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DAUBECHIES WAVELETS ON INTERVALS WITH APPLICATION TO BVPS*

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Abstract. In this paper, Daubechies wavelets on intervals are investigated. An analytic technique for evaluating various types of integrals containing the scaling functions is proposed; they are compared with classical techniques. Finally, these results are applied to two-point boundary value problems.

Keywords: Daubechies wavelets, computing scaling integrals, two-point boundary value problems

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1. INTRODUCTION

In addition to the large theoretical interest in wavelet analysis, wavelets also play an important role in numerical analysis. One of these applications consists in replacing piecewise polynomial test functions in finite element methods by wavelets (cf. [7], [10]).

The wavelets we will consider are those with vanishing moments (see [4]). One of the fundamental problems in applying the wavelets is the construction of numerical quadratures to compute the integrals containing scaling functions. To derive such quadrature formulas, we apply so called moments integrals and derive formulas and recursions for the fast computation of the moments integrals of the form $\int x^n\varphi(x)\,\mathrm{d}x$ and $\int x^n\varphi(x-i)\varphi(x-j)\,\mathrm{d}x$, on any dyadic interval. In [15] this was done for the

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case of $\int_0^\infty x^n \varphi(x)\,\mathrm{d}x$, only. We exploit these moments to derive effective quadrature formulas. Finally, we employ these formulas to solve two-point boundary value problems. As the test functions, we have chosen anti-derivatives of the scaling function. These anti-derivatives are smoother and, moreover, the boundary conditions can be treated easily. This idea was introduced in [16]—the integrals were approximated by Simpson's rule. In our approach, the integrals can be effectively and precisely approximated and "variational crimes" can be easily treated, too. We provide a comparison of these two methods.

The paper is organized as follows. First, we shortly deal with wavelets on finite intervals and with construction of bases for anti-derivatives of wavelets. In the next section, we present a procedure for evaluating the scaling moments. Finally, in the last part we apply this theory to two-point boundary value problems.

Now, we briefly review the key-stones of wavelet theory—definitions and basic properties.

Definition 1. Any function $\psi(\cdot) \in L_2(\mathbb{R})$ which generates an orthonormal basis of the space $L_2(\mathbb{R})$ by the system of translations and dilations

$$\{\psi_{j,k}(\cdot)\}_{j,k\in\mathbb{Z}} = \{2^{j/2}\psi(2^j\cdot - k)\}_{j,k\in\mathbb{Z}}$$

is called an orthonormal wavelet.

If we denote $W_j = \operatorname{span}\{\psi_{j,k}(\cdot);\ k \in \mathbb{Z}\}$, we have: $L_2(\mathbb{R}) = \bigoplus\limits_{j=-\infty}^{\infty} W_j$.

Definition 2. A multiresolution analysis $\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots$ with scaling function $\varphi \in L_2(\mathbb{R})$ is an increasing sequence of subspaces of $L_2(\mathbb{R})$ satisfying the following four conditions:

i) (density) $\bigcup\limits_j V_j$ is dense in $L^2(\mathbb{R})$,

ii) (separation) $\bigcap\limits_j V_j = \{0\}$,

iii) scaling $f(\cdot) \in V_j \Leftrightarrow f(2^{-j}\cdot) \in V_0$,

iv) (orthonormality) $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ forms an orthonormal basis for V_0.

From iii) and iv) it follows that $\{\varphi_{j,k}\}_{k\in\mathbb{Z}}$ forms an orthonormal basis of V_j, where

$$\varphi_{j,k}(\cdot) := 2^{j/2}\varphi(2^j\cdot - k). \tag{1}$$

We will assume that the spaces W_j are orthogonal complements of V_j in V_{j+1}, so that $V_j \bigoplus W_j = V_{j+1}\ \forall\, j \in \mathbb{Z}$. From this fact it follows that the scaling function φ satisfies the scaling equation (identity):

$$\varphi(\cdot) = \sum_{k\in\mathbb{Z}} h_k \varphi(2\cdot - k), \tag{2}$$

where h_k are the corresponding *scaling parameters*, and the wavelet ψ satisfies the wavelet equation:

$$\psi(\cdot) = \sum_{k\in\mathbb{Z}} g_k \varphi(2\cdot -k); \tag{3}$$

g_k are called *wavelet parameters.*

Now, we will summarize the properties of the scaling parameters of the Daubechies wavelets. Let p be some positive integer and let:

i) $h_k = 0 \quad \forall\, k \notin \{0, 1, \dots, 2p-1\}$,

ii) $\delta_{m,0} = 2^{-1} \sum_{j=0}^{2p-1} h_j h_{2m+j}$ for $1-p \leqslant m \leqslant p-1$,

iii) $\sum_{k=0}^{2p-1} h_k = 2$,

iv) $\sum_{k=0}^{2p-1} (-1)^k h_k k^n = 0$ for $0 \leqslant n \leqslant p-1$.

Systems $\{h_k\}$ satisfying the conditions above for $1 \leqslant p \leqslant 10$ can be found in [3]. In [12] it is proved that for a fixed p, there exists only one linear independent scaling function φ which satisfies the scaling equation with these scaling parameters. The support of this function lies in $[0, 2p-1]$.

We choose the corresponding wavelet parameters g_k so that

$$g_k := (-1)^k h_{2p-k-1}.$$

Then the support of the wavelet lies in $[0, 2p-1]$ too and it is well-known that this wavelet has p vanishing moments, i.e.

$$\int_0^{2p-1} x^i \psi(x)\,\mathrm{d}x = 0 \quad \forall\, i = 0, \dots, p-1,$$

integrated over the support of the wavelet.

Further in the text, we will deal only with the Daubechies wavelets and in the fourth section we will use for simplicity the Daubechies wavelet with two vanishing moments. However, the construction of frames and the evaluation of the moments integrals can be easily generalized to other types of orthonormal wavelets too. The only reason for the choice of the Daubechies wavelets are their good approximation properties (which depend on the number of vanishing moments).

2. Wavelets on Intervals

There are several ways how to construct a wavelet basis on intervals. The first possibility is a periodic extension of wavelets (see [4], [13]). In this case, however, the vanishing moments, as well as good approximation properties, are destroyed. The second possibility was described in [14] by Meyer. The idea is the following: We take all the wavelets (and scaling functions) supported in our interval and, instead of wavelets supported outside the interval, we construct new boundary wavelets so that the vanishing moments are preserved.

We will take a closer look at the third possibility (see [16]). For convenience of exposition, we will use the term 'frame' for a 'basis' of a vector space in a weak sense (for more details see [16], [5] and [9]).

Definition 3. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a subset of a Banach space X. Let span $\{\varphi_n\}_{n=1}^{\infty}$ be the set of all elements $\sum \alpha_n \varphi_n$ ($\alpha_n \in \mathbb{R}$) which converge (strongly) in X. Then $\{\varphi_n\}_{n=1}^{\infty}$ is called a frame of X if span $\{\varphi_n\}_{n=1}^{\infty} = X$.

A frame generates the space, but the functions can be linearly dependent. By construction, wavelets form an orthonormal basis of $L_2(\mathbb{R})$, but for the application, a basis on a finite interval is required. First, we construct a frame.

Let $p \in \mathbb{N}$ be the number of vanishing moments. For simplicity, we will consider $\Omega := [0, R] = [0, 2p-1]$ here. We choose $j = 0$ as the starting level. It contains the scaling functions from the space V_0 only. If we restrict ourselves just to wavelets and scaling functions with support intersecting Ω, we obtain a frame of $L_2(\Omega)$.

We denote:

$$I_j := \{k \in \mathbb{Z};\ 1 - R \leqslant k \leqslant 2^j R - 1\},\ \ j \geqslant 0. \tag{4}$$

Now, for given $J \in \mathbb{N}$ we define:

$$V_J(\Omega) := \operatorname{span}\{\psi_{j,k}|_\Omega, \varphi_{0,k}|_\Omega;\ \ 0 \leqslant j < J,\ \ k \in I_j\}. \tag{5}$$

Obviously, we have $V_J(\Omega) \subset V_{J+1}(\Omega)$ and $\bigcup\limits_{J=0}^{\infty} V_J(\Omega)$ is dense in $L^2(\Omega)$.

We can easily see that the wavelets in (5) are linearly dependent in Ω. That is why the number of wavelets in the space W_J and scaling functions in the space V_J is altogether $2^{J+1}R + 2R - 2$ whilst the number of scaling functions in the space V_{J+1} is $2^{J+1}R + R - 1$. It follows that the wavelets do not form a basis for $V_j(\Omega)$. The next Theorem shows how we can construct a basis. We define the index set by

$$D_j := \{k \in \mathbb{Z};\ p - R \leqslant k \leqslant 2^j R - p\},\ \ j \geqslant 0.$$

Theorem 4. *For $J \geqslant 0$, there holds* $\dim V_J(\Omega) = 2^J R + R - 1$. *Furthermore, the sets*

$$\{\varphi_{J,k};\ k \in I_J\} \quad \text{and} \quad \{\psi_{j,k}, \varphi_{0,l};\ 0 \leqslant j < J,\ k \in D_j,\ l \in I_0\} \tag{6}$$

are bases of $V_J(\Omega)$.

Proof. A proof can be found e.g. in [16]. □

3. Anti-derivatives of wavelets

Now, we will use anti-derivatives of wavelets to construct frames of $H_0^1(\Omega)$, where we define:

$$H_0^1(\Omega) := \{v \in H^1(\Omega);\ v(0) = v(R) = 0\},$$

where $H^m(\Omega)$ stands for the Sobolev space of functions from $L_2(\Omega)$ whose distributional derivatives up to order m are square integrable.

Let $f \in L_1(\Omega)$. Then $\overline{f}$ will denote its mean value, i.e. $\overline{f} = \frac{1}{R}\int_0^R f(x)\,\mathrm{d}x$.

Lemma 5. *Let* $\{\theta_n\}_{n=1}^{\infty}$ *be a frame of* $L_2(\Omega)$. *Then*

$$\operatorname{span}\left\{\Delta_n;\ \Delta_n(x) := \int_0^x \theta_n(s)\,\mathrm{d}s - x\overline{\theta}_n,\ 0 \leqslant x \leqslant R\right\}_{n=1}^{\infty} \subset H_0^1(\Omega)$$

forms a frame of the space $H_0^1(\Omega)$.

By Theorem 4, Lemma 5, and because of the fact that integration preserves linear independence, we are able to construct a finite-dimensional basis for the subspaces of $H_0^1(\Omega)$. However, a problem may arise because of the extra term $\overline{\theta}_n$. Thus, we have to verify the linear independence of these functions again.

Let us define

$$\Psi_{j,k}(x) = \int_0^x \psi_{j,k}(s)\,\mathrm{d}s - x\overline{\psi}_{j,k}$$

and

$$\Phi_{J,k}(x) = \int_0^x \varphi_{J,k}(s)\,\mathrm{d}s - x\overline{\varphi}_{J,k} \quad \text{for } x \in [0, R].$$

Using the well-known fact that $\sum_{k\in\mathbb{Z}} \varphi(x-k) = 1$ (see [17]), we obtain

$$\begin{aligned}\sum_{k=1-R}^{R-1} \Phi_{0,k} &= \sum_{k=1-R}^{R-1} \left(\int_0^x \varphi(s-k)\,\mathrm{d}s - \frac{x}{R}\int_0^R \varphi(x-k)\,\mathrm{d}x \right) \\ &= \int_0^x \sum_{k=1-R}^{R-1} \varphi(s-k)\,\mathrm{d}s - \frac{x}{R}\int_0^R \sum_{k=1-R}^{R-1} \varphi(x-k)\,\mathrm{d}x \\ &= x - \frac{x}{R}R = 0.\end{aligned}$$

Thus, the set of $\Phi_{0,k}$ for $k \in I_0$ is linearly dependent. The following lemma shows which functions must be removed.

Lemma 6. *The sets*

$$\{\Psi_{j,k}, \Phi_{0,l};\ 0 \leqslant j < J,\ k \in D_j,\ l \in I_0 \setminus \{1-R\}\} \quad \text{and} \quad \{\Phi_{J,k};\ k \in I_J \setminus \{1-R\}\},$$

respectively, define bases for a subspace of $H_0^1(\Omega)$.

In the following, we will use the functions $\varphi_{j,k}$ and therefore we define the space

$$S_0^J := \operatorname{span}\{\Phi_{J,k};\ k \in I_J \setminus \{1-R\}\}. \tag{7}$$

Furthermore, we need to characterize the approximation properties of the anti-derivatives of wavelets with p vanishing moments. The next theorem deals with this problem.

Theorem 7. *Let $J \geqslant 0$, p be a number of vanishing moments and s be a fixed integer. Then for any function $v \in H_0^1(\Omega) \cap H^{s+1}(\Omega)$ the following inequality holds true:*

$$\inf_{\chi\in S_0^J} |v-\chi|_{1,\Omega} \leqslant O(h^s|v|_{s+1,\Omega}), \quad 0 \leqslant s \leqslant p,$$

where $h = 2^{-J}$.

Proofs of Lemma 5, Lemma 6, and Theorem 7, as well as the construction of finite dimensional subspaces of $H^1(\Omega)$, can be found in [16].

4. Applications to two-point boundary value problems

In this section we will apply the anti-derivatives of wavelets (the order p) to numerical solutions of the two-point boundary value problem

$$-(q(x)u'(x))' = f(x) \quad \text{for } x \in \Omega, \tag{8}$$

with Dirichlet boundary conditions

$$u(0) = u(R) = 0.$$

We assume that $f \in L_2(\Omega)$, that the function $q(\cdot)$ is smooth in $(0, R)$, and that $0 < \underline{q} \leqslant q(\cdot) \leqslant \overline{q}$ in $(0, R)$ for some positive constants $\underline{q}$, $\overline{q}$. The variational form of (8) is

$$\int_0^R q(x)u'(x)v'(x)\,\mathrm{d}x = \int_0^R f(x)v(x)\,\mathrm{d}x \quad \forall\, v \in H_0^1(\Omega). \tag{9}$$

By the Lax-Milgram Lemma, this problem has a unique weak solution $u \in H_0^1(\Omega)$.

In (9), if we substitute S_0^J for $H_0^1(\Omega)$, we obtain a Galerkin-wavelet method. Then we search for an approximation of u expressed by

$$\sum_{k\in I_J\setminus\{1-R\}} u_{J,k}\Phi_{J,k},$$

where $u_{J,k}$ are unknown real parameters, and calculate

$$\sum_{k\in I_J\setminus\{1-R\}} u_{J,k} \int_0^R q(x)\Big(\int_0^x \varphi_{J,k}(s)\,\mathrm{d}s - x\overline{\varphi}_{J,k}\Big)'\Big(\int_0^x \varphi_{J,l}(s)\,\mathrm{d}s - x\overline{\varphi}_{J,l}\Big)'\,\mathrm{d}x$$
$$= \int_0^R f(x)\Big(\int_0^x \varphi_{J,l}(s)\,\mathrm{d}s - x\overline{\varphi}_{J,l}\Big)\,\mathrm{d}x \quad \forall\, l \in I_J \setminus \{1-R\}$$

with I_J defined by (4). After differentiation,

$$\sum_{k\in I_J\setminus\{1-R\}} u_{J,k} \int_0^R q(x)(\varphi_{J,k}(x) - \overline{\varphi}_{J,k})(\varphi_{J,l}(x) - \overline{\varphi}_{J,l})\,\mathrm{d}x$$
$$= \int_0^R f(x)\Big(\int_0^x \varphi_{J,l}(s)\,\mathrm{d}s - x\overline{\varphi}_{J,l}\Big)\,\mathrm{d}x \quad \forall\, l \in I_J \setminus \{1-R\}.$$

We can see that we need to approximate three types of integrals. The first step is to compute the so called scaling moments. For simplicity, we will treat these integrals only for the Daubechies wavelet with two vanishing moments (so $p = 2$) on the interval $[0,3]$. Using (1), we can transform them to integrals with φ. For instance

$$\int_0^3 x^n \varphi_{J,k}(x)\,\mathrm{d}x = \frac{1}{2^J}\int_0^3 \Big(\frac{x+k}{2^J}\Big)^n \varphi(x)\,\mathrm{d}x.$$

Similarly, we can transform the other integrals, too. That is why we derive the formulas for evaluating the scaling moments only for the scaling function φ.

When the support of the function $\varphi_{J,k}$ lies in the interval of integration, we will compute the transformed integral on the interval $[a,b]$, where $a = 0$ and $b = 3$. In the opposite case, when the support of the function $\varphi_{J,k}$ lies partly outside the interval $[0,3]$, we have to compute this integral on the interval $[a,b]$, where $a = 0$ and $b \in \{1,2\}$ or $b = 3$ and $a \in \{1,2\}$.

Consequently, we will study the following three types of integrals:

a) $\displaystyle\int_a^b x^n \varphi(x)\,\mathrm{d}x$,

b) $\displaystyle\int_a^b x^n \Big(\int_a^x \varphi(s)\,\mathrm{d}s - \frac{x}{b-a}\int_a^b \varphi(s)\,\mathrm{d}s\Big)\,\mathrm{d}x$,

c) $\displaystyle\int_a^b x^n \varphi(x)\varphi(x-k)\,\mathrm{d}x$.

4.1. Wavelets and numerical integration

a) Let us use the following notations:

$$M_n := \int_0^3 x^n \varphi(x)\,\mathrm{d}x \quad \text{and} \quad m_n := \sum_{k=0}^3 h_k k^n$$

for the continuous moments and for the discrete moments, respectively. Then the following lemma holds.

Lemma 8. *Let $M_0 = 1$ and let us assume that only finitely many h_k do not vanish. Then for any $n \in \mathbb{N}$, the following relation holds true:*

$$M_n = \frac{1}{2^{n+1}-2}\sum_{j=1}^n \binom{n}{j} m_j M_{n-j}. \tag{10}$$

Proof. By substitution, interchanging summation and integration, and using the binomial formula, we obtain

$$\int_0^3 x^n \varphi(x)\,\mathrm{d}x = \sum_{k=0}^3 h_k \int_0^3 x^n \varphi(2x-k)\,\mathrm{d}x = \frac{1}{2^{n+1}}\sum_{k=0}^3 h_k \int_0^3 (y+k)^n \varphi(y)\,\mathrm{d}y$$
$$= \frac{1}{2^{n+1}}\sum_{k=0}^3 h_k \int_0^3 \sum_{j=0}^n \binom{n}{j} k^j y^{n-j}\varphi(y)\,\mathrm{d}y$$

$$
\begin{aligned}
\int_0^3 x^n \varphi(x)\,\mathrm{d}x &= \frac{1}{2^{n+1}} \sum_{j=0}^{n} \binom{n}{j} \int_0^3 y^{n-j} \varphi(y)\,\mathrm{d}y \sum_{k=0}^{3} h_k k^j \\
&= \frac{1}{2^{n+1}} \sum_{j=0}^{n} \binom{n}{j} m_j M_{n-j}.
\end{aligned}
$$

From this, it follows immediately that

$$
M_n = \frac{1}{2^{n+1}-2} \sum_{j=1}^{n} \binom{n}{j} m_j M_{n-j},
$$

which proves the lemma. □

Remark 9. A study of recursion stability is beyond the scope of this article, because we use only the first four moments, i.e. $n \in \{0, 1, 2, 3\}$.

Unfortunately, the formula for computing the moments integrals we have just derived is not usable for evaluation of these moments in the case when the support of the scaling function exceeds the interval. In the case of Daubechies wavelets with two vanishing moments, we have two integrals on both sides with support out of the interval $[0, 3]$. These integrals must be examined separately. Let us denote

$$
A_n := \int_0^2 x^n \varphi(x+1)\,\mathrm{d}x \quad \text{and} \quad B_n := \int_0^1 x^n \varphi(x+2)\,\mathrm{d}x.
$$

Now, we apply the scaling equation to the scaling function $\varphi(\cdot)$:

$$
\begin{aligned}
A_n = h_3 \int_0^2 x^n \varphi(2x-1)\,\mathrm{d}x &+ h_2 \int_0^{1.5} x^n \varphi(2x)\,\mathrm{d}x \\
+ h_1 \int_0^1 x^n \varphi(2x+1)\,\mathrm{d}x &+ h_0 \int_0^{0.5} x^n \varphi(2x+2)\,\mathrm{d}x, \\
B_n = h_3 \int_0^1 x^n \varphi(2x+1)\,\mathrm{d}x &+ h_2 \int_0^{0.5} x^n \varphi(2x+2)\,\mathrm{d}x.
\end{aligned}
$$

After the transformation, we obtain

$$
A_n = \frac{h_3}{2^{n+1}} M_{n,1} + \frac{h_2}{2^{n+1}} M_{n,0} + \frac{h_1}{2^{n+1}} A_n + \frac{h_0}{2^{n+1}} B_n
$$

and

$$
B_n = \frac{h_3}{2^{n+1}} A_n + \frac{h_2}{2^{n+1}} B_n,
$$

where

$$M_{n,l} := \int_0^3 (x+l)^n \varphi(x)\,\mathrm{d}x = \int_0^3 \sum_{i=0}^n \binom{n}{i} l^i x^{n-i} \varphi(x)\,\mathrm{d}x = \sum_{i=0}^n \binom{n}{i} l^i M_{n-i}.$$

(Here we take $0^0 = 1$.) Together we have:

Lemma 10. *For any $n \in \mathbb{N}$, the following relations hold true:*

$$\begin{aligned}(2^{n+1} - h_1)A_n - h_0 B_n &= h_3 M_{n,1} + h_2 M_{n,0},\\ -h_3 A_n + (2^{n+1} - h_2)B_n &= 0.\end{aligned}$$

Thus, for any $n \in \mathbb{N}$ we obtain a regular system of linear equations. The moments that exceeded the right end of the interval $[0,3]$ can be expressed as a linear combination of the already known moments. Let us denote

$$C_n := \int_1^3 x^n \varphi(x-1)\,\mathrm{d}x \quad \text{and} \quad D_n := \int_2^3 x^n \varphi(x-2)\,\mathrm{d}x.$$

After the transformation, we obtain

$$\begin{aligned} C_n &= \int_{-2}^0 (x+3)^n \varphi(x+2)\,\mathrm{d}x \\ &= \int_{-2}^1 (x+3)^n \varphi(x+2)\,\mathrm{d}x - \int_0^1 (x+3)^n \varphi(x+2)\,\mathrm{d}x \\ &= \int_0^3 (x+1)^n \varphi(x)\,\mathrm{d}x - \int_0^1 (x+3)^n \varphi(x+2)\,\mathrm{d}x = \sum_{i=0}^n \binom{n}{i} (M_{n-i} - 3^i B_{n-i}).\end{aligned}$$

After treating D_n analogously, we can summarize the results.

Lemma 11. *For any $n \in \mathbb{N}$, the following equalities hold true:*

$$C_n = \sum_{i=0}^n \binom{n}{i} (M_{n-i} - 3^i B_{n-i}) \quad \textit{and} \quad D_n = \sum_{i=0}^n \binom{n}{i} (2^i M_{n-i} - 3^i A_{n-i}).$$

Remark 12. We also have:

$$\int_0^1 \varphi(x)\,\mathrm{d}x = D_0, \quad \int_1^2 \varphi(x)\,\mathrm{d}x = 1 - D_0 - B_0, \quad \int_2^3 \varphi(x)\,\mathrm{d}x = B_0. \tag{11}$$

From this, we get the following fast and effective way to compute the integrals on any dyadic interval, for $j \in \mathbb{N}$ and $0 \leqslant k \leqslant 3 \cdot 2^j - 1$:

$$\begin{aligned}\int_{k\,2^{-j}}^{(k+1)\,2^{-j}} \varphi(x)\,\mathrm{d}x &= \sum_{l=0}^{3} h_l \int_{k\,2^{-j}}^{(k+1)\,2^{-j}} \varphi(2x-l)\,\mathrm{d}x \\ &= \frac{1}{2}\sum_{l=0}^{3} h_l \int_{k\,2^{(-j+1)}-l}^{(k+1)\,2^{(-j+1)}-l} \varphi(y)\,\mathrm{d}y.\end{aligned}$$

Repeating this succession j-times, we come to the already known integrals (11).

Let us now focus on integrals b). We have to compute only the first term, because the second term is the integral of the scaling function from a to b, which is already known. For $a, b \in \{0,1,2,3\}$, $a < b$, let us denote

$$I_{a,b}^n := \int_a^b x^n \int_a^x \varphi(s)\,\mathrm{d}s\,\mathrm{d}x - \int_a^b \frac{x^{n+1}}{b-a}\,\mathrm{d}x \int_a^b \varphi(s)\,\mathrm{d}s.$$

By integrating by parts, we obtain

Lemma 13. *For any $n \in \mathbb{N}$, the following equality holds true:*

$$I_{a,b}^n = \frac{\varphi(b)b^{n+1} - \varphi(a)a^{n+1}}{n+1} - \int_a^b \left(\left(\frac{x^{n+1} - a^{n+1}}{n+1}\right)\varphi(x) + \frac{x^{n+1}}{b-a}\int_a^b \varphi(s)\,\mathrm{d}s\right)\mathrm{d}x.$$

Remark 14. How can we compute the values of $\varphi(a)$ from Lemma 13? For $l \in \{0,1,2,3\}$, the scaling equation (2) implies

$$\varphi(l) = \sum_{k=0}^{3} h_k \varphi(2l-k) = \sum_{k=2l-3}^{2l} h_{2l-k}\varphi(k). \tag{12}$$

Furthermore, for given scaling parameters there exists a unique scaling function (up to a multiplicative constant). After a normalization $\sum\limits_k \varphi(k) = 1$, the system of equations (12) has only one solution.

Using results in a) and Remark 14, we see that $I_{a,b}^n$ is computable.

c) Let us again consider the case when the support of both scaling functions lies in the interval of integration. We will use the following notation:

$$L_{n,k} := \begin{cases} \displaystyle\int_0^3 x^n \varphi(x)\varphi(x-k)\,\mathrm{d}x & \text{for } k \in \{-2,-1,0,1,2\}, \\ 0 & \text{otherwise.}\end{cases}$$

Lemma 15. *For any $n \in \mathbb{N}$ and $k \in \{-2,-1,0,1,2\}$, the following relation holds true:*

$$L_{n,k} = 2^{-n-1}\left(\sum_{m,l=0}^{3} h_m h_l L_{n,2k+l-m} + \sum_{m,l=0}^{3} h_m h_l \sum_{i=1}^{n} \binom{n}{i} m^i L_{n-i,2k+l-m}\right).$$

Proof. By substitution, interchanging summation and integration, and using the binomial formula, we obtain

$$\begin{aligned}
L_{n,k} &= \int_0^3 x^n \varphi(x)\varphi(x-k)\,\mathrm{d}x \\
&= \sum_{m,l=0}^{3} h_m h_l \int_0^3 x^n \varphi(2x-m)\varphi(2x-2k-l)\,\mathrm{d}x \\
&= 2^{-n-1} \sum_{m,l=0}^{3} h_m h_l \int_0^3 (y+m)^n \varphi(y)\varphi(y+m-2k-l)\,\mathrm{d}y \\
&= 2^{-n-1} \sum_{m,l=0}^{3} h_m h_l \int_0^3 \sum_{i=0}^{n} \binom{n}{i} m^i y^{n-i} \varphi(y)\varphi(y+m-2k-l)\,\mathrm{d}y \\
&= 2^{-n-1}\left(\sum_{m,l=0}^{3} h_m h_l L_{n,2k+l-m} + \sum_{m,l=0}^{3} h_m h_l \sum_{i=1}^{n} \binom{n}{i} m^i L_{n-i,2k+l-m}\right),
\end{aligned}$$

which proves the lemma. □

So, for any fixed n we obtain a regular system of linear equations (of dimension 5). This system was already considered in [11], [15] and it is well-conditioned. Again, this formula cannot be employed for computing these moments when the support of a scaling function does not lie in the interval. In the case of wavelets with two vanishing moments, we have three integrals at both sides, with their support not in the interval $[0,3]$. Let us denote

$$\begin{aligned}
E_n &:= \int_0^2 x^n \varphi^2(x+1)\,\mathrm{d}x, \\
F_n &:= \int_0^1 x^n \varphi^2(x+2)\,\mathrm{d}x, \\
G_n &:= \int_0^1 x^n \varphi(x+2)\varphi(x+1)\,\mathrm{d}x.
\end{aligned}$$

Again, we apply the scaling equation. Thus

$$\begin{aligned}
E_n = h_3^2 \int_{0.5}^{2} x^n \varphi^2(2x-1)\,\mathrm{d}x + h_2^2 \int_{0}^{1.5} x^n \varphi^2(2x)\,\mathrm{d}x + h_1^2 \int_{0}^{1} x^n \varphi^2(2x+1)\,\mathrm{d}x \\
+ h_0^2 \int_{0}^{0.5} x^n \varphi^2(2x+2)\,\mathrm{d}x + 2h_2h_3 \int_{0.5}^{1.5} x^n \varphi(2x-1)\varphi(2x)\,\mathrm{d}x \\
+ 2h_1h_3 \int_{0.5}^{1} x^n \varphi(2x-1)\varphi(2x+1)\,\mathrm{d}x + 2h_2h_1 \int_{0}^{1} x^n \varphi(2x+1)\varphi(2x)\,\mathrm{d}x \\
+ 2h_1h_0 \int_{0}^{0.5} x^n \varphi(2x+1)\varphi(2x+2)\,\mathrm{d}x + 2h_2h_0 \int_{0}^{0.5} x^n \varphi(2x+2)\varphi(2x)\,\mathrm{d}x,
\end{aligned}$$

$$\begin{aligned}
F_n = h_3^2 \int_{0}^{1} x^n \varphi^2(2x+1)\,\mathrm{d}x + h_2^2 \int_{0}^{0.5} x^n \varphi^2(2x+2)\,\mathrm{d}x \\
+ 2h_2h_3 \int_{0}^{0.5} x^n \varphi(2x+1)\varphi(2x+2)\,\mathrm{d}x,
\end{aligned}$$

$$\begin{aligned}
G_n = h_3^2 \int_{0.5}^{1} x^n \varphi(2x-1)\varphi(2x+1)\,\mathrm{d}x + h_2^2 \int_{0}^{0.5} x^n \varphi(2x)\varphi(2x+2)\,\mathrm{d}x \\
+ h_2h_3 \int_{0}^{1} x^n \varphi(2x+1)\varphi(2x)\,\mathrm{d}x + h_3h_0 \int_{0}^{0.5} x^n \varphi(2x+1)\varphi(2x+2)\,\mathrm{d}x \\
+ h_1h_3 \int_{0}^{1} x^n \varphi^2(2x+1)\,\mathrm{d}x + h_2h_1 \int_{0}^{0.5} x^n \varphi(2x+1)\varphi(2x+2)\,\mathrm{d}x \\
+ h_2h_0 \int_{0}^{0.5} x^n \varphi^2(2x+2)\,\mathrm{d}x.
\end{aligned}$$

After the transformation, we obtain

$$\begin{aligned}
2^{n+1}E_n = h_3^2 \int_{0}^{3} (x+1)^n \varphi^2(x)\,\mathrm{d}x + h_2^2 \int_{0}^{3} x^n \varphi^2(x)\,\mathrm{d}x + h_1^2 \int_{0}^{2} x^n \varphi^2(x+1)\,\mathrm{d}x \\
+ h_0^2 \int_{0}^{1} x^n \varphi^2(x+2)\,\mathrm{d}x + 2h_2h_3 \int_{0}^{2} (x+1)^n \varphi(x)\varphi(x+1)\,\mathrm{d}x \\
+ 2h_1h_3 \int_{0}^{1} (x+1)^n \varphi(x)\varphi(x+2)\,\mathrm{d}x + 2h_2h_1 \int_{0}^{2} x^n \varphi(x)\varphi(x+1)\,\mathrm{d}x \\
+ 2h_1h_0 \int_{0}^{1} x^n \varphi(x+1)\varphi(x+2)\,\mathrm{d}x + 2h_2h_0 \int_{0}^{1} x^n \varphi(x+2)\varphi(x)\,\mathrm{d}x,
\end{aligned}$$

$$\begin{aligned}
2^{n+1}F_n = h_3^2 \int_{0}^{2} x^n \varphi^2(x+1)\,\mathrm{d}x + h_2^2 \int_{0}^{1} x^n \varphi^2(x+2)\,\mathrm{d}x \\
+ 2h_2h_3 \int_{0}^{1} x^n \varphi(x+1)\varphi(x+2)\,\mathrm{d}x,
\end{aligned}$$

$$
\begin{aligned}
2^{n+1}G_n = {} & h_3^2 \int_0^1 (x+1)^n \varphi(x)\varphi(x+2)\,\mathrm{d}x + h_2^2 \int_0^1 x^n \varphi(x)\varphi(x+2)\,\mathrm{d}x \\
& + h_2 h_3 \int_0^2 x^n \varphi(x)\varphi(x+1)\,\mathrm{d}x + h_3 h_0 \int_0^1 x^n \varphi(x+1)\varphi(x+2)\,\mathrm{d}x \\
& + h_1 h_3 \int_0^2 x^n \varphi^2(x+1)\,\mathrm{d}x + h_2 h_1 \int_0^1 x^n \varphi(x+1)\varphi(x+2)\,\mathrm{d}x \\
& + h_2 h_0 \int_0^1 x^n \varphi^2(x+2)\,\mathrm{d}x.
\end{aligned}
$$

Together:

Lemma 16. *For any $n \in \mathbb{N}$, and $k, l \in \{0, 1, 2\}$, the following equalities hold true:*

$$
\begin{gathered}
(2^{n+1} - h_1^2)E_n - h_0^2 F_n - 2h_0 h_1 G_n = h_3^2 L_{n,0,1} + h_2^2 L_{n,0,0} + 2h_2 h_3 L_{n,1,1} \\
+ 2h_1 h_3 L_{n,2,1} + 2h_2 h_1 L_{n,1,0} + 2h_0 h_2 L_{n,2,0}, \\
-h_3^2 E_n + (2^{n+1} - h_2^2)F_n - 2h_2 h_3 G_n = 0, \\
-h_1 h_3 E_n - h_2 h_0 F_n + (2^{n+1} - h_1 h_2 - h_0 h_3)G_n = h_3^2 L_{n,2,1} + h_2^2 L_{n,2,0} + h_2 h_3 L_{n,1,0},
\end{gathered}
$$

where

$$
L_{n,k,l} = \int_0^3 (x+l)^n \varphi(x)\varphi(x+k)\,\mathrm{d}x = \sum_{i=0}^{n} \binom{n}{i} l^i L_{n-i,-k}.
$$

The following lemma shows that moments exceeding the right end of the interval $[0, 3]$ can be again transformed to already known moments.

Lemma 17. *For any $n \in \mathbb{N}$, the following relations hold true:*

$$
\begin{aligned}
\int_1^3 x^n \varphi^2(x-1)\,\mathrm{d}x &= \sum_{i=0}^{n} \binom{n}{i} (L_{n-i,0} - 3^i F_{n-i}), \\
\int_2^3 x^n \varphi^2(x-2)\,\mathrm{d}x &= \sum_{i=0}^{n} \binom{n}{i} (2^i L_{n-i,0} - 3^i E_{n-i}), \\
\int_2^3 x^n \varphi^2(x-1)\varphi^2(x-2)\,\mathrm{d}x &= \sum_{i=0}^{n} \binom{n}{i} (2^i L_{n-i,-1} - 3^i G_{n-i}).
\end{aligned}
$$

Using Lemmas 8, 10, 11, 13, 15, 16, and 17, we are able to compute moment integrals for all three required types of integrals (namely a), b), and c)).

Our next goal is to integrate a general function q weighted by, roughly speeking, the scaling function φ; see the Galerkin-wavelet method at the beginning of Section 4.

To this end, we derive quadrature formulas that are exact for the respective moment integrals a)–c) up to a given order. For example, $\int_0^3 q(x)\varphi(x)\,\mathrm{d}x$ can be approximated by a formula $aq(c) + bq(d)$, where a, b are parameters and c, d are points in $[0,3]$. The values of a, b, c, and d are delivered through solving a system of nonlinear equations determined by the requirement of exact integration of moment integrals up to order three. Treating the scaling function (or functions) as a weight function enables to avoid potential negative effect of lacking smoothness of the scaling function on the accuracy of the quadrature formulas.

The quadrature formulas are derived only once. They are ready for future use, so that the Galerkin-wavelet "stiffness" matrix is calculated efficiently.

4.2. Numerical examples

Using Daubechies wavelet with two vanishing moments, we have an error of order 2^{-3J} where J is the resolution level. Of course, this holds only when the solution is sufficiently smooth. Besides its good approximation properties, the advantage of this wavelet is its relatively small support.

Let us compare our method with the approach used in [16]. In [16], the authors implemented the functions φ and ψ (and also Φ and Ψ) by their values at dyadic points in $[0,3]$ (at $x = k/2^D$, for $0 \leqslant k \leqslant 2^D 3$). The integration was done numerically by Simpson's quadrature formula. The accuracy of numerical quadratures was therefore affected by the choice of D. In their computations, they set D between 8 and 12. Here, in computations using their approach, we set $D = 10$. The need of relatively large D is caused by the lack of smoothness of the scaling function (for $p = 2$, the scaling function $\varphi(\cdot) \in C^{0,\alpha}(\overline{\Omega})$, where $\alpha < 0.55$; for fractional order Sobolev spaces, $\varphi \in H^s(\Omega)$, where $s < 1$).

On the contrary, the scaling function is used as a weight function in our quadrature formulas. Then we can avoid the effect of the lacking smoothness of the scaling function. Thus, to approximate arising integrals with a sufficient accuracy, we need significantly less function values. Furthermore, during computations we actually need no function values of the scaling function; we need only scaling parameters.

To calculate the exact errors, we choose such an example where the analytic solution is known,

$$
\begin{aligned}
-u'' &= -x + 1/2 \quad \text{for } x \in (0,3),\\
u(0) &= u(3) = 0.
\end{aligned}
$$

The computational results are summarized in Tab. 1. The first column contains the results computed by Simpson's rule; the second column shows the results computed by the wavelet quadrature rules.

Another, this time more complicated boundary value problem,

$$-(e^{2x}u')' = 4 - 4x \quad \text{for } x \in (0,3),$$
$$u(0) = u(3) = 0.$$

The computational results are summarized in Tab. 2. Again, the first column contains the results computed by Simpson's rule and the second column presents the results computed by the wavelet quadrature rules.

J	$2^{3J}\|u - u_J\|_0$	
	Method 1	Method 2
0	0.030520	0.030520
1	0.029869	0.029869
2	0.029541	0.029541
3	0.029375	0.029375
4	0.029293	0.029292
5	0.029253	0.029250
6	0.029256	0.029224

Table 1.

J	$2^{3J}\|u - u_J\|_0$	
	Method 1	Method 2
0	0.015550	0.028228
1	0.079219	0.078086
2	0.128423	0.179214
3	0.146847	0.319862
4	0.151953	0.360131
5	0.153100	0.236514
6	0.152963	0.172772
7	0.153117	0.157064

Table 2.

To conclude: In the first, simpler example, the wavelet quadrature rules led to the exact evaluation of the relevant integrals. In the second example, the results computed by Simpson's rule are slightly better. However, the order of error is the same as in the case of the wavelet quadrature rules and, moreover, our approach requires significantly less number of function values.

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DAUBECHIES WAVELETS ON INTERVALS WITH APPLICATION TO BVPS*

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Abstract. In this paper, Daubechies wavelets on intervals are investigated. An analytic technique for evaluating various types of integrals containing the scaling functions is proposed; they are compared with classical techniques. Finally, these results are applied to two-point boundary value problems.

Keywords: Daubechies wavelets, computing scaling integrals, two-point boundary value problems

MSC 2000: 65T60, 65N30, 34B05

1. Introduction

In addition to the large theoretical interest in wavelet analysis, wavelets also play an important role in numerical analysis. One of these applications consists in replacing piecewise polynomial test functions in finite element methods by wavelets (cf. [7], [10]).

The wavelets we will consider are those with vanishing moments (see [4]). One of the fundamental problems in applying the wavelets is the construction of numerical quadratures to compute the integrals containing scaling functions. To derive such quadrature formulas, we apply so called moments integrals and derive formulas and recursions for the fast computation of the moments integrals of the form $\int x^n\varphi(x)\,\mathrm{d}x$ and $\int x^n\varphi(x-i)\varphi(x-j)\,\mathrm{d}x$, on any dyadic interval. In [15] this was done for the

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case of $\int_0^\infty x^n \varphi(x)\,\mathrm{d}x$, only. We exploit these moments to derive effective quadrature formulas. Finally, we employ these formulas to solve two-point boundary value problems. As the test functions, we have chosen anti-derivatives of the scaling function. These anti-derivatives are smoother and, moreover, the boundary conditions can be treated easily. This idea was introduced in [16]—the integrals were approximated by Simpson's rule. In our approach, the integrals can be effectively and precisely approximated and "variational crimes" can be easily treated, too. We provide a comparison of these two methods.

The paper is organized as follows. First, we shortly deal with wavelets on finite intervals and with construction of bases for anti-derivatives of wavelets. In the next section, we present a procedure for evaluating the scaling moments. Finally, in the last part we apply this theory to two-point boundary value problems.

Now, we briefly review the key-stones of wavelet theory—definitions and basic properties.

Definition 1. Any function $\psi(\cdot) \in L_2(\mathbb{R})$ which generates an orthonormal basis of the space $L_2(\mathbb{R})$ by the system of translations and dilations

$$\{\psi_{j,k}(\cdot)\}_{j,k\in\mathbb{Z}} = \{2^{j/2}\psi(2^j\cdot - k)\}_{j,k\in\mathbb{Z}}$$

is called an orthonormal wavelet.

If we denote $W_j = \operatorname{span}\{\psi_{j,k}(\cdot);\ k \in \mathbb{Z}\}$, we have: $L_2(\mathbb{R}) = \bigoplus_{j=-\infty}^{\infty} W_j$.

Definition 2. A multiresolution analysis $\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots$ with scaling function $\varphi \in L_2(\mathbb{R})$ is an increasing sequence of subspaces of $L_2(\mathbb{R})$ satisfying the following four conditions:

i) (density) $\bigcup_j V_j$ is dense in $L^2(\mathbb{R})$,

ii) (separation) $\bigcap_j V_j = \{0\}$,

iii) scaling $f(\cdot) \in V_j \Leftrightarrow f(2^{-j}\cdot) \in V_0$,

iv) (orthonormality) $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ forms an orthonormal basis for V_0.

From iii) and iv) it follows that $\{\varphi_{j,k}\}_{k\in\mathbb{Z}}$ forms an orthonormal basis of V_j, where

$$\varphi_{j,k}(\cdot) := 2^{j/2}\varphi(2^j\cdot - k). \tag{1}$$

We will assume that the spaces W_j are orthogonal complements of V_j in V_{j+1}, so that $V_j \bigoplus W_j = V_{j+1}\ \forall\, j \in \mathbb{Z}$. From this fact it follows that the scaling function φ satisfies the scaling equation (identity):

$$\varphi(\cdot) = \sum_{k\in\mathbb{Z}} h_k \varphi(2\cdot - k), \tag{2}$$

where h_k are the corresponding *scaling parameters*, and the wavelet ψ satisfies the wavelet equation:

$$\psi(\cdot) = \sum_{k\in\mathbb{Z}} g_k \varphi(2\cdot - k); \tag{3}$$

g_k are called *wavelet parameters*.

Now, we will summarize the properties of the scaling parameters of the Daubechies wavelets. Let p be some positive integer and let:

i) $h_k = 0 \quad \forall\, k \notin \{0, 1, \ldots, 2p-1\}$,

ii) $\delta_{m,0} = 2^{-1} \sum\limits_{j=0}^{2p-1} h_j h_{2m+j} \quad \text{for } 1-p \leqslant m \leqslant p-1$,

iii) $\sum\limits_{k=0}^{2p-1} h_k = 2$,

iv) $\sum\limits_{k=0}^{2p-1} (-1)^k h_k k^n = 0 \quad \text{for } 0 \leqslant n \leqslant p-1$.

Systems $\{h_k\}$ satisfying the conditions above for $1 \leqslant p \leqslant 10$ can be found in [3]. In [12] it is proved that for a fixed p, there exists only one linear independent scaling function φ which satisfies the scaling equation with these scaling parameters. The support of this function lies in $[0, 2p-1]$.

We choose the corresponding wavelet parameters g_k so that

$$g_k := (-1)^k h_{2p-k-1}.$$

Then the support of the wavelet lies in $[0, 2p-1]$ too and it is well-known that this wavelet has p vanishing moments, i.e.

$$\int_0^{2p-1} x^i \psi(x)\, \mathrm{d}x = 0 \quad \forall\, i = 0, \ldots, p-1,$$

integrated over the support of the wavelet.

Further in the text, we will deal only with the Daubechies wavelets and in the fourth section we will use for simplicity the Daubechies wavelet with two vanishing moments. However, the construction of frames and the evaluation of the moments integrals can be easily generalized to other types of orthonormal wavelets too. The only reason for the choice of the Daubechies wavelets are their good approximation properties (which depend on the number of vanishing moments).

There are several ways how to construct a wavelet basis on intervals. The first possibility is a periodic extension of wavelets (see [4], [13]). In this case, however, the vanishing moments, as well as good approximation properties, are destroyed. The second possibility was described in [14] by Meyer. The idea is the following: We take all the wavelets (and scaling functions) supported in our interval and, instead of wavelets supported outside the interval, we construct new boundary wavelets so that the vanishing moments are preserved.

We will take a closer look at the third possibility (see [16]). For convenience of exposition, we will use the term 'frame' for a 'basis' of a vector space in a weak sense (for more details see [16], [5] and [9]).

Definition 3. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a subset of a Banach space X. Let $\operatorname{span}\{\varphi_n\}_{n=1}^{\infty}$ be the set of all elements $\sum \alpha_n \varphi_n$ ($\alpha_n \in \mathbb{R}$) which converge (strongly) in X. Then $\{\varphi_n\}_{n=1}^{\infty}$ is called a frame of X if $\operatorname{span}\{\varphi_n\}_{n=1}^{\infty} = X$.

A frame generates the space, but the functions can be linearly dependent. By construction, wavelets form an orthonormal basis of $L_2(\mathbb{R})$, but for the application, a basis on a finite interval is required. First, we construct a frame.

Let $p \in \mathbb{N}$ be the number of vanishing moments. For simplicity, we will consider $\Omega := [0, R] = [0, 2p-1]$ here. We choose $j = 0$ as the starting level. It contains the scaling functions from the space V_0 only. If we restrict ourselves just to wavelets and scaling functions with support intersecting Ω, we obtain a frame of $L_2(\Omega)$.

We denote:

$$I_j := \{k \in \mathbb{Z};\ 1 - R \leqslant k \leqslant 2^j R - 1\},\ \ j \geqslant 0. \tag{4}$$

Now, for given $J \in \mathbb{N}$ we define:

$$V_J(\Omega) := \operatorname{span}\{\psi_{j,k}|_\Omega, \varphi_{0,k}|_\Omega;\ \ 0 \leqslant j < J,\ \ k \in I_j\}. \tag{5}$$

Obviously, we have $V_J(\Omega) \subset V_{J+1}(\Omega)$ and $\bigcup\limits_{J=0}^{\infty} V_J(\Omega)$ is dense in $L^2(\Omega)$.

We can easily see that the wavelets in (5) are linearly dependent in Ω. That is why the number of wavelets in the space W_J and scaling functions in the space V_J is altogether $2^{J+1}R + 2R - 2$ whilst the number of scaling functions in the space V_{J+1} is $2^{J+1}R + R - 1$. It follows that the wavelets do not form a basis for $V_j(\Omega)$. The next Theorem shows how we can construct a basis. We define the index set by

$$D_j := \{k \in \mathbb{Z};\ p - R \leqslant k \leqslant 2^j R - p\},\ \ j \geqslant 0.$$

Theorem 4. *For $J \geqslant 0$, there holds* $\dim V_J(\Omega) = 2^J R + R - 1$. *Furthermore, the sets*

$$\{\varphi_{J,k};\ \ k \in I_J\} \quad \textit{and} \quad \{\psi_{j,k}, \varphi_{0,l};\ \ 0 \leqslant j < J,\ \ k \in D_j,\ \ l \in I_0\} \tag{6}$$

are bases of $V_J(\Omega)$.

Proof. A proof can be found e.g. in [16]. □

3. Anti-derivatives of wavelets

Now, we will use anti-derivatives of wavelets to construct frames of $H_0^1(\Omega)$, where we define:

$$H_0^1(\Omega) := \{v \in H^1(\Omega);\ \ v(0) = v(R) = 0\},$$

where $H^m(\Omega)$ stands for the Sobolev space of functions from $L_2(\Omega)$ whose distributional derivatives up to order m are square integrable.

Let $f \in L_1(\Omega)$. Then $\overline{f}$ will denote its mean value, i.e. $\overline{f} = \frac{1}{R}\int_0^R f(x)\,\mathrm{d}x$.

Lemma 5. *Let* $\{\theta_n\}_{n=1}^{\infty}$ *be a frame of* $L_2(\Omega)$. *Then*

$$\operatorname{span}\left\{\Delta_n;\ \ \Delta_n(x) := \int_0^x \theta_n(s)\,\mathrm{d}s - x\overline{\theta}_n,\ \ 0 \leqslant x \leqslant R\right\}_{n=1}^{\infty} \subset H_0^1(\Omega)$$

forms a frame of the space $H_0^1(\Omega)$.

By Theorem 4, Lemma 5, and because of the fact that integration preserves linear independence, we are able to construct a finite-dimensional basis for the subspaces of $H_0^1(\Omega)$. However, a problem may arise because of the extra term $\overline{\theta}_n$. Thus, we have to verify the linear independence of these functions again.

Let us define

$$\Psi_{j,k}(x) = \int_0^x \psi_{j,k}(s)\,\mathrm{d}s - x\overline{\psi}_{j,k}$$

and

$$\Phi_{J,k}(x) = \int_0^x \varphi_{J,k}(s)\,\mathrm{d}s - x\overline{\varphi}_{J,k} \quad \text{for } x \in [0, R].$$

Using the well-known fact that $\sum\limits_{k\in\mathbb{Z}} \varphi(x-k) = 1$ (see [17]), we obtain

$$\begin{aligned}
\sum_{k=1-R}^{R-1} \Phi_{0,k} &= \sum_{k=1-R}^{R-1} \Bigl(\int_0^x \varphi(s-k)\,\mathrm{d}s - \frac{x}{R}\int_0^R \varphi(x-k)\,\mathrm{d}x\Bigr) \\
&= \int_0^x \sum_{k=1-R}^{R-1} \varphi(s-k)\,\mathrm{d}s - \frac{x}{R}\int_0^R \sum_{k=1-R}^{R-1} \varphi(x-k)\,\mathrm{d}x \\
&= x - \frac{x}{R}R = 0.
\end{aligned}$$

Thus, the set of $\Phi_{0,k}$ for $k \in I_0$ is linearly dependent. The following lemma shows which functions must be removed.

Lemma 6. *The sets*

$$\{\Psi_{j,k}, \Phi_{0,l};\ 0 \leqslant j < J,\ k \in D_j,\ l \in I_0 \setminus \{1-R\}\} \quad \textit{and} \quad \{\Phi_{J,k};\ k \in I_J \setminus \{1-R\}\},$$

respectively, define bases for a subspace of $H_0^1(\Omega)$.

In the following, we will use the functions $\varphi_{j,k}$ and therefore we define the space

$$S_0^J := \operatorname{span}\{\Phi_{J,k};\ k \in I_J \setminus \{1-R\}\}. \tag{7}$$

Furthermore, we need to characterize the approximation properties of the antiderivatives of wavelets with p vanishing moments. The next theorem deals with this problem.

Theorem 7. *Let $J \geqslant 0$, p be a number of vanishing moments and s be a fixed integer. Then for any function $v \in H_0^1(\Omega) \cap H^{s+1}(\Omega)$ the following inequality holds true:*

$$\inf_{\chi\in S_0^J} |v-\chi|_{1,\Omega} \leqslant O(h^s|v|_{s+1,\Omega}), \quad 0 \leqslant s \leqslant p,$$

where $h = 2^{-J}$.

Proofs of Lemma 5, Lemma 6, and Theorem 7, as well as the construction of finite dimensional subspaces of $H^1(\Omega)$, can be found in [16].

4. Applications to two-point boundary value problems

In this section we will apply the anti-derivatives of wavelets (the order p) to numerical solutions of the two-point boundary value problem

$$-(q(x)u'(x))' = f(x) \quad \text{for } x \in \Omega, \tag{8}$$

with Dirichlet boundary conditions

$$u(0) = u(R) = 0.$$

We assume that $f \in L_2(\Omega)$, that the function $q(\cdot)$ is smooth in $(0, R)$, and that $0 < \underline{q} \leqslant q(\cdot) \leqslant \overline{q}$ in $(0, R)$ for some positive constants $\underline{q}$, $\overline{q}$. The variational form of (8) is

$$\int_0^R q(x)u'(x)v'(x)\,\mathrm{d}x = \int_0^R f(x)v(x)\,\mathrm{d}x \quad \forall\, v \in H_0^1(\Omega). \tag{9}$$

By the Lax-Milgram Lemma, this problem has a unique weak solution $u \in H_0^1(\Omega)$.

In (9), if we substitute S_0^J for $H_0^1(\Omega)$, we obtain a Galerkin-wavelet method. Then we search for an approximation of u expressed by

$$\sum_{k \in I_J \setminus \{1-R\}} u_{J,k} \Phi_{J,k},$$

where $u_{J,k}$ are unknown real parameters, and calculate

$$\begin{aligned}\sum_{k \in I_J \setminus \{1-R\}} u_{J,k} \int_0^R q(x) \Big(\int_0^x \varphi_{J,k}(s)\,\mathrm{d}s - x\overline{\varphi}_{J,k} \Big)' \Big(\int_0^x \varphi_{J,l}(s)\,\mathrm{d}s - x\overline{\varphi}_{J,l} \Big)' \,\mathrm{d}x \\ = \int_0^R f(x) \Big(\int_0^x \varphi_{J,l}(s)\,\mathrm{d}s - x\overline{\varphi}_{J,l} \Big) \,\mathrm{d}x \quad \forall\, l \in I_J \setminus \{1-R\}\end{aligned}$$

with I_J defined by (4). After differentiation,

$$\begin{aligned}\sum_{k \in I_J \setminus \{1-R\}} u_{J,k} \int_0^R q(x)(\varphi_{J,k}(x) - \overline{\varphi}_{J,k})(\varphi_{J,l}(x) - \overline{\varphi}_{J,l})\,\mathrm{d}x \\ = \int_0^R f(x) \Big(\int_0^x \varphi_{J,l}(s)\,\mathrm{d}s - x\overline{\varphi}_{J,l} \Big) \,\mathrm{d}x \quad \forall\, l \in I_J \setminus \{1-R\}.\end{aligned}$$

We can see that we need to approximate three types of integrals. The first step is to compute the so called scaling moments. For simplicity, we will treat these integrals only for the Daubechies wavelet with two vanishing moments (so $p = 2$) on the interval $[0, 3]$. Using (1), we can transform them to integrals with φ. For instance

$$\int_0^3 x^n \varphi_{J,k}(x)\,\mathrm{d}x = \frac{1}{2^J}\int_0^3 \Big(\frac{x+k}{2^J}\Big)^n \varphi(x)\,\mathrm{d}x.$$

Similarly, we can transform the other integrals, too. That is why we derive the formulas for evaluating the scaling moments only for the scaling function φ.

When the support of the function $\varphi_{J,k}$ lies in the interval of integration, we will compute the transformed integral on the interval $[a, b]$, where $a = 0$ and $b = 3$. In the opposite case, when the support of the function $\varphi_{J,k}$ lies partly outside the interval $[0, 3]$, we have to compute this integral on the interval $[a, b]$, where $a = 0$ and $b \in \{1, 2\}$ or $b = 3$ and $a \in \{1, 2\}$.

Consequently, we will study the following three types of integrals:

a) $\displaystyle\int_a^b x^n \varphi(x)\,\mathrm{d}x$,

b) $\displaystyle\int_a^b x^n \Big(\int_a^x \varphi(s)\,\mathrm{d}s - \frac{x}{b-a}\int_a^b \varphi(s)\,\mathrm{d}s\Big)\,\mathrm{d}x$,

c) $\displaystyle\int_a^b x^n \varphi(x)\varphi(x-k)\,\mathrm{d}x$.

4.1. Wavelets and numerical integration

a) Let us use the following notations:

$$M_n := \int_0^3 x^n \varphi(x)\,\mathrm{d}x \quad \text{and} \quad m_n := \sum_{k=0}^3 h_k k^n$$

for the continuous moments and for the discrete moments, respectively. Then the following lemma holds.

Lemma 8. *Let $M_0 = 1$ and let us assume that only finitely many h_k do not vanish. Then for any $n \in \mathbb{N}$, the following relation holds true:*

$$M_n = \frac{1}{2^{n+1}-2}\sum_{j=1}^n \binom{n}{j} m_j M_{n-j}. \tag{10}$$

Proof. By substitution, interchanging summation and integration, and using the binomial formula, we obtain

$$\int_0^3 x^n \varphi(x)\,\mathrm{d}x = \sum_{k=0}^3 h_k \int_0^3 x^n \varphi(2x-k)\,\mathrm{d}x = \frac{1}{2^{n+1}}\sum_{k=0}^3 h_k \int_0^3 (y+k)^n \varphi(y)\,\mathrm{d}y$$
$$= \frac{1}{2^{n+1}}\sum_{k=0}^3 h_k \int_0^3 \sum_{j=0}^n \binom{n}{j} k^j y^{n-j} \varphi(y)\,\mathrm{d}y$$

$$\int_0^3 x^n \varphi(x)\,\mathrm{d}x = \frac{1}{2^{n+1}} \sum_{j=0}^{n} \binom{n}{j} \int_0^3 y^{n-j} \varphi(y)\,\mathrm{d}y \sum_{k=0}^{3} h_k k^j$$
$$= \frac{1}{2^{n+1}} \sum_{j=0}^{n} \binom{n}{j} m_j M_{n-j}.$$

From this, it follows immediately that

$$M_n = \frac{1}{2^{n+1}-2} \sum_{j=1}^{n} \binom{n}{j} m_j M_{n-j},$$

which proves the lemma. □

Remark 9. A study of recursion stability is beyond the scope of this article, because we use only the first four moments, i.e. $n \in \{0,1,2,3\}$.

Unfortunately, the formula for computing the moments integrals we have just derived is not usable for evaluation of these moments in the case when the support of the scaling function exceeds the interval. In the case of Daubechies wavelets with two vanishing moments, we have two integrals on both sides with support out of the interval $[0,3]$. These integrals must be examined separately. Let us denote

$$A_n := \int_0^2 x^n \varphi(x+1)\,\mathrm{d}x \quad \text{and} \quad B_n := \int_0^1 x^n \varphi(x+2)\,\mathrm{d}x.$$

Now, we apply the scaling equation to the scaling function $\varphi(\cdot)$:

$$\begin{aligned} A_n &= h_3 \int_0^2 x^n \varphi(2x-1)\,\mathrm{d}x + h_2 \int_0^{1.5} x^n \varphi(2x)\,\mathrm{d}x \\ &\quad + h_1 \int_0^1 x^n \varphi(2x+1)\,\mathrm{d}x + h_0 \int_0^{0.5} x^n \varphi(2x+2)\,\mathrm{d}x, \\ B_n &= h_3 \int_0^1 x^n \varphi(2x+1)\,\mathrm{d}x + h_2 \int_0^{0.5} x^n \varphi(2x+2)\,\mathrm{d}x. \end{aligned}$$

After the transformation, we obtain

$$A_n = \frac{h_3}{2^{n+1}} M_{n,1} + \frac{h_2}{2^{n+1}} M_{n,0} + \frac{h_1}{2^{n+1}} A_n + \frac{h_0}{2^{n+1}} B_n$$

and

$$B_n = \frac{h_3}{2^{n+1}} A_n + \frac{h_2}{2^{n+1}} B_n,$$

where

$$M_{n,l} := \int_0^3 (x+l)^n \varphi(x)\,\mathrm{d}x = \int_0^3 \sum_{i=0}^{n} \binom{n}{i} l^i x^{n-i} \varphi(x)\,\mathrm{d}x = \sum_{i=0}^{n} \binom{n}{i} l^i M_{n-i}.$$

(Here we take $0^0 = 1$.) Together we have:

Lemma 10. *For any $n \in \mathbb{N}$, the following relations hold true:*

$$\begin{aligned}(2^{n+1} - h_1)A_n - h_0 B_n &= h_3 M_{n,1} + h_2 M_{n,0},\\ -h_3 A_n + (2^{n+1} - h_2)B_n &= 0.\end{aligned}$$

Thus, for any $n \in \mathbb{N}$ we obtain a regular system of linear equations. The moments that exceeded the right end of the interval $[0,3]$ can be expressed as a linear combination of the already known moments. Let us denote

$$C_n := \int_1^3 x^n \varphi(x-1)\,\mathrm{d}x \quad \text{and} \quad D_n := \int_2^3 x^n \varphi(x-2)\,\mathrm{d}x.$$

After the transformation, we obtain

$$\begin{aligned}C_n &= \int_{-2}^{0} (x+3)^n \varphi(x+2)\,\mathrm{d}x\\ &= \int_{-2}^{1} (x+3)^n \varphi(x+2)\,\mathrm{d}x - \int_0^1 (x+3)^n \varphi(x+2)\,\mathrm{d}x\\ &= \int_0^3 (x+1)^n \varphi(x)\,\mathrm{d}x - \int_0^1 (x+3)^n \varphi(x+2)\,\mathrm{d}x = \sum_{i=0}^{n} \binom{n}{i} (M_{n-i} - 3^i B_{n-i}).\end{aligned}$$

After treating D_n analogously, we can summarize the results.

Lemma 11. *For any $n \in \mathbb{N}$, the following equalities hold true:*

$$C_n = \sum_{i=0}^{n} \binom{n}{i} (M_{n-i} - 3^i B_{n-i}) \quad \text{and} \quad D_n = \sum_{i=0}^{n} \binom{n}{i} (2^i M_{n-i} - 3^i A_{n-i}).$$

Remark 12. We also have:

$$\int_0^1 \varphi(x)\,\mathrm{d}x = D_0, \quad \int_1^2 \varphi(x)\,\mathrm{d}x = 1 - D_0 - B_0, \quad \int_2^3 \varphi(x)\,\mathrm{d}x = B_0. \tag{11}$$

From this, we get the following fast and effective way to compute the integrals on any dyadic interval, for $j \in \mathbb{N}$ and $0 \leqslant k \leqslant 3 \cdot 2^j - 1$:

$$\begin{aligned}\int_{k\,2^{-j}}^{(k+1)\,2^{-j}} \varphi(x)\,\mathrm{d}x &= \sum_{l=0}^{3} h_l \int_{k\,2^{-j}}^{(k+1)\,2^{-j}} \varphi(2x-l)\,\mathrm{d}x \\ &= \frac{1}{2}\sum_{l=0}^{3} h_l \int_{k\,2^{(-j+1)}-l}^{(k+1)\,2^{(-j+1)}-l} \varphi(y)\,\mathrm{d}y.\end{aligned}$$

Repeating this succession j-times, we come to the already known integrals (11).

Let us now focus on integrals b). We have to compute only the first term, because the second term is the integral of the scaling function from a to b, which is already known. For $a, b \in \{0,1,2,3\}$, $a < b$, let us denote

$$I^n_{a,b} := \int_a^b x^n \int_a^x \varphi(s)\,\mathrm{d}s\,\mathrm{d}x - \int_a^b \frac{x^{n+1}}{b-a}\,\mathrm{d}x \int_a^b \varphi(s)\,\mathrm{d}s.$$

By integrating by parts, we obtain

Lemma 13. *For any $n \in \mathbb{N}$, the following equality holds true:*

$$I^n_{a,b} = \frac{\varphi(b)b^{n+1} - \varphi(a)a^{n+1}}{n+1} - \int_a^b \left(\left(\frac{x^{n+1} - a^{n+1}}{n+1} \right) \varphi(x) + \frac{x^{n+1}}{b-a} \int_a^b \varphi(s)\,\mathrm{d}s \right) \mathrm{d}x.$$

Remark 14. How can we compute the values of $\varphi(a)$ from Lemma 13? For $l \in \{0,1,2,3\}$, the scaling equation (2) implies

$$\varphi(l) = \sum_{k=0}^{3} h_k \varphi(2l-k) = \sum_{k=2l-3}^{2l} h_{2l-k}\varphi(k). \tag{12}$$

Furthermore, for given scaling parameters there exists a unique scaling function (up to a multiplicative constant). After a normalization $\sum\limits_k \varphi(k) = 1$, the system of equations (12) has only one solution.

Using results in a) and Remark 14, we see that $I^n_{a,b}$ is computable.

c) Let us again consider the case when the support of both scaling functions lies in the interval of integration. We will use the following notation:

$$L_{n,k} := \begin{cases} \displaystyle\int_0^3 x^n \varphi(x)\varphi(x-k)\,\mathrm{d}x & \text{for } k \in \{-2,-1,0,1,2\}, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 15. *For any $n \in \mathbb{N}$ and $k \in \{-2,-1,0,1,2\}$, the following relation holds true:*

$$L_{n,k} = 2^{-n-1}\Bigg(\sum_{m,l=0}^{3} h_m h_l L_{n,2k+l-m} + \sum_{m,l=0}^{3} h_m h_l \sum_{i=1}^{n} \binom{n}{i} m^i L_{n-i,2k+l-m}\Bigg).$$

Proof. By substitution, interchanging summation and integration, and using the binomial formula, we obtain

$$\begin{aligned}
L_{n,k} &= \int_0^3 x^n \varphi(x)\varphi(x-k)\,\mathrm{d}x \\
&= \sum_{m,l=0}^{3} h_m h_l \int_0^3 x^n \varphi(2x-m)\varphi(2x-2k-l)\,\mathrm{d}x \\
&= 2^{-n-1} \sum_{m,l=0}^{3} h_m h_l \int_0^3 (y+m)^n \varphi(y)\varphi(y+m-2k-l)\,\mathrm{d}y \\
&= 2^{-n-1} \sum_{m,l=0}^{3} h_m h_l \int_0^3 \sum_{i=0}^{n} \binom{n}{i} m^i y^{n-i} \varphi(y)\varphi(y+m-2k-l)\,\mathrm{d}y \\
&= 2^{-n-1}\Bigg(\sum_{m,l=0}^{3} h_m h_l L_{n,2k+l-m} + \sum_{m,l=0}^{3} h_m h_l \sum_{i=1}^{n} \binom{n}{i} m^i L_{n-i,2k+l-m}\Bigg),
\end{aligned}$$

which proves the lemma. □

So, for any fixed n we obtain a regular system of linear equations (of dimension 5). This system was already considered in [11], [15] and it is well-conditioned. Again, this formula cannot be employed for computing these moments when the support of a scaling function does not lie in the interval. In the case of wavelets with two vanishing moments, we have three integrals at both sides, with their support not in the interval $[0,3]$. Let us denote

$$\begin{aligned}
E_n &:= \int_0^2 x^n \varphi^2(x+1)\,\mathrm{d}x, \\
F_n &:= \int_0^1 x^n \varphi^2(x+2)\,\mathrm{d}x, \\
G_n &:= \int_0^1 x^n \varphi\,(x+2)\varphi\,(x+1)\,\mathrm{d}x.
\end{aligned}$$

Again, we apply the scaling equation. Thus

$$\begin{aligned}
E_n = h_3^2 \int_{0.5}^{2} x^n \varphi^2(2x-1)\,\mathrm{d}x + h_2^2 \int_{0}^{1.5} x^n \varphi^2(2x)\,\mathrm{d}x + h_1^2 \int_{0}^{1} x^n \varphi^2(2x+1)\,\mathrm{d}x \\
+ h_0^2 \int_{0}^{0.5} x^n \varphi^2(2x+2)\,\mathrm{d}x + 2h_2h_3 \int_{0.5}^{1.5} x^n \varphi(2x-1)\varphi(2x)\,\mathrm{d}x \\
+ 2h_1h_3 \int_{0.5}^{1} x^n \varphi(2x-1)\varphi(2x+1)\,\mathrm{d}x + 2h_2h_1 \int_{0}^{1} x^n \varphi(2x+1)\varphi(2x)\,\mathrm{d}x \\
+ 2h_1h_0 \int_{0}^{0.5} x^n \varphi(2x+1)\varphi(2x+2)\,\mathrm{d}x + 2h_2h_0 \int_{0}^{0.5} x^n \varphi(2x+2)\varphi(2x)\,\mathrm{d}x,
\end{aligned}$$

$$\begin{aligned}
F_n = h_3^2 \int_{0}^{1} x^n \varphi^2(2x+1)\,\mathrm{d}x + h_2^2 \int_{0}^{0.5} x^n \varphi^2(2x+2)\,\mathrm{d}x \\
+ 2h_2h_3 \int_{0}^{0.5} x^n \varphi(2x+1)\varphi(2x+2)\,\mathrm{d}x,
\end{aligned}$$

$$\begin{aligned}
G_n = h_3^2 \int_{0.5}^{1} x^n \varphi(2x-1)\varphi(2x+1)\,\mathrm{d}x + h_2^2 \int_{0}^{0.5} x^n \varphi(2x)\varphi(2x+2)\,\mathrm{d}x \\
+ h_2h_3 \int_{0}^{1} x^n \varphi(2x+1)\varphi(2x)\,\mathrm{d}x + h_3h_0 \int_{0}^{0.5} x^n \varphi(2x+1)\varphi(2x+2)\,\mathrm{d}x \\
+ h_1h_3 \int_{0}^{1} x^n \varphi^2(2x+1)\,\mathrm{d}x + h_2h_1 \int_{0}^{0.5} x^n \varphi(2x+1)\varphi(2x+2)\,\mathrm{d}x \\
+ h_2h_0 \int_{0}^{0.5} x^n \varphi^2(2x+2)\,\mathrm{d}x.
\end{aligned}$$

After the transformation, we obtain

$$\begin{aligned}
2^{n+1}E_n = h_3^2 \int_{0}^{3} (x+1)^n \varphi^2(x)\,\mathrm{d}x + h_2^2 \int_{0}^{3} x^n \varphi^2(x)\,\mathrm{d}x + h_1^2 \int_{0}^{2} x^n \varphi^2(x+1)\,\mathrm{d}x \\
+ h_0^2 \int_{0}^{1} x^n \varphi^2(x+2)\,\mathrm{d}x + 2h_2h_3 \int_{0}^{2} (x+1)^n \varphi(x)\varphi(x+1)\,\mathrm{d}x \\
+ 2h_1h_3 \int_{0}^{1} (x+1)^n \varphi(x)\varphi(x+2)\,\mathrm{d}x + 2h_2h_1 \int_{0}^{2} x^n \varphi(x)\varphi(x+1)\,\mathrm{d}x \\
+ 2h_1h_0 \int_{0}^{1} x^n \varphi(x+1)\varphi(x+2)\,\mathrm{d}x + 2h_2h_0 \int_{0}^{1} x^n \varphi(x+2)\varphi(x)\,\mathrm{d}x,
\end{aligned}$$

$$\begin{aligned}
2^{n+1}F_n = h_3^2 \int_{0}^{2} x^n \varphi^2(x+1)\,\mathrm{d}x + h_2^2 \int_{0}^{1} x^n \varphi^2(x+2)\,\mathrm{d}x \\
+ 2h_2h_3 \int_{0}^{1} x^n \varphi(x+1)\varphi(x+2)\,\mathrm{d}x,
\end{aligned}$$

$$
\begin{aligned}
2^{n+1}G_n = {} & h_3^2 \int_0^1 (x+1)^n \varphi(x)\varphi(x+2)\,\mathrm{d}x + h_2^2 \int_0^1 x^n \varphi(x)\varphi(x+2)\,\mathrm{d}x \\
& + h_2 h_3 \int_0^2 x^n \varphi(x)\varphi(x+1)\,\mathrm{d}x + h_3 h_0 \int_0^1 x^n \varphi(x+1)\varphi(x+2)\,\mathrm{d}x \\
& + h_1 h_3 \int_0^2 x^n \varphi^2(x+1)\,\mathrm{d}x + h_2 h_1 \int_0^1 x^n \varphi(x+1)\varphi(x+2)\,\mathrm{d}x \\
& + h_2 h_0 \int_0^1 x^n \varphi^2(x+2)\,\mathrm{d}x.
\end{aligned}
$$

Together:

Lemma 16. *For any $n \in \mathbb{N}$, and $k, l \in \{0, 1, 2\}$, the following equalities hold true:*

$$
\begin{gathered}
(2^{n+1} - h_1^2)E_n - h_0^2 F_n - 2h_0h_1G_n = h_3^2 L_{n,0,1} + h_2^2 L_{n,0,0} + 2h_2h_3L_{n,1,1} \\
+2h_1h_3L_{n,2,1} + 2h_2h_1L_{n,1,0} + 2h_0h_2L_{n,2,0}, \\
-h_3^2 E_n + (2^{n+1} - h_2^2)F_n - 2h_2h_3G_n = 0, \\
-h_1h_3E_n - h_2h_0F_n + (2^{n+1} - h_1h_2 - h_0h_3)G_n = h_3^2 L_{n,2,1} + h_2^2 L_{n,2,0} + h_2h_3L_{n,1,0},
\end{gathered}
$$

where

$$
L_{n,k,l} = \int_0^3 (x+l)^n \varphi(x)\varphi(x+k)\,\mathrm{d}x = \sum_{i=0}^{n} \binom{n}{i} l^i L_{n-i,-k}.
$$

The following lemma shows that moments exceeding the right end of the interval $[0, 3]$ can be again transformed to already known moments.

Lemma 17. *For any $n \in \mathbb{N}$, the following relations hold true:*

$$
\begin{aligned}
\int_1^3 x^n \varphi^2(x-1)\,\mathrm{d}x &= \sum_{i=0}^{n} \binom{n}{i} (L_{n-i,0} - 3^i F_{n-i}), \\
\int_2^3 x^n \varphi^2(x-2)\,\mathrm{d}x &= \sum_{i=0}^{n} \binom{n}{i} (2^i L_{n-i,0} - 3^i E_{n-i}), \\
\int_2^3 x^n \varphi^2(x-1)\varphi^2(x-2)\,\mathrm{d}x &= \sum_{i=0}^{n} \binom{n}{i} (2^i L_{n-i,-1} - 3^i G_{n-i}).
\end{aligned}
$$

Using Lemmas 8, 10, 11, 13, 15, 16, and 17, we are able to compute moment integrals for all three required types of integrals (namely a), b), and c)).

Our next goal is to integrate a general function q weighted by, roughly speeking, the scaling function φ; see the Galerkin-wavelet method at the beginning of Section 4.

To this end, we derive quadrature formulas that are exact for the respective moment integrals a)–c) up to a given order. For example, $\int_0^3 q(x)\varphi(x)\,\mathrm{d}x$ can be approximated by a formula $aq(c) + bq(d)$, where a, b are parameters and c, d are points in $[0, 3]$. The values of a, b, c, and d are delivered through solving a system of nonlinear equations determined by the requirement of exact integration of moment integrals up to order three. Treating the scaling function (or functions) as a weight function enables to avoid potential negative effect of lacking smoothness of the scaling function on the accuracy of the quadrature formulas.

The quadrature formulas are derived only once. They are ready for future use, so that the Galerkin-wavelet "stiffness" matrix is calculated efficiently.

4.2. Numerical examples

Using Daubechies wavelet with two vanishing moments, we have an error of order 2^{-3J} where J is the resolution level. Of course, this holds only when the solution is sufficiently smooth. Besides its good approximation properties, the advantage of this wavelet is its relatively small support.

Let us compare our method with the approach used in [16]. In [16], the authors implemented the functions φ and ψ (and also Φ and Ψ) by their values at dyadic points in $[0, 3]$ (at $x = k/2^D$, for $0 \leqslant k \leqslant 2^D 3$). The integration was done numerically by Simpson's quadrature formula. The accuracy of numerical quadratures was therefore affected by the choice of D. In their computations, they set D between 8 and 12. Here, in computations using their approach, we set $D = 10$. The need of relatively large D is caused by the lack of smoothness of the scaling function (for $p = 2$, the scaling function $\varphi(\cdot) \in C^{0,\alpha}(\overline{\Omega})$, where $\alpha < 0.55$; for fractional order Sobolev spaces, $\varphi \in H^s(\Omega)$, where $s < 1$).

On the contrary, the scaling function is used as a weight function in our quadrature formulas. Then we can avoid the effect of the lacking smoothness of the scaling function. Thus, to approximate arising integrals with a sufficient accuracy, we need significantly less function values. Furthermore, during computations we actually need no function values of the scaling function; we need only scaling parameters.

To calculate the exact errors, we choose such an example where the analytic solution is known,

$$\begin{gathered} -u'' = -x + 1/2 \quad \text{for } x \in (0, 3), \\ u(0) = u(3) = 0. \end{gathered}$$

The computational results are summarized in Tab. 1. The first column contains the results computed by Simpson's rule; the second column shows the results computed by the wavelet quadrature rules.

Another, this time more complicated boundary value problem,

$$-(e^{2x}u')' = 4 - 4x \quad \text{for } x \in (0,3),$$
$$u(0) = u(3) = 0.$$

The computational results are summarized in Tab. 2. Again, the first column contains the results computed by Simpson's rule and the second column presents the results computed by the wavelet quadrature rules.

J	$2^{3J}\|u-u_J\|_0$	
	Method 1	Method 2
0	0.030520	0.030520
1	0.029869	0.029869
2	0.029541	0.029541
3	0.029375	0.029375
4	0.029293	0.029292
5	0.029253	0.029250
6	0.029256	0.029224

Table 1.

J	$2^{3J}\|u-u_J\|_0$	
	Method 1	Method 2
0	0.015550	0.028228
1	0.079219	0.078086
2	0.128423	0.179214
3	0.146847	0.319862
4	0.151953	0.360131
5	0.153100	0.236514
6	0.152963	0.172772
7	0.153117	0.157064

Table 2.

To conclude: In the first, simpler example, the wavelet quadrature rules led to the exact evaluation of the relevant integrals. In the second example, the results computed by Simpson's rule are slightly better. However, the order of error is the same as in the case of the wavelet quadrature rules and, moreover, our approach requires significantly less number of function values.

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