KOLMOGOROV EQUATION AND LARGE-TIME BEHAVIOUR FOR FRACTIONAL BROWNIAN MOTION DRIVEN LINEAR SDE'S*

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Abstract. We consider a stochastic process X_t^x which solves an equation

$$\mathrm{d}X_t^x = AX_t^x\,\mathrm{d}t + \Phi\mathrm{d}B_t^H, \quad X_0^x = x$$

where A and Φ are real matrices and B^H is a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. The Kolmogorov backward equation for the function $u(t,x) = \mathbb{E}f(X_t^x)$ is derived and exponential convergence of probability distributions of solutions to the limit measure is established.

Keywords: fractional Brownian motion, Kolmogorov backwards equation, linear stochastic equation

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1. Introduction

Fractional Brownian motion (fBm) is a family of Gaussian processes indexed by a parameter $H \in (0, 1)$ called the Hurst parameter. The first result on this type of processes goes back to Kolmogorov [13], basic properties of (fBm) have been proved in the pioneering work of Mandelbrot and Van Ness [15]. Hurst [10], [11] used these processes to describe the long term storage capacity of reservoirs along the Nile river, which demonstrated their usefulness as a model for physical phenomena. Later these processes have been used to describe some economic data and most recently to model telecommunication traffic.

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In recent years numerous papers on (fBm)-based stochastic calculus have appeared (e.g. [4] and [1], [3], [5], [6], [7], [20], [21]). Difficulties are coming from the fact that (fBm) is not a semimartingal, so the classical concept of stochastic integral may not be used. On the other hand, the paths of (fBm) do not have bounded variation, hence the classical concept of Lebesgue-Stieltjes integral is useless as well and it is necessary to develop a stochastic calculus. Very recently stochastic differential equations driven by (fBm) have been studied, see e.g. [2], [12], [14], [17] in a finite dimensional state space and [7], [8], [16] in infinite dimensions.

In the present paper we are concerned with a linear stochastic differential equation driven by (fBm). We derive the backward Kolmogorov equation (BKE) associated with the linear stochastic equation. Although our stochastic equation is autonomous, the coefficients of the (BKE) are time-dependent, which reflects the non-Markovian character of the equation. Note that the PDE for the transition density of (fBm) has been derived in [19].

The definition and basic properties of a stochastic integral of a deterministic function with respect to the (fBm) and the formula for integration by parts are in Section 2 of the paper. The main results of the paper are Propositions 3.3, 3.4 and 4.2 where the parabolic equation for $\mathbb{E}f(X_t^x)$ is derived and conditions for exponential convergence in total variation of probability distributions of X_t^x to the limit distribution are established. Note that in [7] strong convergence (but not the speed of convergence) has been shown in an infinite dimensional state space.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathscr{P})$ be a probability space.

Definition 2.1. A fractional Brownian motion $(B^H(t),\, t \geqslant 0)$ with Hurst parameter $H \in (0,1)$ on $(\Omega, \mathcal{F}, \mathscr{P})$ is an $\mathbb{R}^n$-valued Gaussian process which satisfies

(i) $\mathbb{E}B^H(t) = 0$ for all $t \in \mathbb{R}_+$,

(ii) $\mathbb{E}B^H(s)B^H(t)' = 1/2(s^{2H} + t^{2H} - |t-s|^{2H})I$, for all $s, t \in \mathbb{R}_+$, where I is the identity matrix,

(iii) $(B^H(t),\, t \geqslant 0)$ has continuous sample paths $\mathscr{P}$-a.s.

Since the function $F^H(s,t) = 1/2(s^{2H} + t^{2H} - |t-s|^{2H}$ is positive definite for $H \in (0,1)$ there exists a Gaussian process with this covariance function due to the Daniell-Kolmogorov theorem. Continuous modification of this process follows from the Kolmogorov-Tschentsov theorem. From the condition (ii) it is clear that the components of this process are independent. Conditions (i) and (ii) easily imply $B^H(0) = 0$ $\mathscr{P}$-a.s. The (fBm) B^H with $H = 1/2$ is the usual Wiener process. In the rest of the paper we assume that $H > 1/2$.

Now, following [7], we define a stochastic integral with respect to the (fBm), with deterministic integrand. For this purpose we use

Lemma 2.2. *If $p > 1/H$, then for $\varphi \in L^p(0,T;\mathbb{R})$ the inequality*

$$\int_0^T \int_0^T \varphi(u)\varphi(v)\psi(u-v)\,\mathrm{d}u\,\mathrm{d}v \leqslant C|\varphi|^2_{L^p(0,T;\mathbb{R})}$$

holds for some $C > 0$ which depends only on T and p, where

$$\psi(u) = H(2H-1)|u|^{2H-2}. \tag{1}$$

This lemma is an easy consequence of the Young inequality ([18], Proposition 12.25).

First we define the stochastic integral

$$\int_0^T f\,\mathrm{d}\beta^H \tag{2}$$

for $f \in L^p(0,T;\mathbb{R}^n)$ and a scalar (fBm) $(\beta^H(t),\ t \in [0,T])$. Let $\mathcal{E}$ be the family of all $\mathbb{R}^n$-valued step functions, i.e.

$$\mathcal{E} = \Big\{ f\colon\ f(s) = \sum_{i=1}^{n-1} f_i 1_{[t_i,t_{i+1})}(s), \quad 0 = t_1 < t_2 < \ldots < t_n = T \quad \text{and} \\ f_i \in \mathbb{R}^n \quad \text{for } i \in \{1,\ldots,n-1\} \Big\}.$$

For $f \in \mathcal{E}$ we define the stochastic integral (2) as

$$\int_0^T f\,\mathrm{d}\beta^H := \sum_{i=1}^{n-1} f_i(\beta^H(t_{i+1}) - \beta^H(t_i)).$$

The mean of this Gaussian random variable is zero and the covariance matrix is

$$\begin{aligned} &\mathbb{E}\Big(\int_0^T f\,\mathrm{d}\beta^H \int_0^T f'\,\mathrm{d}\beta^H\Big) \\ &\quad= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f_i f_j' \mathbb{E}\big(\beta^H(t_{i+1}) - \beta^H(t_i)\big)\big(\beta^H(t_{j+1}) - \beta^H(t_j)\big) \\ &\quad= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f_i f_j' \mathbb{E}\big(\beta^H(t_{i+1})\beta^H(t_{j+1}) - \beta^H(t_{i+1})\beta^H(t_j) \\ &\qquad - \beta^H(t_i)\beta^H(t_{j+1}) + \beta^H(t_i)\beta(t_j)\big). \end{aligned} \tag{3}$$

It is easy to verify that the last term is equal to

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}} f_i f_j' \psi(u-v)\,\mathrm{d}u\,\mathrm{d}v = \int_0^T\int_0^T f(u)f(v)'\psi(u-v)\,\mathrm{d}u\,\mathrm{d}v. \tag{4}$$

By Lemma 2.2 it follows that

$$\begin{aligned} \Big\|\int_0^T f\,\mathrm{d}\beta^H\Big\|^2_{L^2(\Omega;\mathbb{R}^n)} &= \int_0^T\int_0^T \langle f(u), f(v)\rangle\psi(u-v)\,\mathrm{d}u\,\mathrm{d}v \\ &\leqslant C\Big(\int_0^T |f(s)|^p\,\mathrm{d}s\Big)^{2/p} = C|f|^2_{L^p(0,T;\mathbb{R}^n)}. \end{aligned} \tag{5}$$

Because $\mathcal{E}$ is dense in $L^p(0,T;\mathbb{R}^n)$ there is a sequence $\{f_k\}\subset\mathcal{E}$ such that $|f_k - f|_{L^p(0,T;\mathbb{R}^n)}\to 0$ for $f\in L^p(0,T;\mathbb{R}^n)$. This is a Cauchy sequence in $L^p(0,T;\mathbb{R}^n)$. It follows from inequality (5) that also the sequence of the corresponding stochastic integrals is a Cauchy sequence in the complete space $L^2(\Omega;\mathbb{R}^n)$ and therefore $\int_0^T f_k\,\mathrm{d}\beta^H$ converge to some centered, Gaussian random vector which we denote by $\int_0^T f\,\mathrm{d}\beta^H$.

Now we will define the stochastic integral with respect to an $\mathbb{R}^n$-valued (fBm)

$$\int_0^T G\,\mathrm{d}B^H \tag{6}$$

where $G\colon [0,T]\to\mathbb{R}^{n\times n}$ is a non-random real matrix. We assume that the matrix G satisfies

$$G(\cdot)\in L^p(0,T;\mathbb{R}^{n\times n}) \tag{7}$$

for some $p>1/H$. We define the stochastic integral (6) as

$$I(G;T) := \int_0^T G\,\mathrm{d}B^H := \sum_{i=1}^n \int_0^T G_i\,\mathrm{d}\beta_i^H$$

where $\beta_i^H(\cdot)$, $i=1,\dots,n$, are components of the (fBm) $B^H(\cdot)$ and G_i are the columns of G. Independence of random variables $\int_0^T G_i\,\mathrm{d}\beta_i^H$, conditions (7) and (5) imply

$$\begin{aligned} \mathbb{E}|I(G,T)|^2 &= \sum_{i=1}^n\Big|\int_0^T G_i\,\mathrm{d}\beta^H\Big|^2 \\ &= \sum_{i=1}^n\int_0^T\int_0^T \langle G(s)_i, G(r)_i\rangle\psi(s-r)\,\mathrm{d}s\,\mathrm{d}r<\infty, \end{aligned}$$

therefore the integral (6) is an $\mathbb{R}^n$-valued Gaussian variable. Its probability law is given in the next proposition.

Proposition 2.3. *If $G\colon [0,T] \to \mathbb{R}^{n\times n}$ satisfies condition* (7)*, then $I(G,T)$ is an $\mathbb{R}^n$-valued Gaussian random variable with zero mean and the covariance matrix*

$$\mathbb{E}[I(G,T)I(G,T)'] = \int_0^T \int_0^T G(u)G(v)'\psi(u-v)\,\mathrm{d}u\,\mathrm{d}v \tag{8}$$

where ψ is given by (1)*.*

Proof. $I(G,T)$ is obviously a Gaussian random variable with zero mean. Let C be its covariance matrix. We have

$$\begin{aligned}\langle Cx, y\rangle &= \mathbb{E}[\langle x, I(G,T)\rangle\langle y, I(G,T)\rangle] \\ &= \mathbb{E}\Big[\Big\langle x, \sum_{i=1}^n \int_0^T G_i\,\mathrm{d}\beta_i^H\Big\rangle\Big\langle y, \sum_{j=1}^n \int_0^T G_j\,\mathrm{d}\beta_j^H\Big\rangle\Big] \\ &= \mathbb{E}\Big[\sum_{i=1}^n\Big\langle x, \int_0^T G_i\,\mathrm{d}\beta_i^H\Big\rangle\Big\langle y, \int_0^T G_i\,\mathrm{d}\beta_i^H\Big\rangle\Big]\end{aligned}$$

for all $x, y \in \mathbb{R}^n$. For arbitrary $f \in L^p(0,T;\mathbb{R}^n)$ we have

$$\Big\langle x, \int_0^T f\,\mathrm{d}\beta_i^H\Big\rangle := \int_0^T \langle x, f\rangle\,\mathrm{d}\beta_i^H \quad \text{for } i = 1,\dots,n \text{ and } x \in \mathbb{R}^n.$$

This is true for step functions and therefore for every function from $L^p(0,T;\mathbb{R}^n)$. Consequently,

$$\begin{aligned}\langle Cx, y\rangle &= \sum_{i=1}^n \mathbb{E}\Big[\int_0^T \langle x, G_i\rangle\,\mathrm{d}\beta_i^H \int_0^T \langle y, G_i\rangle\,\mathrm{d}\beta_i^H\Big] \\ &= \sum_{i=1}^n \int_0^T\int_0^T \langle x, G(r)_i\rangle\langle y, G(s)_i\rangle\psi(r-s)\,\mathrm{d}r\,\mathrm{d}s \\ &= \int_0^T\int_0^T \langle G(s)G(r)'x, y\rangle\psi(r-s)\,\mathrm{d}r\,\mathrm{d}s,\end{aligned}$$

and hence $C = \int_0^T\int_0^T G(u)G(v)'\psi(u-v)\,\mathrm{d}u\,\mathrm{d}v$. □

Now it will be shown that the integration by parts is valid. Let $BV(0,T)$ denote the space of functions with bounded variation on $[0,T]$.

Proposition 2.4. *Let $\varphi \in BV(0,T)$ and assume that $\beta^H(t)$ is an $\mathbb{R}$-valued (fBm). Then*

$$\int_0^T \varphi(t)\,\mathrm{d}\beta^H(t) = \varphi(T)\beta^H(T) - \int_0^T \beta^H(t)\,\mathrm{d}\varphi(t) \quad \mathscr{P}\text{-a.s.} \tag{9}$$

where the second term on the right-hand side is the Stieltjes integral.

Proof. Since $\varphi \in BV(0,T)$ there exist nondecreasing functions $\varphi^+, \varphi^-\colon [0,T] \to \mathbb{R}$ such that $\varphi = \varphi^+ - \varphi^-$. We will prove the equality (9) for φ^+ (the proof for φ^- is analogous). Let $\Delta_n(T) := \{0 = t_0^n < t_1^n < \ldots < t_{k_n}^n = T,\, k_n \in \mathbb{N}\}$, $n \in \mathbb{N}$, be a partition of the interval $[0,T]$ such that

$$|\Delta_n(T)| := \max_{0\leqslant k\leqslant k_n} |t_{k+1}^n - t_k^n| \to 0 \quad \text{for } n \to \infty. \tag{10}$$

If we define a sequence of the step functions

$$\varphi_n^+(t) = \sum_{k=0}^{k_n-1} \varphi^+(t_k^n) 1_{[t_k^n, t_{k+1}^n)}(t) \quad \text{for } t \in [0,T), \quad \varphi_n^+(T) = \varphi(T) \text{ for } n \in \mathbb{N},$$

we obtain

$$\begin{aligned}
&\sum_{k=0}^{k_n} \varphi^+(t_k^n)\bigl(\beta^H(t_{k+1}^n) - \beta^H(t_k^n)\bigr) \\
&\quad = \sum_{k=0}^{k_n} \bigl(\varphi^+(t_{k+1}^n)\beta^H(t_{k+1}^n) - \varphi^+(t_k^n)\beta^H(t_k^n) - \beta^H(t_{k+1}^n)(\varphi^+(t_{k+1}^n) - \varphi^+(t_k^n))\bigr)
\end{aligned}$$

and hence

$$\int_0^T \varphi_n^+(t)\,\mathrm{d}\beta^H(t) = \varphi^+(T)\beta^H(T) - \sum_{k=0}^{k_n-1} \beta^H(t_{k+1}^n)\bigl(\varphi^+(t_{k+1}^n) - \varphi^+(t_k^n)\bigr). \tag{11}$$

For every $\omega \in \Omega$ the function $\beta^H(\omega,\cdot)$ is continuous, therefore for $n \to \infty$ the limit on the right-hand side exists and is equal to the Stieltjes integral

$$\int_0^T \beta^H(t)\,\mathrm{d}\varphi^+(t).$$

Functions φ_n^+ converge to φ^+ in $L^p(0,T;\mathbb{R})$, because $1 < 1/H < p$ and

$$\begin{aligned}
\int_0^T |\varphi^+(t) - \varphi_n^+(t)|^p\,\mathrm{d}t &= \sum_{k=0}^{k_n-1} \int_{t_k^n}^{t_{k+1}^n} |\varphi^+(t) - \varphi_n^+(t)|^p\,\mathrm{d}t \\
&\leqslant \sum_{k=0}^{k_n-1} |t_{k+1}^n - t_k^n|\bigl(\varphi^+(t_{k+1}^n) - \varphi^+(t_k^n)\bigr)^p \\
&\leqslant |\Delta_n(T)|\bigl(\varphi^+(T) - \varphi^+(0)\bigr)^p.
\end{aligned}$$

Consequently, we can find a subsequence $\Delta_{n_m}(T)$ such that

$$\int_0^T \varphi_{n_m}^+(t)\,\mathrm{d}\beta^H(t) \to \int_0^T \varphi^+(t)\,\mathrm{d}\beta^H(t) \quad \text{for } m\to\infty \quad \mathscr{P}\text{-a.s.}$$

From the linearity of the stochastic integral we can conclude that $\varphi \in L^p(0,T;\mathbb{R})$ and

$$\begin{aligned}\int_0^T \varphi(t)\,\mathrm{d}\beta^H(t) &:= \int_0^T \varphi^+(t)\,\mathrm{d}\beta^H(t) - \int_0^T \varphi^-(t)\,\mathrm{d}\beta^H(t)\\ &:= \varphi(T)\beta^H(T) - \Big(\int_0^T \beta^H(t)\,\mathrm{d}\varphi^+(t) - \int_0^T \beta^H(t)\,\mathrm{d}\varphi^-(t)\Big)\\ &= \varphi(T)\beta^H(T) - \int_0^T \beta^H(t)\,\mathrm{d}\varphi(t).\end{aligned}$$

□

3. The linear equation

In this section the linear stochastic differential equation

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + \Phi\,\mathrm{d}B_t^H, \quad X_0 = x_0 \tag{12}$$

is studied where $A \in \mathbb{R}^{n\times n}$, $\Phi \in \mathbb{R}^{n\times m}$, $x_0 \in \mathbb{R}^n$ and B^H is an $\mathbb{R}^m$-valued (fBm) defined on the probability space $(\Omega, \mathcal{F}, \mathscr{P})$.

A strong solution of the equation (12) is defined as a continuous process $(X_t,\ t\geqslant 0)$ satisfying

$$X_t = x_0 + \int_0^t AX_s\,\mathrm{d}s + \Phi B_t^H, \quad t\geqslant 0 \ \ \mathscr{P}\text{-a.s..} \tag{13}$$

The strong solution to the equation (12) exists and is given by the so called "mild formula":

$$X_t = \mathrm{e}^{At}x_0 + \int_0^t \mathrm{e}^{A(t-s)}\Phi\,\mathrm{d}B_s^H, \quad t\geqslant 0. \tag{14}$$

As is well known, formula (14) for $H = 1/2$ defines the *Ornstein-Uhlenbeck process* which solves our equation (12) with $H = 1/2$ ($B^{1/2}$ is the Wiener process).

The process defined by (14) is Gaussian with the mean $\mathrm{e}^{At}x_0$ and the covariance matrix

$$Q_t = \int_0^t\int_0^t \mathrm{e}^{Au}\Phi\Phi'\mathrm{e}^{A'v}\psi(u-v)\,\mathrm{d}u\,\mathrm{d}v \tag{15}$$

for $t\geqslant 0$ as follows from Proposition 2.3.

Proposition 3.1. *The strong solution to the equation* (12) *is unique* $\mathscr{P}$*-a.s. and is given by the formula* (14).

P r o o f. First we will prove that the process (14) has a continuous modification. For $n \in \mathbb{N}$ and $t, s \in [n, n+1]$, $t > s$, we have

$$\begin{aligned} \mathbb{E}|X_t - X_s|^2 &\leqslant 2\left|\mathrm{e}^{At}x_0 - \mathrm{e}^{As}x_0\right|^2 + 2\mathbb{E}\left|\int_0^t \mathrm{e}^{A(t-u)}\Phi\,\mathrm{d}B_u^H - \int_0^s \mathrm{e}^{A(s-u)}\Phi\,\mathrm{d}B_u^H\right|^2 \\ &= J_1 + J_2. \end{aligned}$$

There exist constants K_1 and K_2 such that

$$\begin{aligned} J_1 &\leqslant 2\big(|x_0|\,|A|\mathrm{e}^{|A|T}(t-s)\big)^2 \leqslant K_1(t-s)^{2H}, \\ J_2 &= 2\mathbb{E}\left|\int_0^s \big(\mathrm{e}^{A(t-u)} - \mathrm{e}^{A(s-u)}\Phi\big)\,\mathrm{d}B_u^H + \int_s^t \mathrm{e}^{A(t-s)}\Phi\,\mathrm{d}B_u^H\right|^2 \\ &\leqslant 4\mathbb{E}\left|\big(\mathrm{e}^{A(t-s)} - \mathrm{I}\big)\int_0^s \mathrm{e}^{A(s-u)}\Phi\,\mathrm{d}B_u^H\right|^2 + 4\mathbb{E}\left|\int_s^t \mathrm{e}^{A(t-u)}\Phi\,\mathrm{d}B_u^H\right|^2 \\ &\leqslant 4\big(|A|\mathrm{e}^{|A|}(t-s)\big)^2\,\mathrm{Tr}\int_0^s\int_0^s \mathrm{e}^{Au}\Phi\Phi'\mathrm{e}^{A'v}\psi(u-v)\,\mathrm{d}u\,\mathrm{d}v \\ &\quad + 4\,\mathrm{Tr}\int_s^t\int_s^t \mathrm{e}^{Au}\Phi\Phi'\mathrm{e}^{A'v}\psi(u-v)\,\mathrm{d}u\,\mathrm{d}v \\ &\leqslant 4|A|\mathrm{e}^{2|A|}\sup_{s\in[n,n+1]}|\,\mathrm{Tr}\,Q_s|(t-s)^{2H} + 4\,\mathrm{Tr}\sup_{n\leqslant u,\,v\leqslant n+1}\left|\mathrm{e}^{Au}\Phi\Phi'\mathrm{e}^{A'v}\right|(t-s)^{2H} \\ &\leqslant K_2(t-s)^{2H}. \end{aligned}$$

It follows from the Kolmogorov-Tchentsov Theorem that there exists a continuous modification X_t^n of the process X_t in the interval $[n, n+1]$. Therefore we can find sets U_n, $n \in \mathbb{N}$, such that

$$X_{n+1}^n(\omega) = X_{n+1}(\omega) = X_{n+1}^{n+1}(\omega), \quad \omega \in \Omega \setminus U_n, \;\; P(U_n) = 0.$$

It follows that $P\Big(\bigcup\limits_{n=0}^{\infty} U_n\Big) = 0$ and for $\omega \in \Omega \setminus \bigcup\limits_{n=0}^{\infty} U_n$ the process $\tilde{X}_t(\omega) := X_t^n(\omega)$ is a continuous modification of X_t on $[n, n+1]$ (we will write X_t for this modification).

Now we verify that the process (14) satisfies equation (13). We use the equation

$$\int_0^t \left(\int_0^s A\mathrm{e}^{A(s-r)}\Phi\,\mathrm{d}B^H(r)\right)\mathrm{d}s = \int_0^t \left(\int_r^t A\mathrm{e}^{A(s-r)}\,\mathrm{d}s\right)\Phi\,\mathrm{d}B^H(r) \tag{16}$$

which we have got by integration by parts:

$$\int_0^s A\mathrm{e}^{A(s-r)}\Phi\,\mathrm{d}B^H(r) = A\Phi B^H(s) + \int_0^s A^2\mathrm{e}^{A(s-r)}\Phi B^H(r)\,\mathrm{d}r$$

$$\int_0^t \left(\int_r^t A\mathrm{e}^{A(s-r)}\,\mathrm{d}s\right)\Phi\,\mathrm{d}B^H(r) = \int_0^t \left(A + \int_r^t A^2\mathrm{e}^{A(s-r)}\,\mathrm{d}s\right)\Phi B^H(r)\,\mathrm{d}r.$$

Using (16) we have for the left-side of (13)

$$
\begin{aligned}
x_0 &+ \int_0^t AX(s)\,\mathrm{d}s + \Phi B^H(t) \\
&= x_0 + \int_0^t A\Big(\mathrm{e}^{As}x_0 + \int_0^s \mathrm{e}^{A(s-r)}\Phi\,\mathrm{d}B^H(r)\Big)\,\mathrm{d}s + \Phi B^H(t) \\
&= \mathrm{e}^{At}x_0 + \int_0^t \Big(\int_r^t A\mathrm{e}^{A(s-r)}\,\mathrm{d}s\Big)\Phi\,\mathrm{d}B^H(r) + \Phi B^H(t) \\
&= \mathrm{e}^{At}x_0 + \int_0^t \big(\mathrm{e}^{A(t-r)} - I\big)\Phi\,\mathrm{d}B^H(r) + \Phi B^H(t) = X(t)
\end{aligned}
$$

and hence X_t is a strong solution. Let Y_t be any strong solution, then

$$
|X_t - Y_t| = \left|\int_0^t A(X_s - Y_s)\,\mathrm{d}s\right| \leqslant \int_0^t |A|\,|X_s - Y_s|\,\mathrm{d}s \quad \text{for } t \geqslant 0 \ \mathscr{P}\text{-a.s.},
$$

and using the Gronwall lemma we have $P[|X_t - Y_t| = 0,\, t \geqslant 0] = 1$. □

Proposition 3.2. *Let matrices $S(t), Q_t \in \mathbb{R}^{n\times n}$ be differentiable, Q_t symmetric and positive definite, $S(t)$ invertible and let*

$$
p(t,x,y) = \frac{1}{(2\pi)^{n/2}(\det Q_t)^{1/2}} \exp\Big\{-\frac{1}{2}(S(t)x - y)' Q_t^{-1}(S(t)x - y)\Big\} \tag{17}
$$

be the density of the n-dimensional Gaussian distribution $N_n(S(t)x, Q_t)$. Then

$$
\begin{aligned}
\frac{\partial p(t,x,y)}{\partial t} &= \langle S(t)^{-1}\dot{S}(t)x, D_x p(t,x,y)\rangle \\
&\quad + \frac{1}{2}\operatorname{Tr}(S(t)^{-1}\dot{Q}_t(S(t)^{-1})' D_x^2 p(t,x,y)).
\end{aligned} \tag{18}
$$

P r o o f. We are using the following formulae for the time derivatives of the determinant and the inverse matrix:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\det Q_t = \det Q_t \operatorname{Tr}(Q_t^{-1}\dot{Q}_t), \quad \frac{\mathrm{d}}{\mathrm{d}t}Q_t^{-1} = -Q_t^{-1}\dot{Q}_t Q_t^{-1}.
$$

Thus for the first factor in the formula (17) we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\det Q_t)^{-1/2} = -\frac{1}{2}\frac{\det Q_t \operatorname{Tr} Q_t^{-1}\dot{Q}_t}{(\det Q_t)^{3/2}} = -\frac{1}{2}\frac{\operatorname{Tr} Q_t^{-1}\dot{Q}_t}{(\det Q_t)^{1/2}}
$$

and setting $z := S(t)x - y$ we have

$$\begin{aligned}\frac{\partial p(t,x,y)}{\partial t} &= -\,p(t,x,y)\Big\{\frac{1}{2}\,\mathrm{Tr}(Q_t^{-1}\dot{Q}_t) + (\dot{S}(t)x)'Q_t^{-1}z\Big\}\\ &\quad + \frac{1}{2}p(t,x,y)\big\{z'Q_t^{-1}\dot{Q}_tQ_t^{-1}z\big\}.\end{aligned}$$

We have

$$\begin{aligned}D_xp(t,x,y) &= -\,p(t,x,y)S(t)'Q_t^{-1}z,\\ D_x^2p(t,x,y) &= p(t,x,y)S(t)'Q_t^{-1}zz'Q_t^{-1}S(t) - p(t,x,y)S(t)'Q_t^{-1}S(t),\end{aligned}$$

and therefore

$$\begin{aligned}\langle S(t)^{-1}\dot{S}(t)x, D_xp(t,x,y)\rangle &= -\,p(t,x,y)(\dot{S}(t)x)'Q_t^{-1}z,\\ (S(t)^{-1})'D_x^2p(t,x,y)S(t)^{-1}\dot{Q}_t &= p(t,x,y)Q_t^{-1}zz'Q_t^{-1}\dot{Q}_t - p(t,x,y)Q_t^{-1}\dot{Q}_t,\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\,\mathrm{Tr}((S(t)^{-1})'D_x^2p(t,x,y)S(t)^{-1}\dot{Q}_t)\\ &\quad = \frac{1}{2}p(t,x,y)z'Q_t^{-1}\dot{Q}_tQ_t^{-1}z - \frac{1}{2}p(t,x,y)\mathrm{Tr}(Q_t^{-1}\dot{Q}_t).\end{aligned}$$

Now it is easy to see that equation (18) is satisfied. □

Let X_t^x denote the solution of equation (12) with an initial condition $X_0^x = x$.

Proposition 3.3. *Suppose $\Phi\Phi' > 0$ and let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a Borel-measurable function satisfying the condition*

$$\int_{-\infty}^{+\infty} |f(y)|\mathrm{e}^{-a|y|^2}\,\mathrm{d}y < \infty \tag{19}$$

for some $a > 0$. Then there exists $t_0 > 0$ such that the function

$$u(t,x) := \mathbb{E}f(X_t^x) \tag{20}$$

is well defined, differentiable with respect to t for $t \in (0, t_0)$, $u(t,\cdot) \in C^\infty(\mathbb{R}^n)$, and that

$$\frac{\partial u(t,x)}{\partial t} = \langle Ax, D_xu(t,x)\rangle + \mathrm{Tr}(\Phi\Phi'C(t)D_x^2u(t,x)) \tag{21}$$

holds for $(t,x) \in (0,t_0) \times \mathbb{R}^n$ *where* $C(t) = \int_0^t \mathrm{e}^{-A's}\psi(s)\,\mathrm{d}s$ *and* $\psi(s) = H(2H-1)|s|^{2H-2}$. *If* f *is bounded and continuous, then*

$$f(x) = \lim_{t\to 0+} u(t,x) \quad \forall\, x \in \mathbb{R}^n. \tag{22}$$

P r o o f. Let $p(t,x,y)$ be given by (17), where we substitute $S(t) = \mathrm{e}^{At}$ and Q_t according to (15). Note that

$$S(t)^{-1}\dot{S}(t) = \mathrm{e}^{-At}A\mathrm{e}^{At} = A$$

and due to the symmetry of the covariance matrix we have

$$\begin{aligned}
Q_t &= \int_0^t\int_0^t \mathrm{e}^{Au}\Phi\Phi'\mathrm{e}^{A'v}\psi(u-v)\,\mathrm{d}u\,\mathrm{d}v\\
&= 2\int_0^t \mathrm{e}^{Au}\int_0^u \Phi\Phi'\mathrm{e}^{A'v}\psi(u-v)\,\mathrm{d}v\,\mathrm{d}u,
\end{aligned}$$

$$\dot{Q}_t = 2\mathrm{e}^{At}\Phi\Phi'\int_0^t \mathrm{e}^{A'v}\psi(t-v)\,\mathrm{d}v, \tag{23}$$

$$\begin{aligned}
S(t)^{-1}\dot{Q}_t(S(t)^{-1})' &= 2\Phi\Phi'\int_0^t \mathrm{e}^{A'(v-t)}\psi(t-v)\,\mathrm{d}v \qquad (24)\\
&= 2\Phi\Phi'\int_0^t \mathrm{e}^{-A'u}\psi(u)\,\mathrm{d}u.
\end{aligned}$$

Therefore, we have

$$\frac{\partial p(t,x,y)}{\partial t} = \langle Ax, D_x p(t,x,y)\rangle + \mathrm{Tr}(\Phi\Phi'C(t)D_x^2 p(t,x,y)). \tag{25}$$

Now we will prove that the function $u(t,x) = \int_{\mathbb{R}^n} f(y)p(t,x,y)\,\mathrm{d}y$ is differentiable. We bound eigenvalues Q_t from above and from below to find an integrable majorant to derivatives of $p(t,x,y)$. Matrices Q_t are positive definite for $t>0$, therefore there exist orthonormal matrices U_t and diagonal matrices $\Lambda_t > 0$ such that $Q_t = U_t'\Lambda_t U_t$. If we denote the diagonal entries of the matrix Λ_t by $\lambda_i(t)$, then $\lim_{t\to 0}\lambda_i(t) = 0$ for $i \in \{1,\ldots,n\}$. It follows that we can find $t_0 > 0$ such that

$$0 < \lambda_i(t) < (4a)^{-1} \quad \text{for } 0 < t < t_0,\ \ i \in \{1,\ldots,n\}. \tag{26}$$

The function $\det Q_t$ is continuous, positive and so for every $t_1 \in (0,t_0)$

$$\inf_{t_1\leqslant t\leqslant t_0} \det Q_t =: \inf_{t_1\leqslant t\leqslant t_0} \prod_{i=1}^n \lambda_i(t) =: \delta_1 > 0 \tag{27}$$

and for $t \in (t_1, t_0)$ and for all i

$$\lambda_i(t) \geqslant \delta_1 \Big(\prod_{j \neq i} \lambda_j(t) \Big)^{-1} > \delta_1 (4a)^{n-1} =: \delta > 0.$$

Therefore $|Q_t^{-1}|^2 = \sum_{i=1}^{n} \lambda_i(t)^{-2} < n\delta^{-2}$ and we can estimate the normalizing constant in $p(t,x,y)$ by $(2\pi)^{-n/2} (\det Q_t)^{-1/2} < (2\pi)^{-n/2} \delta_1^{-1/2}$ for $t \geqslant t_1$. Further we estimate the exponent in $p(t,x,y)$. Let $|x| < k$ for some constant k. Then for $0 < t < t_0$

$$\begin{aligned}
(28)\quad -\frac{1}{2}\big(\mathrm{e}^{At}x - y\big)' Q_t^{-1} \big(\mathrm{e}^{At}x - y\big) &= -\frac{1}{2}\big(\mathrm{e}^{At}x - y\big)' U_t' \Lambda_t^{-1} U_t \big(\mathrm{e}^{At}x - y\big) \\
&= -\frac{1}{2}\big(U_t \mathrm{e}^{At}x - U_t y\big)' \Lambda_t^{-1} \big(U_t \mathrm{e}^{At}x - U_t y\big) \\
&\leqslant 4a \big(U_t \mathrm{e}^{At}\big)' U_t y - 2a (U_t y)' U_t y \\
&\leqslant 4a |U_t|^2 \mathrm{e}^{|A|t} |x|\,|y| - 2a|y|^2 \\
(29)\qquad &\leqslant 4an\, \mathrm{e}^{|A|t_0} k|y| - 2a|y|^2 \\
&= -2a|y|^2 \bigg(1 - \frac{2nk\, \mathrm{e}^{|A|t_0}}{|y|}\bigg) \leqslant -a|y|^2
\end{aligned}$$

for $|y| \geqslant y_0$, y_0 sufficiently large. If we take $J = (t_1, t_0) \times B_k(0)$ ($B_k(0)$ is the open ball with center 0 and radius k), then we have

$$(30)\qquad \Big| \frac{\partial f(y) p(t,x,y)}{\partial t} \Big| \vee \Big| \frac{\partial^{k+l} f(y) p(t,x,y)}{\partial x_i^k \partial x_j^l} \Big| \leqslant K |f(y)| \mathrm{e}^{-a|y|^2}$$
$$\forall\, y \in \mathbb{R}^n,\ \ \forall\, (t,x) \in J,\ \ i,j \in \{1,\dots,n\},\ \ k,l \in \{0,1\}$$

where K is a constant independent of t and x. Our statement follows from (25) and the theorem about differentiation of an integral with respect to a parameter.

If the function f is bounded and continuous then for almost all ω we have $f(X_t^x(\omega)) \to f(x) < \infty$ for all x and the equation (22) follows from the dominated convergence theorem. $\square$

Corollary 3.4. *If $\Phi\Phi' > 0$, $f\colon \mathbb{R}^n \to \mathbb{R}$ is a Borel-measurable function and there exists $b > 0$ such that*

$$(31)\qquad \int_{-\infty}^{+\infty} |f(y)| \mathrm{e}^{-b|y|} \,\mathrm{d}y < \infty$$

then the statement of Proposition 3.4 is valid for all $t_0 > 0$.

P r o o f. For arbitrary $t_0 > 0$ we have $\sup_{t\in(0,t_0)} \lambda_i(t) < \infty$. Therefore we can find $a > 0$ which satisfies (26) and also (19) because

$$\int_{-\infty}^{+\infty} |f(y)|\mathrm{e}^{-a|y|^2}\,\mathrm{d}y = \int_{-\infty}^{+\infty} \mathrm{e}^{-a|y|^2+b|y|}|f(y)|\mathrm{e}^{-b|y|}\,\mathrm{d}y < \infty$$

for every $a > 0$. $\square$

4. Limit measure for the solution of the equation

In this section we are interested in weak and strong convergence of the probability laws of the solutions to the equation (12). In the case when the matrix A is negatively definite there exists a limit measure in the weak sense. Let $N_n(a, Q)$ denote the n-dimensional Gaussian distribution with mean a and a covariance matrix Q and let μ_t^x denote the distribution of the solution X_t^x to the equation (12) with an initial condition $X_0^x = x$.

Proposition 4.1. *If A is negatively definite then the matrix*

$$Q_\infty = \int_0^\infty \int_0^\infty \mathrm{e}^{Au}\Phi\Phi'\mathrm{e}^{A'v}\psi(u-v)\,\mathrm{d}u\,\mathrm{d}v$$

is well defined and for $\mu_\infty := N(0, Q_\infty)$ we have

$$\lim_{t\to\infty} \mu_t^x = \mu_\infty, \quad w^\star\text{-weakly}$$

for every x.

P r o o f. If A is negatively definite then there exist constants $M, \omega > 0$ such that

$$|\mathrm{e}^{At}| \leqslant M\mathrm{e}^{-\omega t}, \quad t \geqslant 0.$$

We define a random process $Z(t)$ by

$$Z(t) := \int_0^t \mathrm{e}^{Au}\Phi\,\mathrm{d}B^H(u)$$

which has the distribution $\mu_t = N(0, Q_t)$, where Q_t is the covariance matrix from Proposition 3.2. Now we show that $(Z(t),\, t \geqslant 0)$ converge in $L^2(\Omega;\mathbb{R}^n)$ for $t \to \infty$.

Let $t > s \geqslant 1$. There exist constants (all of them are denoted by the same K) such that

$$\begin{aligned}
\mathbb{E}|Z(t) - Z(s)|^2 &= \mathbb{E}\left|\int_s^t \mathrm{e}^{Au}\Phi \,\mathrm{d}B^H(u)\right|^2 \\
&= \operatorname{Tr}\int_s^t\int_s^t \mathrm{e}^{Au}\Phi\Phi'\mathrm{e}^{A'v}\psi(u-v)\,\mathrm{d}u\,\mathrm{d}v \\
&\leqslant \int_s^t\int_s^t \left|\mathrm{e}^{Au}\Phi\right|\left|\mathrm{e}^{A'v}\Phi\right|\psi(u-v)\,\mathrm{d}u\,\mathrm{d}v \\
&\leqslant K\int_s^\infty\int_s^\infty \mathrm{e}^{-\omega u}\mathrm{e}^{-\omega v}\psi(u-v)\,\mathrm{d}u\,\mathrm{d}v \\
&= 2K\int_s^\infty\int_s^v \mathrm{e}^{-\omega u}\mathrm{e}^{-\omega v}\psi(u-v)\,\mathrm{d}u\,\mathrm{d}v \\
&\leqslant K\int_s^\infty \mathrm{e}^{-\omega v}\int_s^v \mathrm{e}^{-\omega u}(v-u)^{2H-2}\,\mathrm{d}u\,\mathrm{d}v \\
&\leqslant K\int_s^\infty \mathrm{e}^{-\omega v}v^{2H-1}\int_0^1 (1-u)^{2H-2}\,\mathrm{d}u\,\mathrm{d}v \\
&\leqslant K\int_s^\infty \mathrm{e}^{-\omega v}v^{2H-1}\,\mathrm{d}v.
\end{aligned}$$

The last integral on the right-hand side converges to 0 for $s \to \infty$. Therefore every sequence $Z(t)$, $t \to \infty$ is Cauchy sequence in $L^2(\Omega;\mathbb{R}^n)$ and hence there exists a random vector $Z(\infty)$ such that $Z(t) \to Z(\infty)$ in $L^2(\Omega)$. The probability law of $Z(\infty)$ is $\mu_\infty = N(0, Q_\infty)$, because $Q_t \to Q_\infty$. For $x \in \mathbb{R}^n$ and $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ a bounded and lipschitz continuous function we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^n}\varphi\,\mathrm{d}\mu_t^x - \int_{\mathbb{R}^n}\varphi\,\mathrm{d}\mu_\infty\right| &= \left|\mathbb{E}\varphi\big(\mathrm{e}^{At}x + Z(t)\big) - \mathbb{E}\varphi(Z(\infty))\right| \\
&\leqslant k_\varphi\big(\left|\mathrm{e}^{At}x\right| + \mathbb{E}|Z(t) - Z(\infty)|\big) \\
&\leqslant k_\varphi(M|x|\mathrm{e}^{-\omega t} + \mathbb{E}|Z(t) - Z(\infty)|^2)^{1/2})
\end{aligned}$$

where k_φ is the Lipschitz constant of φ. Our assertion easily follows. □

In what follows we consider the metric of total variation $\|P_1 - P_2\|_{\mathrm{var}} := \sup\limits_{B\in\mathcal{B}} |P_1(B) - P_2(B)|$ as the "distance" between two probability measures P_1 and P_2 on $\mathcal{B} \equiv \mathcal{B}(\mathbb{R}^n)$. If these measures are absolutely continuous with respect to the Lebesgue measure with densities f_1 and f_2 we have

$$\|P_1 - P_2\|_{\mathrm{var}} = \frac{1}{2}\int_{\mathbb{R}^n} |f_1(x) - f_2(x)|\,\mathrm{d}x. \tag{32}$$

Suppose moreover that $A = A'$ and $\Phi\Phi' A = A\Phi\Phi'$. Then there exists an orthonormal basis (e_i) and numbers $-\alpha_n < -\alpha_{n-1} < \ldots < -\alpha_1 < 0$ and $0 < \lambda_i$ such that

$$Ae_i = -\alpha_i e_i \quad \text{and} \quad \Phi\Phi' e_i = \lambda_i e_i \tag{33}$$

(see [9]). In this case the eigenvalues of Q_t are

$$\begin{aligned} q_i(t) &= \lambda_i \int_0^t \int_0^t \mathrm{e}^{-\alpha_i u} \mathrm{e}^{-\alpha_i v} \psi(u-v) \,\mathrm{d}u \,\mathrm{d}v \\ &= \frac{\lambda_i}{\alpha_i^{2H}} \int_0^{t\alpha_i} \int_0^{t\alpha_i} \mathrm{e}^{-u} \mathrm{e}^{-v} \psi(u-v) \,\mathrm{d}u \,\mathrm{d}v \end{aligned} \tag{34}$$

and the eigenvalues of Q_∞ are

$$q_i = \frac{\lambda_i}{\alpha_i^{2H}} \int_0^\infty \int_0^\infty \mathrm{e}^{-u} \mathrm{e}^{-v} \psi(u-v) \,\mathrm{d}u \,\mathrm{d}v. \tag{35}$$

If we denote

$$f(y,z,t) = (\pi)^{-n/2} (\det Q_t)^{-1/2} \exp\{-(z-y)' Q_t^{-1} (z-y)\}$$

for $y, z \in \mathbb{R}^n$, $t \in \mathbb{R}_+ \cup \{\infty\}$ then the measure μ_t^x has the density $f(\cdot, \mathrm{e}^{At}x, t)$ and the measure μ_∞ has the density $f(\cdot, 0, \infty)$.

Proposition 4.2. *If A and $\Phi\Phi'$ satisfy the above conditions then*

$$\|\mu_t^x - \mu_\infty\|_{\mathrm{var}} \leqslant K(|x|+1)\mathrm{e}^{-t\alpha}, \quad t \geqslant 1, \tag{36}$$

for an arbitrary $0 < \alpha < \alpha_1$ and a constant $K = K(H, \alpha, \lambda_i)$.

Proof. The measures μ_t^x and μ_∞ are absolutely continuous with respect to the Lebesgue measure, therefore

$$\begin{aligned} \|\mu_t^x - \mu_\infty\|_{\mathrm{var}} &= \frac{1}{2} \int_{\mathbb{R}^n} \left| f\big(y, \mathrm{e}^{At}x, t\big) - f(y,0,\infty) \right| \mathrm{d}y \\ &\leqslant \int_{\mathbb{R}^n} \left| f\big(y, \mathrm{e}^{At}x, t\big) - f(y,0,t) \right| \mathrm{d}y \\ &\quad + \int_{\mathbb{R}^n} |f(y,0,t) - (\det Q_t^{-1} \det Q_\infty)^{1/2} f(y,0,\infty)| \,\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^n} |(\det Q_t^{-1} \det Q_\infty)^{1/2} f(y,0,\infty) - f(y,0,\infty)| \,\mathrm{d}y. \end{aligned} \tag{37}$$

Due to the mean value theorem and to the Fubini theorem we can estimate the first term on the right-hand side:

$$\begin{aligned}
(38)\qquad &\int_{\mathbb{R}^n} \bigl|f\bigl(y, \mathrm{e}^{At}x, t\bigr) - f(y,0,t)\bigr| \,\mathrm{d}y \\
&\qquad = \int_{\mathbb{R}^n} \left| \int_0^1 \bigl\langle D_z f\bigl(y, u\mathrm{e}^{At}x, t\bigr), \mathrm{e}^{At}x\bigr\rangle \,\mathrm{d}u \right| \mathrm{d}y \\
&\qquad = \int_{\mathbb{R}^n} \left| \int_0^1 \bigl\langle Q_t^{-1}\bigl(u\mathrm{e}^{At}x - y\bigr), \mathrm{e}^{At}x\bigr\rangle f\bigl(y, u\mathrm{e}^{At}x, t\bigr) \,\mathrm{d}u \right| \mathrm{d}y \\
&\qquad \leqslant |Q_t^{-1}|\,\bigl|\mathrm{e}^{At}x\bigr| \int_0^1 \int_{\mathbb{R}^n} \bigl|u\mathrm{e}^{At}x - y\bigr| f\bigl(y, u\mathrm{e}^{At}x, t\bigr) \,\mathrm{d}y \,\mathrm{d}u.
\end{aligned}$$

Now we substitute $z = u\mathrm{e}^{At}x - y$ and use the Hölder inequality, which yields

$$\begin{aligned}
(39)\qquad |Q_t^{-1}|\,|\mathrm{e}^{At}x| \int_{\mathbb{R}^n} |z| f(z,0,t) \,\mathrm{d}z &\leqslant |Q_t^{-1}|\,|Q_t|\,\bigl|\mathrm{e}^{At}x\bigr| \\
&\leqslant |Q_1^{-1}|\,|Q_\infty|\,|x|\,\mathrm{e}^{-t\alpha_1}.
\end{aligned}$$

The second term on the right-hand side in (37) we estimate as follows:

$$\begin{aligned}
(40)\quad &\int_{\mathbb{R}^n} |f(y,0,t) - (\det Q_t^{-1} \det Q_\infty)^{1/2} f(y,0,\infty)| \,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} (\pi^n \det Q_t)^{-1/2} \left| \exp\Bigl\{-\sum_{i=1}^n q_i(t)^{-1} y_i^2\Bigr\} - \exp\Bigl\{-\sum_{i=1}^n q_i^{-1} y_i^2\Bigr\} \right| \mathrm{d}y \\
&= (\pi^n \det Q_t)^{-1/2} \int_{\mathbb{R}^n} \left| \int_0^1 \frac{\partial}{\partial u} \exp\Bigl\{-\sum_{i=1}^n (q_i^{-1} + u(q_i(t)^{-1} - q_i^{-1})) y_i^2\Bigr\} \,\mathrm{d}u \right| \mathrm{d}y \\
&= (\pi^n \det Q_t)^{-1/2} \sum_{i=1}^n \int_{\mathbb{R}^n} \left(\frac{q_i - q_i(t)}{q_i(t) q_i}\right) y_i^2 \exp\Bigl\{-\sum_{i=1}^n q_i^{-1} y_i^2\Bigr\} \\
&\qquad\qquad \times \int_0^1 \exp\Bigl\{-u \sum_{i=1}^n (q_i(t)^{-1} - q_i^{-1}) y_i^2\Bigr\} \,\mathrm{d}u \,\mathrm{d}y \\
&\leqslant \max_{i \in \{1,\dots,n\}} \left\{\frac{q_i - q_i(t)}{q_i q_i(t)}\right\} (\pi^n \det Q_t)^{-1/2} \int_{\mathbb{R}^n} |y|^2 \exp\Bigl\{-\sum_{i=1}^n q_i^{-1} y_i^2\Bigr\} \,\mathrm{d}y \\
&= \max_{i \in \{1,\dots,n\}} \left\{\frac{q_i - q_i(t)}{q_i q_i(t)}\right\} (\det Q_t^{-1} Q_\infty)^{1/2} |Q_\infty| \\
&\leqslant \sum_{i=1}^n \left(\frac{q_i - q_i(t)}{q_i q_i(t)}\right) (\det Q_1^{-1} Q_\infty)^{1/2} |Q_\infty|,
\end{aligned}$$

because $0 < q(1) < q_i(t)$ for all i and for all $t \geqslant 1$. Let $\alpha < \alpha_1$, then there exist constants k_i (their values may differ from line to line) depending only on H, λ_i, α_i

and α such that

$$
\begin{aligned}
(41)\qquad q_i - q_i(t) &= k_i \int_t^\infty \mathrm{e}^{-\alpha_i u} \int_0^u \mathrm{e}^{-\alpha_i v} |u-v|^{2H-2}\,\mathrm{d}v\,\mathrm{d}u \\
&\leqslant k_i \int_t^\infty \mathrm{e}^{-\alpha_i u} \int_0^\infty \mathrm{e}^{-\alpha_i v} |u-v|^{2H-2}\,\mathrm{d}v\,\mathrm{d}u \\
&= k_i \int_t^\infty \mathrm{e}^{-\alpha_i u} \int_0^\infty \mathrm{e}^{-\alpha_i v} u^{2H-2} |1-v/u|^{2H-2}\,\mathrm{d}v\,\mathrm{d}u \\
&= k_i \int_{t\alpha_i}^\infty \mathrm{e}^{-u} u^{2H-1} \int_0^\infty \mathrm{e}^{-uv} |v-1|^{2H-2}\,\mathrm{d}v\,\mathrm{d}u \\
&\leqslant k_i \int_{t\alpha_i}^\infty \mathrm{e}^{-u} u^{2H-1}\,\mathrm{d}u = k_i \int_t^\infty \mathrm{e}^{-\alpha_i u} u^{2H-1}\,\mathrm{d}u \\
&= k_i \mathrm{e}^{-t\alpha} \int_t^\infty \mathrm{e}^{-(\alpha_i-\alpha)u} u^{2H-2}\,\mathrm{d}u \leqslant k_i \mathrm{e}^{-t\alpha}.
\end{aligned}
$$

Now we can estimate the expression (40) by a negative exponential, because

$$
(42)\qquad \sum_{i=1}^n \biggl(\frac{q_i - q_i(t)}{q_i q_i(t)} \biggr) \leqslant \mathrm{e}^{-t\alpha} \sum_{i=1}^n \frac{k_i}{q_i q_i(1)}.
$$

For the third term in (37) we have

$$
\begin{aligned}
(43)\qquad &\int_{\mathbb{R}^n} |(\det Q_t^{-1} \det Q_\infty)^{1/2} f(y,0,\infty) - f(y,0,\infty)|\,\mathrm{d}y \\
&\qquad = |(\det Q_t^{-1} \det Q_\infty)^{1/2} - 1| = \biggl| \prod_{i=1}^n \sqrt{q_i q_i(t)^{-1}} - 1 \biggr|.
\end{aligned}
$$

As we have

$$
\begin{aligned}
\sqrt{\frac{q_i}{q_i(t)}} = \frac{\sqrt{q_i} - \sqrt{q_i(t)}}{\sqrt{q_i(t)}} + 1 &= \frac{q_i - q_i(t)}{\sqrt{q_i(t)}(\sqrt{q_i} + \sqrt{q_i(t)})} + 1 \\
&\leqslant \frac{q_i - q_i(t)}{q_i(t)} + 1 \leqslant \exp\biggl\{ \frac{q_i - q_i(t)}{q_i(t)} \biggr\}
\end{aligned}
$$

and $0 < q_i - q_i(t) \to 0$ for $t \to +\infty$ we get

$$
(44)\qquad \biggl| \prod_{i=1}^n \sqrt{q_i q_i(t)^{-1}} - 1 \biggr| \leqslant \exp\biggl\{ \sum_{i=1}^n \frac{q_i - q_i(t)}{q_i(t)} \biggr\} - 1 \leqslant K_1 \sum_{i=1}^n \frac{q_i - q_i(t)}{q_i(t)}
$$

for some constant K_1. Similarly to (42) there exists a constant K such that $K\mathrm{e}^{-t\alpha}$, $t \geqslant 1$ is a majorant for the right-hand side of (44). $\square$

R e m a r k 4.3. In particular, if A and Φ are diagonal, Proposition 4.2 may be applied.

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