorname{trace}_{\Gamma_c} u_{h1}^\star|)$, where $u_h^\star = (u_{h1}^\star, u_{h2}^\star) \in K_h$ solves

$$a(u_h^\star, v_h - u_h^\star) + \int_{\Gamma_c} \mathcal{F}(r_h|u_{h1}^\star|)(|v_{h1}| - |u_{h1}^\star|)\,\mathrm{d}x_1 \geqslant L(v_h - u_h^\star) \quad \forall\, v_h \in K_h. \tag{2.6}$$

Let $\bar{v} \in K$ be given. Then one can find a sequence $\{\bar{v}_h\}$, $\bar{v}_h \in K_h$ such that

$$\bar{v}_h \to \bar{v} \quad \text{in } (H^1(\Omega))^2. \tag{2.7}$$

Since $\{u_h^\star\}$ is bounded in $(H^1(\Omega))^2$ one can pass to a subsequence $\{u_{h'}^\star\}$ and find an element $u^\star \in K$ such that

$$u_{h'}^\star \rightharpoonup u^\star \quad \text{in } (H^1(\Omega))^2. \tag{2.8}$$

From this and (2.7) we obtain

$$\begin{cases} \limsup\limits_{h'\to 0+} a(u_{h'}^\star, \bar{v}_{h'} - u_{h'}^\star) \leqslant a(u^\star, \bar{v} - u^\star), \\ \lim\limits_{h'\to 0+} L(\bar{v}_{h'} - u_{h'}^\star) = L(\bar{v} - u^\star). \end{cases} \tag{2.9}$$

It is easy to verify that

$$r_{h''}|u_{h''1}^\star| \to |u| \quad \text{a.e. on } \Gamma_c \tag{2.10}$$

for an appropriate subsequence $\{u_{h''}^\star\} \subset \{u_{h'}^\star\}$. Indeed

$$\begin{aligned} \|r_{h'}|u_{h'1}^\star| - |u_1^\star|\,\|_{0,\Gamma_c} &\leqslant \|r_{h'}|u_{h'1}^\star| - |u_{h'1}^\star|\,\|_{0,\Gamma_c} + \|\,|u_{h'1}^\star| - |u_1^\star|\,\|_{0,\Gamma_c} \\ &\leqslant c(h')^{1/2}\|u_{h'1}^\star\|_{1/2,\Gamma_c} + \|\,|u_{h'1}^\star| - |u_1^\star|\,\|_{0,\Gamma_c} \overset{h'\to 0+}{\longrightarrow} 0. \end{aligned}$$

Using (2.7)–(2.9) we obtain

$$\lim_{h''\to 0+} \int_{\Gamma_c} \mathcal{F}(r_{h''}|u_{h''1}^\star|)(|\bar{v}_{h''1}| - |u_{h''1}^\star|)\,\mathrm{d}x_1 = \int_{\Gamma_c} \mathcal{F}(|u_1^\star|)(|\bar{v}_1| - |u_1^\star|)\,\mathrm{d}x_1.$$

From this and (2.9) we see that $u^\star \in K$ satisfies

$$a(u^\star, \bar{v} - u^\star) + \int_{\Gamma_c} \mathcal{F}(|u_1^\star|)(|\bar{v}_1| - |u_1^\star|)\,\mathrm{d}x_1 \geqslant L(\bar{v} - u^\star).$$

Since $\bar{v} \in K$ was arbitrarily chosen this means that $u^\star$ solves the respective variational inequality and in addition $\varphi^\star := \operatorname{trace}_{\Gamma_c}|u_1^\star|$ is a fixed point of Ψ.

We have proved the following theorem.

Theorem 2.1. *Let $\{\varphi_h^\star\}$, $h \to 0+$ be a sequence of fixed points of Ψ_h in $\Lambda_h \cap B_R$. Then there exists a subsequence $\{\varphi_{h'}^\star\} \subset \{\varphi_h^\star\}$ such that*

$$\varphi_{h'}^\star \rightharpoonup \varphi^\star \quad \text{in } H^{1/2}(\Gamma_c), \quad h' \to 0+\,.$$

In addition, $\varphi^\star \in H_+^{1/2}(\Gamma_c) \cap B_R$ is a fixed point of Ψ.

We now prove that Ψ is Lipschitz continuous in the $L^2(\Gamma_c)$-norm provided that $\mathcal{F}$ is so. Indeed, we have

Theorem 2.2. *Let* $\Psi\colon H_+^{1/2}(\Gamma_c) \mapsto H_+^{1/2}(\Gamma_c)$ *be defined by* (1.8) *and* $q > 0$ *be such that*

$$|\mathcal{F}(x_1) - \mathcal{F}(\bar{x}_1)| \leqslant q|x_1 - \bar{x}_1| \quad \forall\, x_1, \bar{x}_1 \in \mathbb{R}^1_+. \tag{2.11}$$

Then there exists a positive constant c *such that*

$$\|\Psi(\varphi) - \Psi(\overline{\varphi})\|_{0,\Gamma_c} \leqslant cq\|\varphi - \overline{\varphi}\|_{0,\Gamma_c} \tag{2.12}$$

holds for any $\varphi, \overline{\varphi} \in H_+^{1/2}(\Gamma_c)$.

Proof. Let $\varphi, \overline{\varphi} \in H_+^{1/2}(\Gamma_c)$ be given and let u, $\bar{u}$ be the solutions of $(\mathcal{P}(\varphi))$, $(\mathcal{P}(\overline{\varphi}))$, respectively, i.e.

$$\begin{aligned} a(u, v-u) + \int_{\Gamma_c} \mathcal{F}(\varphi)(|v_1| - |u_1|)\,\mathrm{d}x_1 &\geqslant L(v-u) \quad \forall\, v \in K,\\ a(\bar{u}, v-\bar{u}) + \int_{\Gamma_c} \mathcal{F}(\overline{\varphi})(|v_1| - |\bar{u}_1|)\,\mathrm{d}x_1 &\geqslant L(v-\bar{u}) \quad \forall\, v \in K. \end{aligned}$$

Inserting $v := \bar{u}$ and u respectively into the first and the second inequality and then summing them we obtain

$$a(u-\bar{u}, u-\bar{u}) \leqslant \int_{\Gamma_c} (\mathcal{F}(\varphi) - \mathcal{F}(\overline{\varphi}))(|\bar{u}_1| - |u_1|)\,\mathrm{d}x_1.$$

Hence

$$\begin{aligned} \alpha\|u-\bar{u}\|_1^2 \leqslant a(u-\bar{u}, u-\bar{u}) &\leqslant q\|\varphi - \overline{\varphi}\|_{0,\Gamma_c}\|\bar{u}_1 - u_1\|_{0,\Gamma_c}\\ &\leqslant qc_1\|\varphi - \overline{\varphi}\|_{0,\Gamma_c}\|u - \bar{u}\|_{1,\Omega}, \end{aligned}$$

so that

$$\|\,|u_1| - |\bar{u}_1|\,\|_{0,\Gamma_c} \leqslant \|u_1 - \bar{u}_1\|_{0,\Gamma_c} \leqslant c_1\|u-\bar{u}\|_{1,\Omega} \leqslant \frac{qc_1^2}{\alpha}\|\varphi - \overline{\varphi}\|_{0,\Gamma_c},$$

where c_1 is a positive constant. Setting $c := c_1^2/\alpha$ we arrive at (2.12). □

Corollary 2.1. *If $cq < 1$, the mapping $\Psi\colon H_+^{1/2}(\Gamma_c) \mapsto H_+^{1/2}(\Gamma_c)$ is contractive in the $L^2(\Gamma_c)$-norm. Consequently, Ψ has a unique fixed point and the method of successive approximations is convergent.*

Next we present the dual formulation of the Signorini problem with a coefficient of friction depending on the solution. For more detail we refer to [7]. To this end we introduce some notation.

Let $H^{-1/2}(\Gamma_c) := (H^{1/2}(\Gamma_c))'$ be the dual space of $H^{1/2}(\Gamma_c)$ and $\Lambda_2 := H_+^{-1/2}(\Gamma_c)$ the cone of positive functionals, i.e.

$$\mu_2 \in \Lambda_2 \quad \text{iff} \quad \langle \mu_2, \varphi\rangle \geqslant 0 \ \ \forall\, \varphi \in H_+^{1/2}(\Gamma_c),$$

where $\langle\,,\rangle$ stands for the respective duality pairing. With any $\varphi \in H_+^{1/2}(\Gamma_c)$ we associate the convex set

$$\Lambda_1(\varphi) = \{\mu_1 \in L^2(\Gamma_c) \mid |\mu_1| \leqslant \mathcal{F}(\varphi) \ \text{ a.e. on } \Gamma_c\}.$$

The *dual formulation* of the auxiliary problem $(\mathcal{P}(\varphi))$ reads as follows:

$$(\mathcal{D}(\varphi)) \qquad \begin{cases} \text{Find } \lambda := (\lambda_1(\varphi), \lambda_2(\varphi)) \in \Lambda_1(\varphi) \times \Lambda_2 \text{ such that} \\ b(\lambda, \mu - \lambda) \geqslant f(\mu - \lambda) \quad \forall\, \mu = (\mu_1, \mu_2) \in \Lambda_1(\varphi) \times \Lambda_2. \end{cases}$$

The bilinear form $b\colon (H^{-1/2}(\Gamma_c))^2 \times (H^{-1/2}(\Gamma_c))^2 \mapsto \mathbb{R}^1$ and the linear form $f\colon (H^{-1/2}(\Gamma_c))^2 \mapsto \mathbb{R}^1$ are given respectively by

$$b(\mu, \nu) = \langle \mu_1, G_1(\nu_1, \nu_2)\rangle + \langle \mu_2, G_2(\nu_1, \nu_2)\rangle, \tag{2.13}$$

$$f(\mu) = -\langle \mu_1, G_1(L)\rangle - \langle \mu_2, G_2(L)\rangle, \tag{2.14}$$

where $\mu = (\mu_1, \mu_2)$, $\nu = (\nu_1, \nu_2) \in (H^{-1/2}(\Gamma_c))^2$. Here $G = (G_1, G_2) \in \mathcal{L}(\mathbb{V}', \mathbb{V})$ is the Green operator corresponding to the bilinear form a and the functions

$$\hat{z} := G(L) = (G_1(L), G_2(L)),$$
$$\tilde{z} := G(\nu_1, \nu_2) = (G_1(\nu_1, \nu_2), G_2(\nu_1, \nu_2))$$

are solutions to the linear elasticity problems

$$\hat{z} \in \mathbb{V}\colon\ a(\hat{z}, v) = L(v) \quad \forall\, v \in \mathbb{V}, \tag{2.15}$$

$$\tilde{z} \in \mathbb{V}\colon\ a(\tilde{z}, v) = \langle \nu_1, v_1\rangle + \langle \nu_2, v_2\rangle \quad \forall\, v = (v_1, v_2) \in \mathbb{V}. \tag{2.16}$$

The relation between $(\mathcal{P}(\varphi))$ and $(\mathcal{D}(\varphi))$ follows from the next theorem.

Theorem 2.3. *For any $\varphi \in H^{1/2}_+(\Gamma_c)$ there exists a unique solution $\lambda := (\lambda_1(\varphi), \lambda_2(\varphi))$ of $(\mathcal{D}(\varphi))$. In addition,*

$$\lambda_1 = T_1(u(\varphi)), \quad \lambda_2 = T_2(u(\varphi)),$$

where $u(\varphi) \in K$ is the solution to $(\mathcal{P}(\varphi))$.

For the proof we refer to [7].

Denote $\Lambda(\varphi) := \Lambda_1(\varphi) \times \Lambda_2$. Then $(\mathcal{D}(\varphi))$ is equivalent to the following minimalization problem:

$$(\mathcal{D}(\varphi))' \qquad \begin{cases} \text{Find } \lambda := \lambda(\varphi) \in \Lambda(\varphi) \text{ such that} \\ \mathcal{S}(\lambda) \leqslant \mathcal{S}(\mu) \ \ \forall\, \mu \in \Lambda(\varphi), \end{cases}$$

where $\mathcal{S}$ is the dual functional defined by

$$\mathcal{S}(\mu) = \frac{1}{2} b(\mu, \mu) - f(\mu).$$

In order to release the constraint $\mu = (\mu_1, \mu_2) \in \Lambda(\varphi)$ we use again the duality approach. It is readily seen that

$$\min_{\mu \in \Lambda(\varphi)} \mathcal{S}(\mu) = \min_{\substack{\mu_1 \in L^2(\Gamma_c) \\ \mu_2 \in H^{-1/2}(\Gamma_c)}} \ \sup_{\substack{v_1^1, v_1^2 \in L^2_+(\Gamma_c) \\ v_2 \in H^{1/2}_+(\Gamma_c)}} \mathcal{L}(\mu, v_1^1, v_1^2, v_2),$$

where $\mathcal{L}\colon\ L^2(\Gamma_c) \times H^{-1/2}(\Gamma_c) \times (L^2_+(\Gamma_c))^2 \times H^{1/2}_+(\Gamma_c) \mapsto \mathbb{R}^1$ is the Lagrangian defined by

$$\begin{aligned} \mathcal{L}(\mu, v_1^1, v_1^2, v_2) = \frac{1}{2} b(\mu, \mu) - f(\mu) &- \int_{\Gamma_c} (\mathcal{F}(\varphi) - \mu_1) v_1^1 \, \mathrm{d}x_1 \\ &- \int_{\Gamma_c} (\mathcal{F}(\varphi) + \mu_1) v_1^2 \, \mathrm{d}x_1 - \langle \mu_2, v_2 \rangle, \quad \mu = (\mu_1, \mu_2). \end{aligned}$$

Let $(\lambda^\star, w_1^1, w_1^2, w_2)$ be a saddle-point of $\mathcal{L}$ on $L^2(\Gamma_c) \times H^{-1/2}(\Gamma_c) \times (L^2_+(\Gamma_c))^2 \times H^{1/2}_+(\Gamma_c)$, i.e.

$$\begin{gathered} \mathcal{L}(\lambda^\star, v_1^1, v_1^2, v_2) \leqslant \mathcal{L}(\lambda^\star, w_1^1, w_1^2, w_2) \leqslant \mathcal{L}(\mu, w_1^1, w_1^2, w_2) \\ \forall\, \mu \in L^2(\Gamma_c) \times H^{-1/2}(\Gamma_c), \quad \forall\, (v_1^1, v_1^2, v_2) \in (\mathcal{L}^2_+(\Gamma_c))^2 \times H^{1/2}_+(\Gamma_c). \end{gathered}$$

It is known that this problem is equivalent to

$$
(\mathcal{M}(\varphi))\quad
\begin{cases}
\text{Find } (\lambda^\star, w_1^1, w_1^2, w_2) \in L^2(\Gamma_c) \times H^{-1/2}(\Gamma_c) \times (L^2_+(\Gamma_c))^2 \times H^{1/2}_+(\Gamma_c)\colon \\
b(\lambda^\star, \mu) = f(\mu) + \displaystyle\int_{\Gamma_c} (w_1^2 - w_1^1)\mu_1 \, \mathrm{d}x_1 + \langle \mu_2, w_2\rangle \\
\hfill \forall\, \mu \in L^2(\Gamma_c) \times H^{-1/2}(\Gamma_c), \\
\displaystyle\int_{\Gamma_c} (\mathcal{F}(\varphi) - \lambda_1^\star)(v_1^1 - w_1^1)\,\mathrm{d}x_1 + \int_{\Gamma_c} (\mathcal{F}(\varphi) + \lambda_1^\star)(v_1^2 - w_1^2)\,\mathrm{d}x_1 \geqslant 0 \\
\hfill \forall\, (v_1^1, v_1^2) \in (L^2_+(\Gamma_c))^2 \\
\langle \lambda_2^\star, v_2 - w_2\rangle \geqslant 0 \ \ \forall\, v_2 \in H^{1/2}_+(\Gamma_c).
\end{cases}
$$

We now give the interpretation of the saddle-point. First of all $\lambda^\star = \lambda(\varphi)$, where $\lambda(\varphi) \in \Lambda(\varphi)$ solves $\mathcal{D}(\varphi))'$. From $(\mathcal{M}(\varphi))_1$, (2.13) and (2.14) we see that

$$
\langle G_1(\lambda^\star + L), \mu_1\rangle + \langle G_2(\lambda^\star + L), \mu_2\rangle = \int_{\Gamma_c} (w_1^2 - w_1^1)\mu_1 \,\mathrm{d}x_1 + \langle \mu_2, w_2\rangle
$$

for all $\mu \in L^2(\Gamma_c) \times H^{-1/2}(\Gamma_c)$ so that

$$
\begin{cases}
w_1^2 - w_1^1 = G_1(\lambda^\star + L) = u_1(\varphi) & \text{on } \Gamma_c, \\
w_2 = G_2(\lambda^\star + L) = u_2(\varphi) & \text{on } \Gamma_c,
\end{cases}
\tag{2.17}
$$

where $u(\varphi) := (u_1(\varphi), u_2(\varphi)) \in K$ is the solution to $(\mathcal{P}(\varphi))$. Further, from $(\mathcal{M}(\varphi))_2$ and (2.17) we see that

$$
\begin{aligned}
(\mathcal{F}(\varphi) - \lambda_1^\star) w_1^1 = (\mathcal{F}(\varphi) + \lambda_1^\star) w_1^2 = 0 \qquad & \text{a.e.} \quad \text{on } \Gamma_c, \\
w_1^1 = u_1^-(\varphi), \quad w_1^2 = u_1^+(\varphi) \qquad & \qquad\quad \text{on } \Gamma_c.
\end{aligned}
$$

Conversely, it is easy to show that the element $(\lambda(\varphi), u_1^-(\varphi), u_1^+(\varphi), u_2(\varphi))$, where $\lambda(\varphi)$, $u(\varphi) = (u_1(\varphi), u_2(\varphi))$, solves $(\mathcal{D}(\varphi))$, $(\mathcal{P}(\varphi))$ or, in other words, it is a saddle-point of $\mathcal{L}$ on $L^2(\Gamma_c) \times H^{-1/2}(\Gamma_c) \times (L^2_+(\Gamma_c))^2 \times H^{1/2}_+(\Gamma_c)$. Denote by $w := \max\{w_1^1, w_1^2\} \in H^{1/2}_+(\Gamma_c)$. We now present an alternative to Definition 1.1.

Definition 2.1. By a weak solution to the Signorini problem with a coefficient of friction depending on the solution we mean an element $(\lambda^\star, w_1^1, w_1^2, w_2)$ which is the solution of $(\mathcal{M}(w))$, where $w = \max\{w_1^1, w_1^2\}$.

The relation between these two definitions is readily seen:

$$
\lambda^\star = (T_1(u), T_2(u)), \quad (w_1^1, w_1^2, w_2) = (u_1^-, u_1^+, u_2) \ \text{ on } \Gamma_c,
$$

where $u \in K$ is the solution to the Signorini problem in the sense of Definition 1.1.

In this section we shortly describe algorithms for numerical realization of the Signorini problem with the coefficient of friction depending on the solution. These algorithms will be based on the method of successive approximations combined with the dual formulation $(\mathcal{D}(\varphi))$. Two variants which depend on the realization of $(\mathcal{D}(\varphi))$ will be shown.

Variant I

$\varphi^{(0)} \in H_+^{1/2}(\Gamma_c)$ given;
for $\varphi^{(k)} \in H_+^{1/2}(\Gamma_c)$ known compute $\lambda^{(k)}$, $\varphi^{(k+1)}$ by:
(i) $\lambda^{(k)} \in \Lambda(\varphi^{(k)})\colon\ \mathcal{S}(\lambda^{(k)}) \leqslant \mathcal{S}(\mu)\ \ \forall\, \mu \in \Lambda(\varphi^{(k)})$,
i.e. $\lambda^{(k)}$ solves $(\mathcal{D}(\varphi^{(k)}))'$;
(ii) $\varphi^{(k+1)} := w^{(k)} = \max\{w_1^{1(k)}, w_1^{2(k)}\}$ is a part of the solution to $(\mathcal{M}(\varphi^{(k)}))$;
Repeat until stopping criterion.

From its definition we see that $(\mathcal{D}(\varphi^{(k)}))'$ is a *quadratic programming* problem with *simple (box) constraints* which can be efficiently solved by conjugate gradient methods. To obtain the update $\varphi^{(k+1)}$ one does not need to solve the saddle-point problem $(\mathcal{M}(\varphi^{(k)}))$. Indeed, $\varphi^{(k+1)}$ is defined by the Lagrange multipliers releasing the constraint $\mu_1 \in \Lambda_1(\varphi^{(k)})$. These multipliers are at our disposal as soon as $\lambda^{(k)}$ is known. Let us mention that Variant I is convergent provided that the conditions of Corollary 2.1 are satisfied. The next variant consists in a separate minimization of the dual functional $\mathcal{S}$ on the Cartesian product $\Lambda_1(\varphi) \times \Lambda_2$ followed immediately by the update of $\varphi^{(k)}$. It reads as follows:

Variant II

$\varphi^{(0)} \in H_+^{1/2}(\Gamma_c)$, $\lambda_1^{(0)} \in \Lambda_1(\varphi^{(0)})$ given;
for $\varphi^{(k)} \in H_+^{1/2}(\Gamma_c)$, $\lambda_1^{(k)} \in \Lambda_1(\varphi^{(k)})$ known compute:
(i) $\lambda_2^{(k)} \in \Lambda_2\colon\ \mathcal{S}(\lambda_1^{(k)}, \lambda_2^{(k)}) \leqslant \mathcal{S}(\lambda_1^{(k)}, \mu_2)\ \ \forall\, \mu_2 \in \Lambda_2$,
(ii) $\lambda_1^{(k+1)} \in \Lambda_1(\varphi^{(k)})\colon\ \mathcal{S}(\lambda_1^{(k+1)}, \lambda_2^{(k)}) \leqslant \mathcal{S}(\mu_1, \lambda_2^{(k)})\ \ \forall\, \mu_1 \in \Lambda_1(\varphi^{(k)})$,
(iii) $\varphi^{(k+1)} := w^{(k)} = \max\{\tilde{w}_1^{1(k)}, \tilde{w}_1^{2(k)}\}$, where $\tilde{w}_1^{1(k)}$, $\tilde{w}_1^{2(k)}$ are the Lagrange multipliers from step (ii) releasing the constraint $\mu_1 \in \Lambda_1(\varphi^{(k)})$.
Repeat until stopping criterion.

Let us observe that Variant II is formally the same as the splitting type algorithm for numerical realization of contact problems with Coulomb friction (see [4], [1]).

We now present numerical results of several model examples. The elastic body Ω is represented by the rectangle $\Omega = (0,5) \times (0,1)$. The material is characterized by

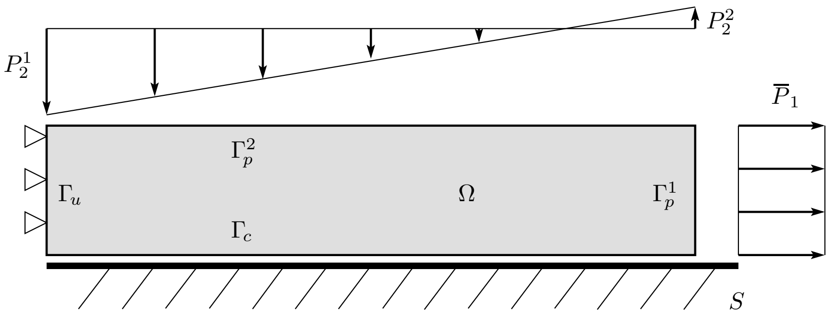

Figure 3.1.

the Young modulus $E = 21.19\mathrm{e}10\,[\mathrm{Pa}]$ and Poisson's ratio $\sigma = 0.277$. The partition of $\partial\Omega$ into Γ_u, Γ_p and Γ_c is seen from Fig. 3.1.

On $\Gamma_p = \Gamma_p^1 \cup \Gamma_p^2$ the body is subject to the surface tractions $P = (P_1, P_2)$:

$$\begin{aligned} &P_1 = 0, \quad P_2 = (1-\lambda)P_2^1 + \lambda P_2^2 && \text{on } \Gamma_p^1, \\ &P_1 = \overline{P}_1, \quad P_2 = 0 && \text{on } \Gamma_p^2, \end{aligned}$$

where $\lambda \in [0,1]$, $P_2^1 = -4.0\mathrm{e}6\,[\mathrm{N{\cdot}m^{-1}}]$, $P_2^2 = 1.0\mathrm{e}6\,[\mathrm{N{\cdot}m^{-1}}]$, $\overline{P}_1 = 4.0\mathrm{e}6\,[\mathrm{N{\cdot}m^{-1}}]$.

The coefficient of friction $\mathcal{F}$ is given by

$$\mathcal{F}(t) = 0.2 + 0.3/(\mathrm{param}) * t + 3), \quad t \geqslant 0,$$

where three different values of param are considered: param $= 5.0\mathrm{e}4$, $1.0\mathrm{e}5$, $1.5\mathrm{e}5$ (see Fig. 3.2). The slip bound was chosen to be $g = 15.0\mathrm{e}6\,[\mathrm{N{\cdot}m^{-1}}]$. The uniform grid consists of 40×8 points; each square was divided into two triangles by its diagonal (left bottom, right top).

Both the variants presented above use the same stopping criterion, namely

$$\frac{\|w^{(k+1)} - w^{(k)}\|}{\|w^{(k)}\|} \leqslant 10^{-6},$$

where $w^{(k)}$ stands for the kth iteration of the modulus of the tangential component of the displacements on Γ_c and $\|x\|$ is the Euclidean norm.

The quadratic programming problems with simple constraints were solved by the conjugate gradient method with proportioning analyzed in [2].

The number of fixed point and conjugate gradient iterations is shown in Tab. 3.1 and 3.2.

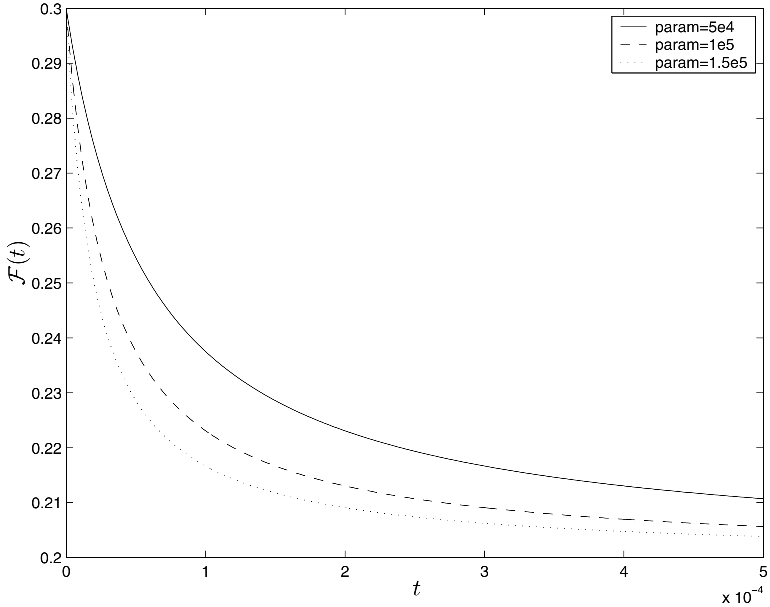

Figure 3.2.

param	fixed point iterations	CG iterations
5.0e4	8	890
10.0e4	9	896
15.0e4	9	920

Table 3.1. Variant I.

param	fixed point iterations	CG iterations
5.0e4	20	480
10.0e4	18	456
15.0e4	18	411

Table 3.2. Variant II.

The convergence history of the method of successive iterations for both variants is depicted in Figs. 3.3 and 3.4.

The behavior of the normal contact stress and the normal contact displacement is shown in Figs. 3.5 and 3.6, respectively. The same for the tangential components is shown in Figs. 3.7 and 3.8.

The graphs corresponding to param = const. illustrate the distribution of the respective physical quantities for the coefficient of friction $\mathcal{F} = 0.3$ which does not depend on the solution.

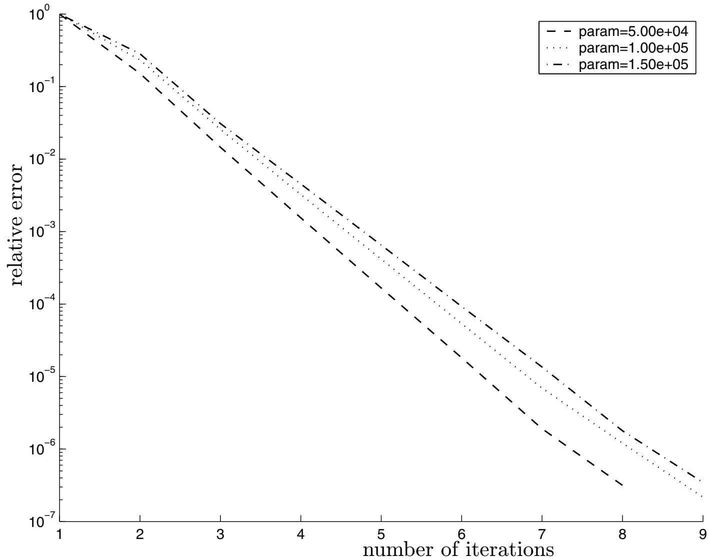

Figure 3.3. Variant I.

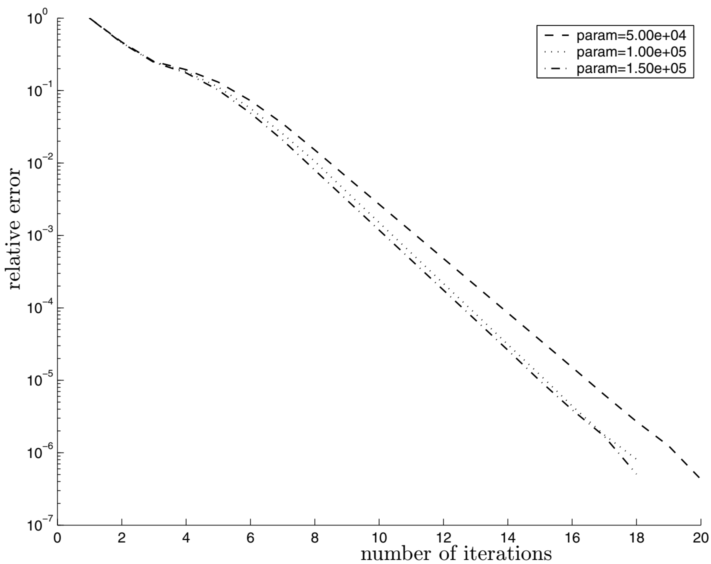

Figure 3.4. Variant II.

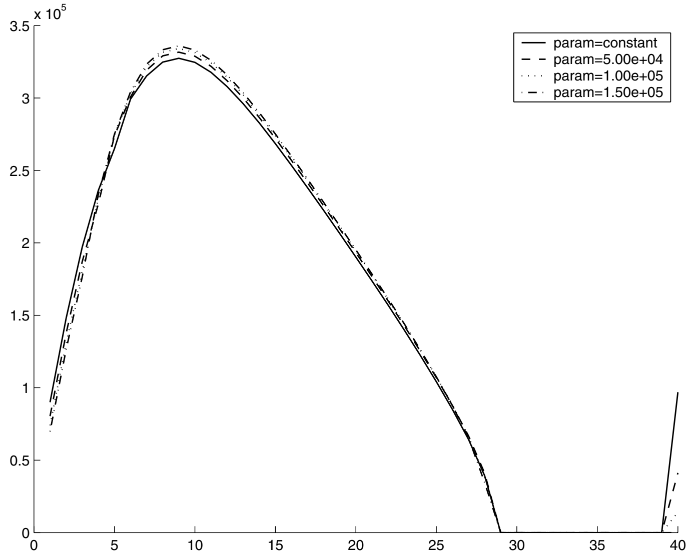

Figure 3.5. Normal contact stress.

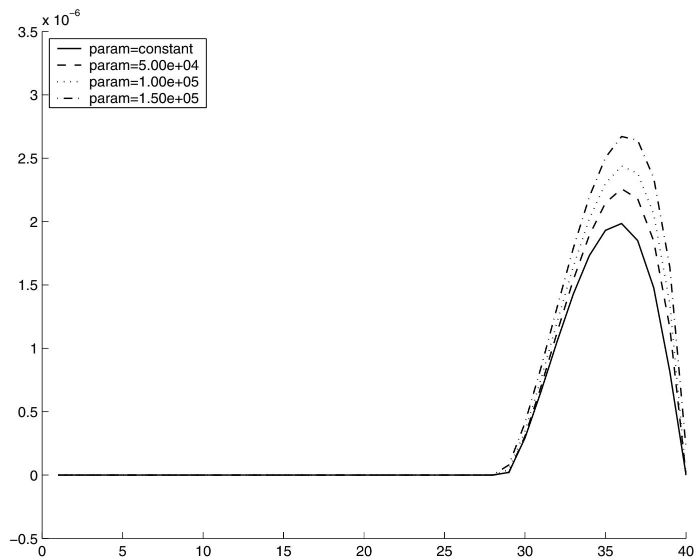

Figure 3.6. Normal contact displacement.

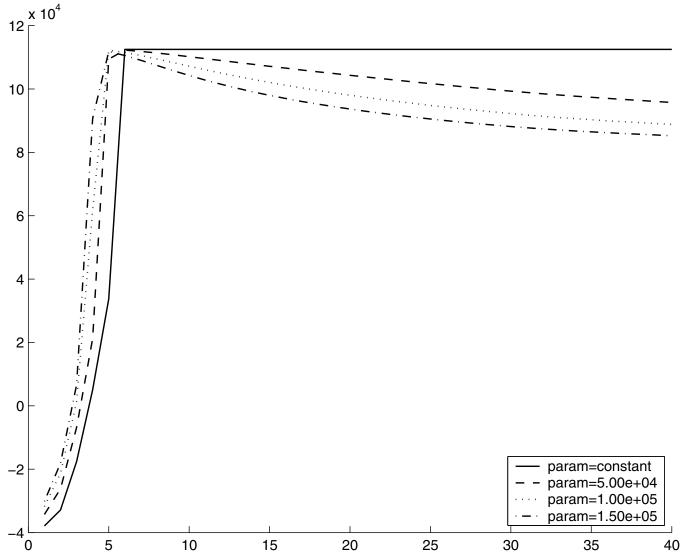

Figure 3.7. Tangential contact stress.

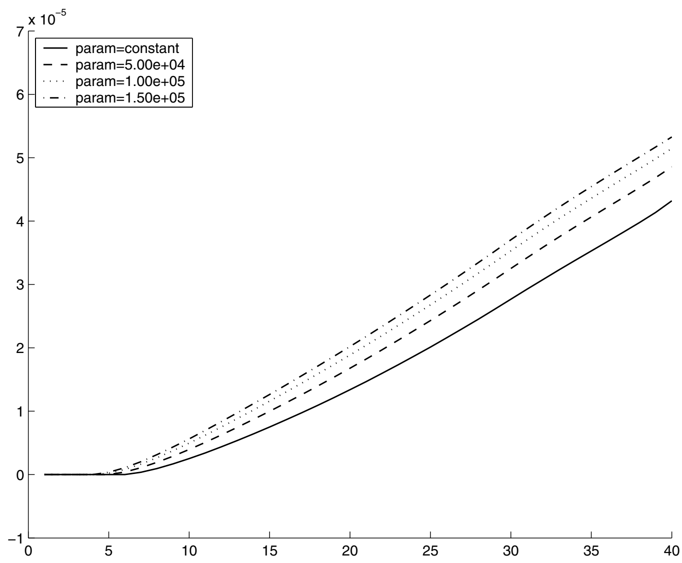

Figure 3.8. Tangential contact displacement.

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