Applications of Mathematics

Alexander Ženíšek

Extensions from the Sobolev spaces H^1 satisfying prescribed Dirichlet boundary conditions

Applications of Mathematics, Vol. 49 (2004), No. 5, 405–413

Persistent URL: http://dml.cz/dmlcz/134576

EXTENSIONS FROM THE SOBOLEV SPACES H^1 SATISFYING PRESCRIBED DIRICHLET BOUNDARY CONDITIONS*

ALEXANDER ŽENÍŠEK, Brno

(Received May 20, 2002, in revised version Novermber 11, 2003)

Abstract. Extensions from $H^1(\Omega_P)$ into $H^1(\Omega)$ (where $\Omega_P \subset \Omega$) are constructed in such a way that extended functions satisfy prescribed boundary conditions on the boundary $\partial\Omega$ of Ω. The corresponding extension operator is linear and bounded.

Keywords: extensions satisfying prescribed boundary conditions, Nikolskij extension theorem

MSC 2000: 65N99

This note completes the considerations and results of [4] where a completely discretized variational problem corresponding to a two-dimensional nonlinear second order parabolic-elliptic initial-boundary value problem was analyzed.

Our problem reads as follows: Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz continuous boundary in the sense of Nečas (see [3] or [6, Definition 1]). Let

$$\overline{\Omega} = \overline{\Omega}_E \cup \overline{\Omega}_P, \quad \Omega_E \cap \Omega_P = \emptyset, \tag{1}$$

where the subset Ω_M ($M = E$, P) is either a domain or a union of a finite number of mutually disjoint domains (all domains considered are assumed to have a Lipschitz continuous boundary)**—see, for example, Figs. 1–3. (Ω_P and Ω_E denote the domains (or sets) where the problem studied in [4] is described by parabolic and elliptic

* This work was supported by the grants No 201/00/0557 and 201/03/0570 of the Grant Agency of the Czech Republic and and by the grant MSM: 262100001.

** The fact that a bounded domain Ω has a Lipschitz continuous boundary will be denoted by the symbol $\Omega \in \mathcal{C}^{0,1}$.

equations, respectively.) We define

$$V = \{v \in H^1(\Omega)\colon\ v = 0 \text{ on } \Gamma_1\} \quad (\Gamma_1 \subset \partial\Omega,\ \text{meas}_{N-1}\Gamma_1 > 0), \tag{2}$$

$$V_M = \{v_M \in H^1(\Omega_M)\colon\ v_M = 0 \text{ on } \Gamma_1 \cap \partial\Omega_M\} \quad (M = E,\ P). \tag{3}$$

We have to find a bounded linear extension operator $\mathcal{P}\colon V_P \to V$; this means an operator $\mathcal{P}$ with the following properties:

$$\mathcal{P}(c_1 u_P + c_2 v_P) = c_1 \mathcal{P} u_P + c_2 \mathcal{P} v_P \quad \forall c_1, c_2 \in \mathbb{R},\ \forall u_P, v_P \in V_P, \tag{4}$$

$$\|\mathcal{P} u_P\|_{H^1(\Omega)} \leqslant C \|u_P\|_{H^1(\Omega_P)} \quad \forall u_P \in V_P, \tag{5}$$

$$\mathcal{P} u_P\big|_{\Omega_P} = u_P \quad \forall u_P \in V_P. \tag{6}$$

In [4, Lemma 3.9] the existence of such an extension operator was proved under the restrictive assumption

$$\partial\Omega \cap \partial\Omega_E \cap \partial\Omega_P \subset \Gamma_1 \quad \text{or} \quad \partial\Omega_E \cap \partial\Omega_P \cap \Gamma_1 = \emptyset; \tag{7}$$

in [5, Theorem 44.3] the two-dimensional situation with $\partial\Omega_E \cap \partial\Omega_P \cap \Gamma_1$ being a one point set was also studied. (It should be noted that assumption (7) and [4, Lemma 3.9] do not depend on the dimension N.)

In this paper the two-dimensional considerations are completed and generalized to the three-dimensional case.

In our considerations we shall need first of all the following form of the Nikolskij extension theorem (formulated first with this name in [2]):

1. Lemma. *Let $G \in \mathcal{C}^{0,1}$ be an N-dimensional domain (for applications, $N = 2$ and $N = 3$ is sufficient) and let $G_0 \in \mathcal{C}^{0,1}$ be such a domain that $\overline{G} \subset G_0$. Then there exists a bounded linear operator $\mathcal{E}\colon H^1(G) \to H_0^1(G_0)$ such that*

$$(\mathcal{E}u)(X) = u(X) \quad \forall X \in G,$$

where

$$H_0^1(G_0) = \{v \in H^1(G_0),\ \mathfrak{T}v = 0 \text{ on } G_0\},$$

$\mathfrak{T}\colon H^1(G_0) \to L_2(\partial G_0)$ being the trace operator.

We note that we use the usual brief notation $H^1(G) = H^{1,2}(G)$ and $H_0^1(G_0) = H_0^{1,2}(G_0)$ for the corresponding Sobolev spaces (see [1]).

The proof of Lemma 1 is a special case (for $k = 1$) of the proof of [6, Theorem 1.4 and Lemma 1.6]. The following lemma can be obtained by a simple modification of this proof:

2. Lemma. *Let $G \in \mathcal{C}^{0,1}$ be an N-dimensional domain ($N = 2$ or $N = 3$) which is multiply connected. Let $\overline{H}_1, \ldots, \overline{H}_n$ be the "holes" in G with boundaries $\partial H_1, \ldots, \partial H_n$. Let ∂L_0 be such a closed simple curve (or surface) that $\partial G = \partial L_0 \cup \partial H_1 \cup \ldots \cup \partial H_n$. Further, let $\partial L_1, \ldots, \partial L_n$ be such closed simple curves (or surfaces) that $\partial S_i = \partial H_i \cup \partial L_i$ form the boundary of a strip (or layer) $\overline{S}_i \subset \overline{H}_i$ with a positive width ($S_i \in \mathcal{C}^{0,1}$). Let us define a closed domain $\overline{D} = \overline{G} \cup \overline{S}_1 \cup \ldots \cup \overline{S}_n$. Then there exists a bounded linear operator $\mathcal{F}\colon\ H^1(G) \to H^1(D)$ such that*

$$(\mathcal{F}u)(X) = u(X) \quad \forall X \in G,\ \forall u \in H^1(G),$$
$$\mathcal{F}u\big|_{\partial L_i} = 0 \quad \forall u \in H^1(G)\ \ (i = 1, \ldots, n).$$

The following theorem is valid for both $N = 2$ and $N = 3$.

3. Theorem. *Let $N = 2$ or $N = 3$. Let $\Omega \in \mathcal{C}^{0,1}$, $\Omega_E \in \mathcal{C}^{0,1}$, $\Omega_P \in \mathcal{C}^{0,1}$ be domains satisfying* (1). *Then there exists a bounded linear extension operator $\mathcal{P}\colon\ V_P \to V$, i.e., an operator satisfying* (4)–(6).

Proof. First we note that part A3a of the proof of [4, Lemma 3.9] is not correct; thus we choose a quite different and more general way of proving. We shall consider several situations, most of them being indicated in Figs. 1a–3b. (Shadowed parts of the boundary $\partial\Omega$ denote the set $\Gamma_1 \subset \partial\Omega$.) In parts A–E of this proof the two-dimensional case is studied. Changes in the proof when $N = 3$ are introduced in part F.

A) In the case of Fig. 1a we apply Lemma 1 with $G = \Omega_P$ and $G_0 = \overline{\Omega}_P \cup \Omega_E$.

B) In the case of Fig. 1b we apply Lemma 2 with $G = \Omega_P$, $\overline{H}_1 = \overline{\Omega}_E$ and $n = 1$. By Lemma 2 we have

$$\|\mathcal{F}u_P\|_{1,D} \leqslant C\|u_P\|_{1,\Omega_P}. \tag{8}$$

We define

$$\mathcal{P}u_P = \begin{cases} u_P & \text{in } \Omega_P, \\ \mathcal{F}u_P & \text{in } \overline{S}_1, \\ 0 & \text{in } \Omega_E \setminus \overline{S}_1. \end{cases}$$

Hence by (8)

$$\|\mathcal{P}u_P\|_{1,\Omega} \leqslant C\|u_P\|_{1,\Omega_P}.$$

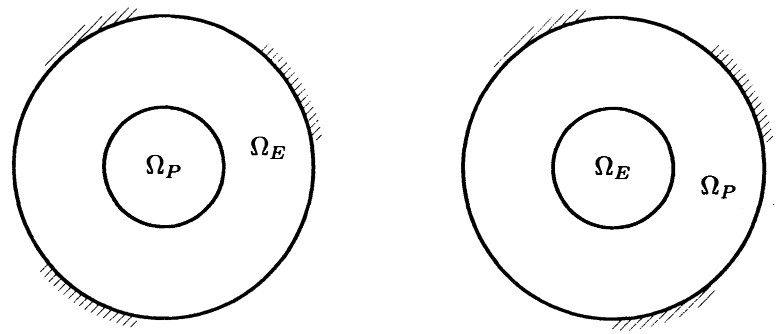

Figure 1a and Figure 1b.

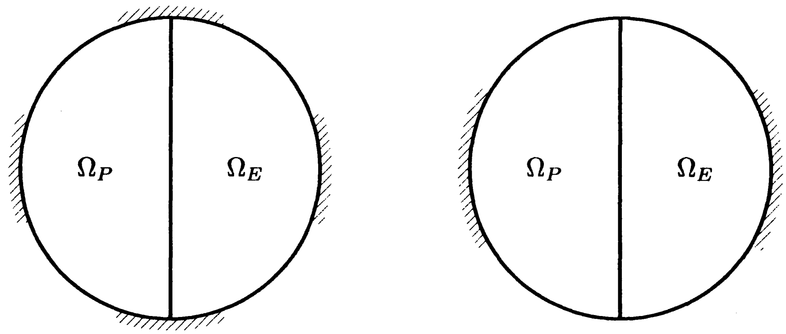

Figure 2a and Figure 2b.

C) In the case of Fig. 2b, where $\partial\Omega_E \cap \partial\Omega_P \cap \Gamma_1 = \emptyset$, we use Lemma 1 with $G = \Omega_P$ and choose a domain $G_0 \supset \overline{\Omega}_P$ such that $G_0 \cap \partial\Omega_E \cap \Gamma_1 = \emptyset$. For $\mathcal{E}u_P \in H_0^1(G_0)$ we have by Lemma 1

(9) $$\|\mathcal{E}u_P\|_{1,G_0} \leqslant C\|u_P\|_{1,\Omega_P}.$$

We define

$$\mathcal{P}u_P = \begin{cases} u_P & \text{in } \Omega_P, \\ \mathcal{E}u_P & \text{in } \overline{G}_0 \setminus \Omega_P, \\ 0 & \text{in } \Omega_E \setminus \overline{G}_0. \end{cases}$$

Hence by (9)

$$\|\mathcal{P}u_P\|_{1,\Omega} \leqslant C\|u_P\|_{1,\Omega_P}.$$

D) Now we shall consider the cases where $\partial\Omega_E \cap \partial\Omega_P \cap \Gamma_1 \neq \emptyset$. Let $\Omega^* = \Omega \cup \overline{H}_1 \cup \ldots \cup \overline{H}_m$, $\overline{H}_i$ being the "holes" in Ω. Let ∂K be a closed simple curve with the property $\partial K \cap \Omega = \emptyset$ and such that ∂K and $\partial\Omega^*$ form the boundary of a strip Ω_1 with a positive width: $\partial\Omega_1 = \partial K \cup \partial\Omega^*$.

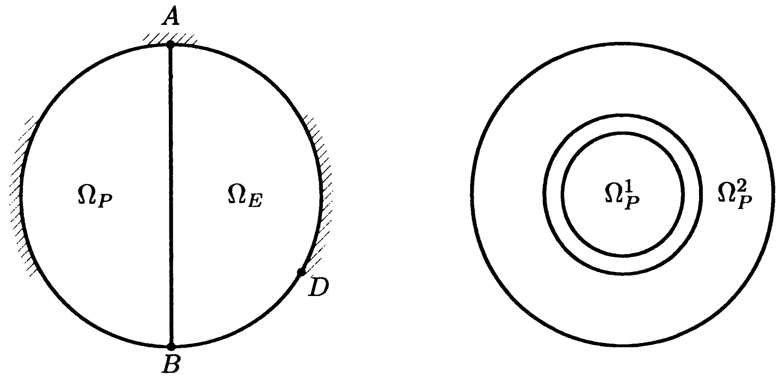

Figure 3a and Figure 3b.

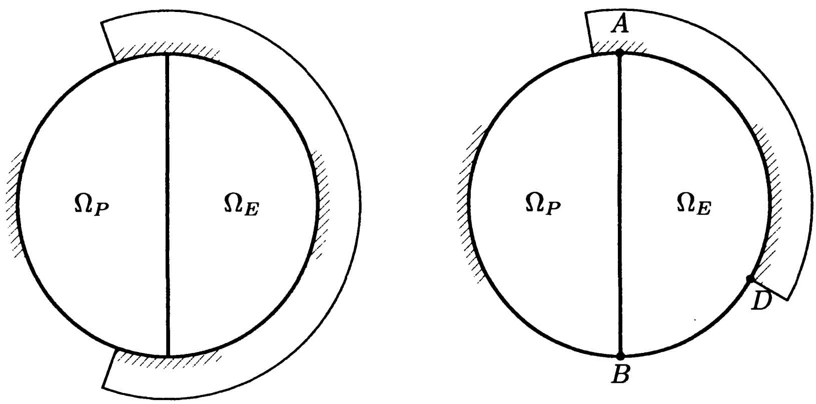

Figure 4a and Figure 4b.

1. First, let us consider the case $\Gamma_1 = \partial\Omega$ (or at least $\Gamma_1 = \partial\Omega^*$). Let us define a closed domain $\overline{G}$ by the relation

$$\overline{G} = \overline{\Omega}_P \cup \overline{\Omega}_1$$

and let the function $v \in H^1(G)$ satisfy

$$v = \begin{cases} u_P & \text{in } \Omega_P, \\ 0 & \text{in } \Omega_1. \end{cases}$$

We have

(10) $$\|v\|_{1,G} = \|u_P\|_{1,\Omega_P}.$$

Let G_0 be such a domain that $\overline{G} \subset G_0$. Moreover, if Ω_E is not simply connected then we choose G_0 in such a way that $G_0 \cap \overline{H}_i = \emptyset$, where $\overline{H}_i$ $(i = 1, \ldots, n)$ are the "holes" in Ω_E. Applying Lemma 1 to the function v we obtain a function $\mathcal{E}v \in H^1(G_0)$ satisfying

$$\|\mathcal{E}v\|_{1,G_0} \leqslant C\|v\|_{1,G}. \tag{11}$$

Let us set

$$\tilde{u}_E = \begin{cases} \mathcal{E}v & \text{in } G_0 \cap \Omega_E, \\ 0 & \text{in } \Omega_E \setminus G_0. \end{cases}$$

Then the function

$$\tilde{u} = \begin{cases} u_P & \text{in } \Omega_P, \\ \tilde{u}_E & \text{in } \Omega_E \end{cases} \tag{12}$$

satisfies, according to (11) and (10),

$$\begin{aligned} \|\tilde{u}\|^2_{1,\Omega} &= \|u_P\|^2_{1,\Omega_P} + \|\tilde{u}_E\|^2_{1,\Omega_E} = \|u_P\|^2_{1,\Omega_P} + \|\mathcal{E}v\|^2_{1,G_0\cap\Omega_E} \\ &\leqslant \|u_P\|^2_{1,\Omega_P} + C^2\|v\|^2_{1,G} = (1+C^2)\|u_P\|^2_{1,\Omega_P}. \end{aligned}$$

Hence the function $\tilde{u}$ given by (12) is the desired extension, $\tilde{u} = \mathcal{P}u_P$.

2. Let now $\Gamma_1 \neq \partial\Omega^*$ and $\partial\Omega \cap \partial\Omega_E \cap \partial\Omega_P \subset \Gamma_1$; see, for example, Fig. 2a. It suffices to explain the idea of the proof for the circle and boundary conditions from Fig. 2a. Let the center of this circle coincide with the origin of the given Cartesian coordinate system and let $\partial\Omega_P \cap \partial\Omega_E$ be the segment lying on the axis x_2. Let $A = [0, R]$ and $B = [0, -R]$ be the end-points of $\partial\Omega_P \cap \partial\Omega_E$, where R is the radius of the circle considered. Let γ_A and γ_B be the parts of Γ_1 containing the points A and B, respectively. Let A_1 be the end-point of γ_A which lies on $\partial\Omega_P$. Similarly, let B_1 be the end-point of γ_B which lies on $\partial\Omega_P$. Finally, let A_1^* and B_1^* be the points of ∂K which are closest to A_1 and B_1, respectively. Let us cut the domain Ω_1 into two parts Q_2 and Ω_2 by the segments $A_1^*A_1$ and $B_1^*B_1$: $\overline{\Omega}_1 = \overline{\Omega}_2 \cup \overline{Q}_2$, $\Omega_2 \cap Q_2 = \emptyset$, where Q_2 lies along a part of $\partial\Omega_P$. The domain Ω_2 is sketched together with Ω in Fig. 4a.

Let us define a closed domain $\overline{G}$ by the relation

$$\overline{G} = \overline{\Omega}_P \cup \overline{\Omega}_2 \tag{13}$$

and let the function $v \in H^1(G)$ satisfy

$$v = \begin{cases} u_P & \text{in } \Omega_P, \\ 0 & \text{in } \Omega_2. \end{cases}$$

Again, as in part 1, relation (10) holds and we can repeat all considerations from that part and obtain also in this case the desired extension $\mathcal{P}u_P$.

3. Now let us consider the case that the set $\partial\Omega \cap \partial\Omega_E \cap \partial\Omega_P$ consists only of the point A (see, e.g., Fig. 3a). Let the point A_1 have the same meaning as in part 2 and let $D^* \in \partial K$ be the point of ∂K closest to the point D (which is sketched in Fig. 4b). Let us cut the domain Ω_1 into two parts Q_3 and Ω_3 by segments $A_1^*A_1$ and D^*D: $\overline{\Omega}_1 = \overline{Q}_3 \cup \overline{\Omega}_3$, $Q_3 \cap \Omega_3 = \emptyset$, where the closed strip $\overline{Q}_3$ contains the point B. The domain Ω_3 is sketched together with Ω in Fig. 4b.

Let us define a closed domain $\overline{G}$ by the relation

$$\overline{G} = \overline{\Omega}_P \cup \overline{\Omega}_3 \tag{14}$$

and let the function $v \in H^1(G)$ satisfy

$$v = \begin{cases} u_P & \text{in } \Omega_P, \\ 0 & \text{in } \Omega_3. \end{cases}$$

Relation (10) again holds.

Let G_0 be such a domain that $\overline{G} \subset G_0$ and $\overline{G}_0 \cap \overline{H}_i = \emptyset$, where $\overline{H}_i$ are the "holes" in Ω_E. Now we can apply Lemma 1 to the function v and repeat the construction of $\mathcal{P}u_P$ introduced in part 1.

3a. Let us note that we could use the segment $A_2^*A_2$ instead of the segment D^*D, where A_2 is the second end-point of the arc γ_A and $A_2^* \in \partial K$. This approach has a modification (whose three-dimensional generalization will be useful in part F of this proof): Let $\omega \in \mathcal{C}^{0,1}$ be a domain with the following properties: $\omega \cap \Omega = \emptyset$ and $\overline{\omega} \cap \overline{\Omega} = \gamma_A$. We define

$$G = \omega \cup \gamma_A \cup \Omega_P. \tag{15}$$

Hence $\overline{G} = \overline{\omega} \cup \overline{\Omega}_P$ and we can construct the extension $\mathcal{P}u_P$ in the same way as in part 3.

4. We should mention also the case $\partial\Omega_P \setminus (\partial\Omega_E \cap \partial\Omega_P) \subset \Gamma_1$: it suffices to exchange the notation of the subdomains Ω_E and Ω_P in Fig. 4a; the rest is clear.

5. If $\gamma_A \cap (\partial\Omega_E \setminus \overline{I}) = \emptyset$, $\gamma_B \cap (\partial\Omega_E \setminus \overline{I}) = \emptyset$, where I is the relative interior of $\partial\Omega_E \cap \partial\Omega_P$, then we proceed in the same way as in part C.

6. At the end of part D of the proof let us consider the case $\gamma_A \cap (\partial\Omega_P \setminus \overline{I}) = \emptyset$, $\gamma_B \cap (\partial\Omega_P \setminus \overline{I}) = \emptyset$, where $I = \partial\Omega_P \cap \partial\Omega_E$. For a greater simplicity, let $\gamma_B = \emptyset$ and $\Gamma_1 = \gamma_A \cup \lambda$ where $\lambda \subset \partial\Omega_P \setminus I$ ($\lambda \cap \gamma_A = \emptyset$); further, let $\partial\Omega$ be again a circle. We obtain a modification of Fig. 3a with $A_1 \equiv A$, where A_1 again denotes the left-hand end-point of the arc γ_A. Let R_1 be the upper end-point of λ and R_2 the lower

end-point of λ. Let τ_1 be a (piecewise) smooth arc connecting the points R_1, A_1 and τ_2 a (piecewise) smooth arc connecting the points R_2, A_2. Let τ_1, τ_2 have no common points with $\partial\Omega$ (except for the end-points R_i, A_i). Let S be the strip with $\partial S = \tau_1 \cup \tau_2 \cup \lambda \cup \gamma_A \cup \partial K$. Let us set $\overline{G} = \overline{\Omega}_P \cup \overline{S}$. It is always possible to choose the arcs τ_1, τ_2 such that $G \in \mathcal{C}^{0,1}$. The domain G is a domain with two "holes". Let the function $v \in H^1(G)$ satisfy

$$v = \begin{cases} u_P & \text{in } \Omega_P, \\ 0 & \text{in } S. \end{cases}$$

Relation (10) again holds. Applying Lemma 2 we obtain the required result in this case.

E) It remains to analyze the case from Fig. 3b, where we sketch the situation which appears in applications very often: the domain Ω_P^1 is the rotor of an electromachine, the domain Ω_P^2 is the stator of an electromachine and the narrow domain between them (the domain Ω_E) represents an air crevice. In this case

$$\Omega_P = \Omega_P^1 \cup \Omega_P^2$$

is not a domain. Also in this case we can define the space $H^1(\Omega)$: For $v_1 \in H^1(\Omega_P^1)$, $v_2 \in H^1(\Omega_P^2)$ we set

$$Fv = F(v_1, v_2) = \begin{cases} v_1 & \text{on } \Omega_P^1, \\ v_2 & \text{on } \Omega_P^2. \end{cases}$$

Then F is a bounded linear mapping from $H^1(\Omega_P^1) \times H^1(\Omega_P^2)$ into $H^1(\Omega_P) = H^1(\Omega_P^1 \cup \Omega_P^2)$ and $v = (v_1, v_2) \in H^1(\Omega_P)$ satisfies

$$v\big|_{\Omega_P^1} = v_1, \quad v\big|_{\Omega_P^2} = v_2.$$

Let δ be the width of the domain Ω_E. Let us use Lemma 1 with $G = \Omega_P^1$ in such a way that $G_0 \cap \Omega_E$ is a strip of the width $\delta/3$. Further, let us use Lemma 2 with $G = \Omega_P^2$ in such a way that $D \cap \Omega_E$ is a strip of the width $\delta/3$. Let us set

$$u_E = \begin{cases} \mathcal{E} u_P^1 & \text{in } G_0 \cap \Omega_E, \\ 0 & \text{in } \Omega_E \setminus \{(G_0 \cap \Omega_E) \cup (D \cap \Omega_E)\}, \\ \mathcal{F} u_P^2 & \text{in } D \cap \Omega_E. \end{cases} \tag{16}$$

By (16) and Lemmas 1, 2 we have

$$\begin{aligned} \|u_E\|_{1,\Omega_E}^2 &= \|\mathcal{E} u_P^1\|_{1,G_0\cap\Omega_E}^2 + \|\mathcal{F} u_P^2\|_{1,D\cap\Omega_E}^2 \\ &\leqslant C_1^2 \|u_P^1\|_{1,\Omega_P^1}^2 + C_2^2 \|u_P^2\|_{1,\Omega_P^2} \leqslant \max\{C_1^2, C_2^2\} \|u_P\|_{1,\Omega_P}^2, \end{aligned}$$

which we wanted to prove.

F) The above presented method of proving can be easily extended to three dimensions. Only the case not covered by assumption (7) deserves a special attention: Let $\sigma_1, \ldots, \sigma_n$ with $\operatorname{meas}_2 \sigma_i > 0$ $(i = 1, \ldots, n)$ and $\sigma_j \cap \sigma_k = \emptyset$ be the parts of Γ_1 such that $\sigma_i \cap (\partial\Omega_E \cap \partial\Omega_P) \neq \emptyset$ $(i = 1, \ldots, n)$. Let Δ_i $(i = 1, \ldots, n)$ be parts of a three-dimensional layer (which is a three-dimensional generalization of the strip Ω_1 appearing at the beginning of part D) such that $\overline{\Delta}_j \cap \overline{\Delta}_k = \emptyset$ and

$$\overline{\Delta}_i \cap \overline{\Omega} = \sigma_i \quad (i = 1, \ldots, n).$$

We define

$$G = \Delta \cup \sigma \cup \Omega_P,$$

where

$$\Delta = \bigcup_{i=1}^{n} \Delta_i, \quad \sigma = \bigcup_{i=1}^{n} \sigma_i.$$

Hence $\overline{G} = \overline{\Delta} \cup \overline{\Omega}_P$ and the construction of the extension $\mathcal{P}u_P$ is a straightforward modification of part D3. □

4. Remark. The results presented in this paper play an important role connected with the theory of electromagnetic fields in electromachines and generally in the theory of parabolic-elliptic equations (see, e.g., [4]). Without using them one cannot present correct proofs of some related results (as it happened, for example, in [7] and [8]).

References

[1] *A. Kufner, O. John, and S. Fučík*: Function Spaces. Academia, Prague, 1977.

[2] *J. Nečas*: Les Méthodes Directes en Théorie des Equations Elliptiques. Academia/Masson, Prague/Paris, 1967.

[3] *J. Nečas, I. Hlaváček*: Mathematical Theory of Elastic and Elasto-Plastic Bodies: An Introduction. Elsevier, Amsterdam, 1981.

[4] *A. Ženíšek*: Finite element variational crimes in parabolic-elliptic problems. Part I. Nonlinear schemes. Numer. Math. *55* (1989), 343–376.

[5] *A. Ženíšek*: Nonlinear Elliptic and Evolution Problems and their Finite Element Approximations. Academic Press, London, 1990.

[6] *A. Ženíšek*: On a generalization of Nikolskij's extension theorem in the case of two variables. Appl. Math. *48* (2003), 367–404.

[7] *M. Zlámal*: Finite element solution of quasistationary nonlinear magnetic field. RAIRO Anal. Numér. *16* (1982), 161–191.

[8] *M. Zlámal*: A linear scheme for the numerical solution of nonlinear quasistationary magnetic fields. Math. Comput. *41* (1983), 425–440.

Author's address: *A. Ženíšek*, Department of Mathematics, Faculty of Mechanical Engineering VUT, Technická 2, 616 69 Brno, Czech Republic, e-mail: `zenisek@um.fme.vutbr.cz`.

EXTENSIONS FROM THE SOBOLEV SPACES H^1 SATISFYING PRESCRIBED DIRICHLET BOUNDARY CONDITIONS*

Alexander Ženíšek, Brno

(Received May 20, 2002, in revised version Novermber 11, 2003)

Abstract. Extensions from $H^1(\Omega_P)$ into $H^1(\Omega)$ (where $\Omega_P \subset \Omega$) are constructed in such a way that extended functions satisfy prescribed boundary conditions on the boundary $\partial\Omega$ of Ω. The corresponding extension operator is linear and bounded.

Keywords: extensions satisfying prescribed boundary conditions, Nikolskij extension theorem

MSC 2000: 65N99

This note completes the considerations and results of [4] where a completely discretized variational problem corresponding to a two-dimensional nonlinear second order parabolic-elliptic initial-boundary value problem was analyzed.

Our problem reads as follows: Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz continuous boundary in the sense of Nečas (see [3] or [6, Definition 1]). Let

$$\overline{\Omega} = \overline{\Omega}_E \cup \overline{\Omega}_P, \quad \Omega_E \cap \Omega_P = \emptyset, \tag{1}$$

where the subset Ω_M ($M = E,\ P$) is either a domain or a union of a finite number of mutually disjoint domains (all domains considered are assumed to have a Lipschitz continuous boundary)**—see, for example, Figs. 1–3. (Ω_P and Ω_E denote the domains (or sets) where the problem studied in [4] is described by parabolic and elliptic

* This work was supported by the grants No 201/00/0557 and 201/03/0570 of the Grant Agency of the Czech Republic and and by the grant MSM: 262100001.

** The fact that a bounded domain Ω has a Lipschitz continuous boundary will be denoted by the symbol $\Omega \in \mathcal{C}^{0,1}$.

equations, respectively.) We define

$$V = \{v \in H^1(\Omega)\colon\ v = 0 \text{ on } \Gamma_1\} \quad (\Gamma_1 \subset \partial\Omega,\ \operatorname{meas}_{N-1}\Gamma_1 > 0), \tag{2}$$

$$V_M = \{v_M \in H^1(\Omega_M)\colon\ v_M = 0 \text{ on } \Gamma_1 \cap \partial\Omega_M\} \quad (M = E,\ P). \tag{3}$$

We have to find a bounded linear extension operator $\mathcal{P}\colon V_P \to V$; this means an operator $\mathcal{P}$ with the following properties:

$$\mathcal{P}(c_1 u_P + c_2 v_P) = c_1 \mathcal{P} u_P + c_2 \mathcal{P} v_P \quad \forall c_1, c_2 \in \mathbb{R},\ \forall u_P, v_P \in V_P, \tag{4}$$

$$\|\mathcal{P} u_P\|_{H^1(\Omega)} \leqslant C \|u_P\|_{H^1(\Omega_P)} \quad \forall u_P \in V_P, \tag{5}$$

$$\mathcal{P} u_P\big|_{\Omega_P} = u_P \quad \forall u_P \in V_P. \tag{6}$$

In [4, Lemma 3.9] the existence of such an extension operator was proved under the restrictive assumption

$$\partial\Omega \cap \partial\Omega_E \cap \partial\Omega_P \subset \Gamma_1 \quad \text{or} \quad \partial\Omega_E \cap \partial\Omega_P \cap \Gamma_1 = \emptyset; \tag{7}$$

in [5, Theorem 44.3] the two-dimensional situation with $\partial\Omega_E \cap \partial\Omega_P \cap \Gamma_1$ being a one point set was also studied. (It should be noted that assumption (7) and [4, Lemma 3.9] do not depend on the dimension N.)

In this paper the two-dimensional considerations are completed and generalized to the three-dimensional case.

In our considerations we shall need first of all the following form of the Nikolskij extension theorem (formulated first with this name in [2]):

1. Lemma. *Let $G \in \mathcal{C}^{0,1}$ be an N-dimensional domain (for applications, $N = 2$ and $N = 3$ is sufficient) and let $G_0 \in \mathcal{C}^{0,1}$ be such a domain that $\overline{G} \subset G_0$. Then there exists a bounded linear operator $\mathcal{E}\colon H^1(G) \to H^1_0(G_0)$ such that*

$$(\mathcal{E}u)(X) = u(X) \quad \forall X \in G,$$

where

$$H^1_0(G_0) = \{v \in H^1(G_0),\ \mathfrak{T}v = 0 \text{ on } G_0\},$$

$\mathfrak{T}\colon H^1(G_0) \to L_2(\partial G_0)$ being the trace operator.

We note that we use the usual brief notation $H^1(G) = H^{1,2}(G)$ and $H^1_0(G_0) = H^{1,2}_0(G_0)$ for the corresponding Sobolev spaces (see [1]).

The proof of Lemma 1 is a special case (for $k = 1$) of the proof of [6, Theorem 1.4 and Lemma 1.6]. The following lemma can be obtained by a simple modification of this proof:

2. Lemma. *Let $G \in \mathcal{C}^{0,1}$ be an N-dimensional domain ($N = 2$ or $N = 3$) which is multiply connected. Let $\overline{H}_1, \ldots, \overline{H}_n$ be the "holes" in G with boundaries $\partial H_1, \ldots, \partial H_n$. Let ∂L_0 be such a closed simple curve (or surface) that $\partial G = \partial L_0 \cup \partial H_1 \cup \ldots \cup \partial H_n$. Further, let $\partial L_1, \ldots, \partial L_n$ be such closed simple curves (or surfaces) that $\partial S_i = \partial H_i \cup \partial L_i$ form the boundary of a strip (or layer) $\overline{S}_i \subset \overline{H}_i$ with a positive width ($S_i \in \mathcal{C}^{0,1}$). Let us define a closed domain $\overline{D} = \overline{G} \cup \overline{S}_1 \cup \ldots \cup \overline{S}_n$. Then there exists a bounded linear operator $\mathcal{F}\colon H^1(G) \to H^1(D)$ such that*

$$(\mathcal{F}u)(X) = u(X) \quad \forall X \in G, \ \forall u \in H^1(G),$$
$$\mathcal{F}u\big|_{\partial L_i} = 0 \quad \forall u \in H^1(G) \ \ (i = 1, \ldots, n).$$

The following theorem is valid for both $N = 2$ and $N = 3$.

3. Theorem. *Let $N = 2$ or $N = 3$. Let $\Omega \in \mathcal{C}^{0,1}$, $\Omega_E \in \mathcal{C}^{0,1}$, $\Omega_P \in \mathcal{C}^{0,1}$ be domains satisfying* (1)*. Then there exists a bounded linear extension operator $\mathcal{P}\colon V_P \to V$, i.e., an operator satisfying* (4)–(6).

Proof. First we note that part A3a of the proof of [4, Lemma 3.9] is not correct; thus we choose a quite different and more general way of proving. We shall consider several situations, most of them being indicated in Figs. 1a–3b. (Shadowed parts of the boundary $\partial\Omega$ denote the set $\Gamma_1 \subset \partial\Omega$.) In parts A–E of this proof the two-dimensional case is studied. Changes in the proof when $N = 3$ are introduced in part F.

A) In the case of Fig. 1a we apply Lemma 1 with $G = \Omega_P$ and $G_0 = \overline{\Omega}_P \cup \Omega_E$.

B) In the case of Fig. 1b we apply Lemma 2 with $G = \Omega_P$, $\overline{H}_1 = \overline{\Omega}_E$ and $n = 1$. By Lemma 2 we have

$$\|\mathcal{F}u_P\|_{1,D} \leqslant C\|u_P\|_{1,\Omega_P}. \tag{8}$$

We define

$$\mathcal{P}u_P = \begin{cases} u_P & \text{in } \ \Omega_P, \\ \mathcal{F}u_P & \text{in } \ \overline{S}_1, \\ 0 & \text{in } \ \Omega_E \setminus \overline{S}_1. \end{cases}$$

Hence by (8)

$$\|\mathcal{P}u_P\|_{1,\Omega} \leqslant C\|u_P\|_{1,\Omega_P}.$$

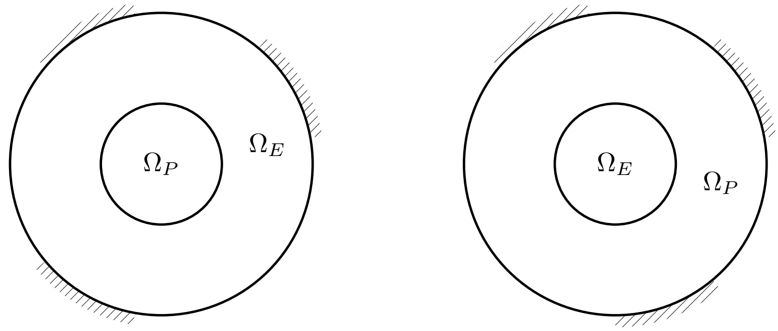

Figure 1a and Figure 1b.

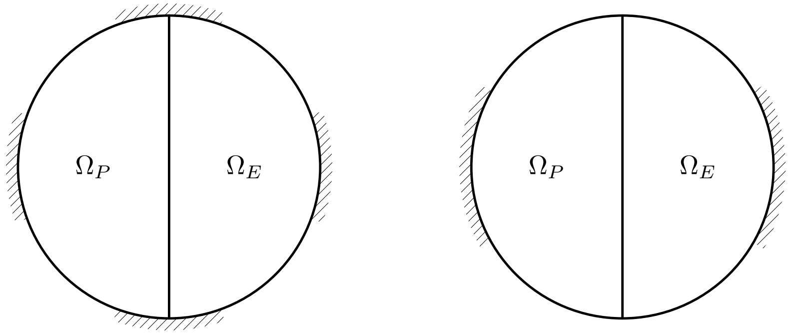

Figure 2a and Figure 2b.

C) In the case of Fig. 2b, where $\partial\Omega_E \cap \partial\Omega_P \cap \Gamma_1 = \emptyset$, we use Lemma 1 with $G = \Omega_P$ and choose a domain $G_0 \supset \overline{\Omega}_P$ such that $G_0 \cap \partial\Omega_E \cap \Gamma_1 = \emptyset$. For $\mathcal{E}u_P \in H_0^1(G_0)$ we have by Lemma 1

$$\|\mathcal{E}u_P\|_{1,G_0} \leqslant C\|u_P\|_{1,\Omega_P}. \tag{9}$$

We define

$$\mathcal{P}u_P = \begin{cases} u_P & \text{in } \Omega_P, \\ \mathcal{E}u_P & \text{in } \overline{G}_0 \setminus \Omega_P, \\ 0 & \text{in } \Omega_E \setminus \overline{G}_0. \end{cases}$$

Hence by (9)

$$\|\mathcal{P}u_P\|_{1,\Omega} \leqslant C\|u_P\|_{1,\Omega_P}.$$

D) Now we shall consider the cases where $\partial\Omega_E \cap \partial\Omega_P \cap \Gamma_1 \neq \emptyset$. Let $\Omega^* = \Omega \cup \overline{H}_1 \cup \ldots \cup \overline{H}_m$, $\overline{H}_i$ being the "holes" in Ω. Let ∂K be a closed simple curve with the property $\partial K \cap \Omega = \emptyset$ and such that ∂K and $\partial\Omega^*$ form the boundary of a strip Ω_1 with a positive width: $\partial\Omega_1 = \partial K \cup \partial\Omega^*$.

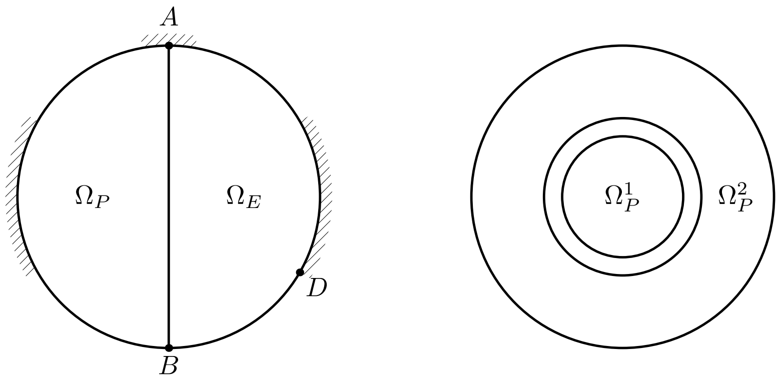

Figure 3a and Figure 3b.

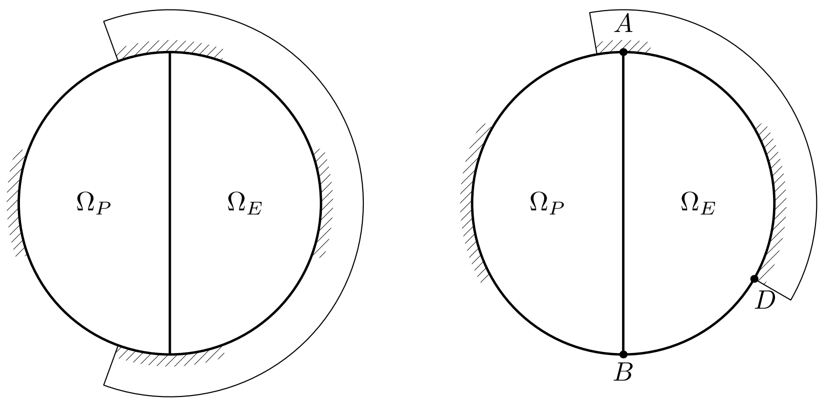

Figure 4a and Figure 4b.

1. First, let us consider the case $\Gamma_1 = \partial\Omega$ (or at least $\Gamma_1 = \partial\Omega^*$). Let us define a closed domain $\overline{G}$ by the relation

$$\overline{G} = \overline{\Omega}_P \cup \overline{\Omega}_1$$

and let the function $v \in H^1(G)$ satisfy

$$v = \begin{cases} u_P & \text{in } \Omega_P, \\ 0 & \text{in } \Omega_1. \end{cases}$$

We have

$$\|v\|_{1,G} = \|u_P\|_{1,\Omega_P}. \tag{10}$$

Let G_0 be such a domain that $\overline{G} \subset G_0$. Moreover, if Ω_E is not simply connected then we choose G_0 in such a way that $G_0 \cap \overline{H}_i = \emptyset$, where $\overline{H}_i$ $(i = 1, \ldots, n)$ are the "holes" in Ω_E. Applying Lemma 1 to the function v we obtain a function $\mathcal{E}v \in H^1(G_0)$ satisfying

$$\|\mathcal{E}v\|_{1,G_0} \leqslant C\|v\|_{1,G}. \tag{11}$$

Let us set

$$\tilde{u}_E = \begin{cases} \mathcal{E}v & \text{in } G_0 \cap \Omega_E, \\ 0 & \text{in } \Omega_E \setminus G_0. \end{cases}$$

Then the function

$$\tilde{u} = \begin{cases} u_P & \text{in } \Omega_P, \\ \tilde{u}_E & \text{in } \Omega_E \end{cases} \tag{12}$$

satisfies, according to (11) and (10),

$$\begin{aligned} \|\tilde{u}\|^2_{1,\Omega} &= \|u_P\|^2_{1,\Omega_P} + \|\tilde{u}_E\|^2_{1,\Omega_E} = \|u_P\|^2_{1,\Omega_P} + \|\mathcal{E}v\|^2_{1,G_0\cap\Omega_E} \\ &\leqslant \|u_P\|^2_{1,\Omega_P} + C^2\|v\|^2_{1,G} = (1+C^2)\|u_P\|^2_{1,\Omega_P}. \end{aligned}$$

Hence the function $\tilde{u}$ given by (12) is the desired extension, $\tilde{u} = \mathcal{P}u_P$.

2. Let now $\Gamma_1 \neq \partial\Omega^*$ and $\partial\Omega \cap \partial\Omega_E \cap \partial\Omega_P \subset \Gamma_1$; see, for example, Fig. 2a. It suffices to explain the idea of the proof for the circle and boundary conditions from Fig. 2a. Let the center of this circle coincide with the origin of the given Cartesian coordinate system and let $\partial\Omega_P \cap \partial\Omega_E$ be the segment lying on the axis x_2. Let $A = [0, R]$ and $B = [0, -R]$ be the end-points of $\partial\Omega_P \cap \partial\Omega_E$, where R is the radius of the circle considered. Let γ_A and γ_B be the parts of Γ_1 containing the points A and B, respectively. Let A_1 be the end-point of γ_A which lies on $\partial\Omega_P$. Similarly, let B_1 be the end-point of γ_B which lies on $\partial\Omega_P$. Finally, let A_1^* and B_1^* be the points of ∂K which are closest to A_1 and B_1, respectively. Let us cut the domain Ω_1 into two parts Q_2 and Ω_2 by the segments $A_1^*A_1$ and $B_1^*B_1$: $\overline{\Omega}_1 = \overline{\Omega}_2 \cup \overline{Q}_2$, $\Omega_2 \cap Q_2 = \emptyset$, where Q_2 lies along a part of $\partial\Omega_P$. The domain Ω_2 is sketched together with Ω in Fig. 4a.

Let us define a closed domain $\overline{G}$ by the relation

$$\overline{G} = \overline{\Omega}_P \cup \overline{\Omega}_2 \tag{13}$$

and let the function $v \in H^1(G)$ satisfy

$$v = \begin{cases} u_P & \text{in } \Omega_P, \\ 0 & \text{in } \Omega_2. \end{cases}$$

Again, as in part 1, relation (10) holds and we can repeat all considerations from that part and obtain also in this case the desired extension $\mathcal{P}u_P$.

3. Now let us consider the case that the set $\partial\Omega \cap \partial\Omega_E \cap \partial\Omega_P$ consists only of the point A (see, e.g., Fig. 3a). Let the point A_1 have the same meaning as in part 2 and let $D^* \in \partial K$ be the point of ∂K closest to the point D (which is sketched in Fig. 4b). Let us cut the domain Ω_1 into two parts Q_3 and Ω_3 by segments $A_1^*A_1$ and D^*D: $\overline{\Omega}_1 = \overline{Q}_3 \cup \overline{\Omega}_3$, $Q_3 \cap \Omega_3 = \emptyset$, where the closed strip $\overline{Q}_3$ contains the point B. The domain Ω_3 is sketched together with Ω in Fig. 4b.

Let us define a closed domain $\overline{G}$ by the relation

$$\overline{G} = \overline{\Omega}_P \cup \overline{\Omega}_3 \tag{14}$$

and let the function $v \in H^1(G)$ satisfy

$$v = \begin{cases} u_P & \text{in } \Omega_P, \\ 0 & \text{in } \Omega_3. \end{cases}$$

Relation (10) again holds.

Let G_0 be such a domain that $\overline{G} \subset G_0$ and $\overline{G}_0 \cap \overline{H}_i = \emptyset$, where $\overline{H}_i$ are the "holes" in Ω_E. Now we can apply Lemma 1 to the function v and repeat the construction of $\mathcal{P}u_P$ introduced in part 1.

3a. Let us note that we could use the segment $A_2^*A_2$ instead of the segment D^*D, where A_2 is the second end-point of the arc γ_A and $A_2^* \in \partial K$. This approach has a modification (whose three-dimensional generalization will be useful in part F of this proof): Let $\omega \in \mathcal{C}^{0,1}$ be a domain with the following properties: $\omega \cap \Omega = \emptyset$ and $\overline{\omega} \cap \overline{\Omega} = \gamma_A$. We define

$$G = \omega \cup \gamma_A \cup \Omega_P. \tag{15}$$

Hence $\overline{G} = \overline{\omega} \cup \overline{\Omega}_P$ and we can construct the extension $\mathcal{P}u_P$ in the same way as in part 3.

4. We should mention also the case $\partial\Omega_P \setminus (\partial\Omega_E \cap \partial\Omega_P) \subset \Gamma_1$: it suffices to exchange the notation of the subdomains Ω_E and Ω_P in Fig. 4a; the rest is clear.

5. If $\gamma_A \cap (\partial\Omega_E \setminus \overline{I}) = \emptyset$, $\gamma_B \cap (\partial\Omega_E \setminus \overline{I}) = \emptyset$, where I is the relative interior of $\partial\Omega_E \cap \partial\Omega_P$, then we proceed in the same way as in part C.

6. At the end of part D of the proof let us consider the case $\gamma_A \cap (\partial\Omega_P \setminus \overline{I}) = \emptyset$, $\gamma_B \cap (\partial\Omega_P \setminus \overline{I}) = \emptyset$, where $I = \partial\Omega_P \cap \partial\Omega_E$. For a greater simplicity, let $\gamma_B = \emptyset$ and $\Gamma_1 = \gamma_A \cup \lambda$ where $\lambda \subset \partial\Omega_P \setminus I$ ($\lambda \cap \gamma_A = \emptyset$); further, let $\partial\Omega$ be again a circle. We obtain a modification of Fig. 3a with $A_1 \equiv A$, where A_1 again denotes the left-hand end-point of the arc γ_A. Let R_1 be the upper end-point of λ and R_2 the lower

end-point of λ. Let τ_1 be a (piecewise) smooth arc connecting the points R_1, A_1 and τ_2 a (piecewise) smooth arc connecting the points R_2, A_2. Let τ_1, τ_2 have no common points with $\partial\Omega$ (except for the end-points R_i, A_i). Let S be the strip with $\partial S = \tau_1 \cup \tau_2 \cup \lambda \cup \gamma_A \cup \partial K$. Let us set $\overline{G} = \overline{\Omega}_P \cup \overline{S}$. It is always possible to choose the arcs τ_1, τ_2 such that $G \in \mathcal{C}^{0,1}$. The domain G is a domain with two "holes". Let the function $v \in H^1(G)$ satisfy

$$v = \begin{cases} u_P & \text{in } \Omega_P, \\ 0 & \text{in } S. \end{cases}$$

Relation (10) again holds. Applying Lemma 2 we obtain the required result in this case.

E) It remains to analyze the case from Fig. 3b, where we sketch the situation which appears in applications very often: the domain Ω_P^1 is the rotor of an electromachine, the domain Ω_P^2 is the stator of an electromachine and the narrow domain between them (the domain Ω_E) represents an air crevice. In this case

$$\Omega_P = \Omega_P^1 \cup \Omega_P^2$$

is not a domain. Also in this case we can define the space $H^1(\Omega)$: For $v_1 \in H^1(\Omega_P^1)$, $v_2 \in H^1(\Omega_P^2)$ we set

$$Fv = F(v_1, v_2) = \begin{cases} v_1 & \text{on } \Omega_P^1, \\ v_2 & \text{on } \Omega_P^2. \end{cases}$$

Then F is a bounded linear mapping from $H^1(\Omega_P^1) \times H^1(\Omega_P^2)$ into $H^1(\Omega_P) = H^1(\Omega_P^1 \cup \Omega_P^2)$ and $v = (v_1, v_2) \in H^1(\Omega_P)$ satisfies

$$v\big|_{\Omega_P^1} = v_1, \quad v\big|_{\Omega_P^2} = v_2.$$

Let δ be the width of the domain Ω_E. Let us use Lemma 1 with $G = \Omega_P^1$ in such a way that $G_0 \cap \Omega_E$ is a strip of the width $\delta/3$. Further, let us use Lemma 2 with $G = \Omega_P^2$ in such a way that $D \cap \Omega_E$ is a strip of the width $\delta/3$. Let us set

$$u_E = \begin{cases} \mathcal{E}u_P^1 & \text{in } G_0 \cap \Omega_E, \\ 0 & \text{in } \Omega_E \setminus \{(G_0 \cap \Omega_E) \cup (D \cap \Omega_E)\}, \\ \mathcal{F}u_P^2 & \text{in } D \cap \Omega_E. \end{cases} \tag{16}$$

By (16) and Lemmas 1, 2 we have

$$\begin{aligned} \|u_E\|_{1,\Omega_E}^2 &= \|\mathcal{E}u_P^1\|_{1,G_0\cap\Omega_E}^2 + \|\mathcal{F}u_P^2\|_{1,D\cap\Omega_E}^2 \\ &\leqslant C_1^2\|u_P^1\|_{1,\Omega_P^1}^2 + C_2^2\|u_P^2\|_{1,\Omega_P^2} \leqslant \max\{C_1^2, C_2^2\}\|u_P\|_{1,\Omega_P}^2, \end{aligned}$$

which we wanted to prove.

F) The above presented method of proving can be easily extended to three dimensions. Only the case not covered by assumption (7) deserves a special attention: Let $\sigma_1, \ldots, \sigma_n$ with $\operatorname{meas}_2 \sigma_i > 0$ $(i = 1, \ldots, n)$ and $\sigma_j \cap \sigma_k = \emptyset$ be the parts of Γ_1 such that $\sigma_i \cap (\partial\Omega_E \cap \partial\Omega_P) \neq \emptyset$ $(i = 1, \ldots, n)$. Let Δ_i $(i = 1, \ldots, n)$ be parts of a three-dimensional layer (which is a three-dimensional generalization of the strip Ω_1 appearing at the beginning of part D) such that $\overline{\Delta}_j \cap \overline{\Delta}_k = \emptyset$ and

$$\overline{\Delta}_i \cap \overline{\Omega} = \sigma_i \quad (i = 1, \ldots, n).$$

We define

$$G = \Delta \cup \sigma \cup \Omega_P,$$

where

$$\Delta = \bigcup_{i=1}^{n} \Delta_i, \quad \sigma = \bigcup_{i=1}^{n} \sigma_i.$$

Hence $\overline{G} = \overline{\Delta} \cup \overline{\Omega}_P$ and the construction of the extension $\mathcal{P}u_P$ is a straightforward modification of part D3. □

4. Remark. The results presented in this paper play an important role connected with the theory of electromagnetic fields in electromachines and generally in the theory of parabolic-elliptic equations (see, e.g., [4]). Without using them one cannot present correct proofs of some related results (as it happened, for example, in [7] and [8]).

References

[1] *A. Kufner, O. John, and S. Fučík*: Function Spaces. Academia, Prague, 1977.
[2] *J. Nečas*: Les Méthodes Directes en Théorie des Equations Elliptiques. Academia/Masson, Prague/Paris, 1967.
[3] *J. Nečas, I. Hlaváček*: Mathematical Theory of Elastic and Elasto-Plastic Bodies: An Introduction. Elsevier, Amsterdam, 1981.
[4] *A. Ženíšek*: Finite element variational crimes in parabolic-elliptic problems. Part I. Nonlinear schemes. Numer. Math. *55* (1989), 343–376.
[5] *A. Ženíšek*: Nonlinear Elliptic and Evolution Problems and their Finite Element Approximations. Academic Press, London, 1990.
[6] *A. Ženíšek*: On a generalization of Nikolskij's extension theorem in the case of two variables. Appl. Math. *48* (2003), 367–404.
[7] *M. Zlámal*: Finite element solution of quasistationary nonlinear magnetic field. RAIRO Anal. Numér. *16* (1982), 161–191.
[8] *M. Zlámal*: A linear scheme for the numerical solution of nonlinear quasistationary magnetic fields. Math. Comput. *41* (1983), 425–440.

Author's address: *A. Ženíšek*, Department of Mathematics, Faculty of Mechanical Engineering VUT, Technická 2, 616 69 Brno, Czech Republic, e-mail: zenisek@um.fme.vutbr.cz.