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EXTENDED HASHIN-SHTRIKMAN VARIATIONAL PRINCIPLES*

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Abstract. Internal parameters, eigenstrains, or eigenstresses, arise in functionally graded materials, which are typically present in particulate, layered, or rock bodies. These parameters may be realized in different ways, e.g., by prestressing, temperature changes, effects of wetting, swelling, they may also represent inelastic strains, etc. In order to clarify the use of eigenparameters (eigenstrains or eigenstresses) in physical description, the classical formulation of elasticity is presented, and the two most important Lagrange's and Castigliano's variational principles are formulated in the sequel. Then the classical Hashin-Shtrikman principles are recalled and the involvement of eigenparameters is studied in more detail.

Keywords: extended Hashin-Shtrikman variational principle, eigenparameter, transformation field analysis

MSC 2000: 74E30, 74B10, 49S05

1. INTRODUCTION

Eigenstresses and eigenstrains play a very important role in many branches of applied mechanics, e.g., in composites, geotechnics, concrete structures, etc. In previous papers, [13], [14], the authors have formulated an effective approach to the analysis and optimization of nonhomogeneous bodies with prescribed boundary displacements or tractions and have used the transformation field analysis for relating the components of stress or strain tensors and of eigenstrains or eigenstresses. The transformation field analysis established by Dvorak in [2] has been applied to localization of stresses and strains in two-phase composites. The eigenstresses stood for relaxation stresses while eigenstrains represented plastic strains. This idea was extended in [3], [4], [15], where applications of a large scale of combinations of internal material situations together with prestress of composite structures were considered.

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In [3], thick-walled cylindrical structures were studied while in [4] and [15] submerged cylindrical laminates with different properties in combination with prestress were discussed.

R. Hill in [10] presented one of the first comprehensive approaches on how to solve elastic problems with sudden change of material parameters in terms of variational principles. An introduction of special material constants belongs also to Hill, who enabled researchers to split three-dimensional problems into pure shear and pure compression (tension) problems.

In [12] and [17] an interesting attempt at obtaining effective material properties of a nonlinear isotropic composite has been made. A new variational approach was proposed that provides the effective energy potentials of nonlinear composites in terms of the corresponding energy potentials for linear composites with the same microstructural distributions. When using the eigenparameters in the sense of [2] and generalize it to the macrostructure (localization) of composites, one can obtain procedures that involve a very wide scale of nonlinear problems (plasticity, viscoplasticity, damage, etc.). This is why we have been interested in such a variational formulation which is naturally valid for composites and allows us to extend the well-known variational principles using eigenparameters. To this end the most appropriate means are Hashin-Shtrikman variational principles [7], [9], which have been applied to estimation of material bounds in [8]. Using Eshelby's trick [6], an integral formulation can be stated [14], and the boundary element method is then applicable [1]. In comparison with the finite element method the boundary element method appears to be far more efficient in this case.

It is worth noting that the eigenparameters are an extension of, among other, the influence of change of temperature (eigenstrain); this has been discussed in the well-known paper by Levin [11].

Our approach is based on the idea of augmented Hashin-Shtrikman variational principles. This paper deals with extended primary and dual variational principles for nonhomogeneous bodies. By means of internal parameters, eigenstrains or eigenstresses, involved in H-S principles, it is possible to obtain new bounds on mechanical properties of the trial material, increase the bearing capacity of structures, and to minimize the stress excesses.

The paper deals with the deterministic solution of overall properties of composite materials. Randomly distributed phases (fibers) in connection with H-S principles have recently been studied by Willis [16], and Drugan and Willis [5].

1. Basic relations

We start with basic relations which are valid in mechanics of continuum and are appropriate for our next considerations.

Denote by $\Omega \in \mathbb{R}_3 \equiv 0x_1x_2x_3$ a bounded domain, $\Gamma = \Gamma_u \cup \Gamma_p$ $(\Gamma_u \cap \Gamma_p = 0)$ being its Lipschitz's boundary, both representing the trial body. On Γ_u the displacement vector $\bar{\mathbf{u}} \equiv \{\bar{u}_1, \bar{u}_2, \bar{u}_3\} \in [H^{\frac{1}{2}}(\Gamma_u)]^3$ is prescribed, and on Γ_p the vector of tractions $\bar{\mathbf{p}} \equiv \{\bar{p}_1, \bar{p}_2, \bar{p}_3\} \in [L^2(\Gamma_p)]^3$ is given. Recall the relation stresses-tractions on the boundary Γ_p: $p_i(\boldsymbol{\xi}) = \sigma_{ij}(\boldsymbol{\xi})n_j(\boldsymbol{\xi})$, where $\mathbf{n} \equiv \{n_1, n_2, n_3\}$ is the outward unit normal to the boundary Γ, $\boldsymbol{\xi} \equiv \{\xi_1, \xi_2, \xi_3\} \in \Gamma_p$.

Hooke's law for anisotropic and nonhomogeneous field is introduced in the form

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{L}(\mathbf{x})\colon \boldsymbol{\varepsilon}(\mathbf{x}) + \boldsymbol{\lambda}(\mathbf{x}), \quad \boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{M}(\mathbf{x})\colon \boldsymbol{\sigma}(\mathbf{x}) + \boldsymbol{\mu}(\mathbf{x}) \tag{2.1}$$

or

$$\sigma_{ij}(\mathbf{x}) = L_{ijkl}(\mathbf{x})\varepsilon_{kl}(\mathbf{x}) + \lambda_{ij}(\mathbf{x}), \quad \varepsilon_{ij}(\mathbf{x}) = M_{ijkl}(\mathbf{x})\sigma_{kl}(\mathbf{x}) + \mu_{ij}(\mathbf{x}),$$

where $\boldsymbol{\sigma} \equiv [\sigma_{ij}] \in H^{\mathrm{sym}}_{\mathrm{div}}(\Omega)$ is the stress tensor, $\boldsymbol{\varepsilon} \equiv [\varepsilon_{ij}] \in H^{\mathrm{sym}}_{\mathrm{div}}(\Omega)$ is the strain tensor, $\boldsymbol{\lambda} \equiv [\lambda_{ij}] \in H^{\mathrm{sym}}_{\mathrm{div}}(\Omega)$ is the eigenstress tensor, $\boldsymbol{\mu} \equiv [\mu_{ij}] \in H^{\mathrm{sym}}_{\mathrm{div}}(\Omega)$ is the eigenstrain tensor, $\mathbf{x} \equiv \{x_1, x_2, x_3\} \in \Omega$ is a position at which the material relations are studied, $\mathbf{L} \equiv L_{ijkl}$, $L_{ijkl} \in L^\infty(\Omega)$ is the material stiffness tensor and $\mathbf{M} \equiv M_{ijkl}$, $M_{ijkl} \in L^\infty(\Omega)$ is its compliance material tensor, both with the standard symmetry; the subscripts run the set $\{1,2,3\}$,

$$[\sigma_{ij}] \in H^{\mathrm{sym}}_{\mathrm{div}}(\Omega) \equiv \left((\sigma_{ij})^3_{i,j=1} \in L_2(\Omega), \ \frac{\partial \sigma_{ij}}{\partial x_j} \in L_2(\Omega), \ \sigma_{ij} = \sigma_{ji}\right).$$

Moreover, we have

$$L_{ijkl}M_{klmn} = I_{ijmn}, \quad I_{ijmn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}), \tag{2.2}$$

where $\mathbf{I} \equiv I_{ijkl}$ is the fourth-order unit tensor, δ_{ij} is the Kronecker delta.

Note that for a homogeneous and isotropic material the tensor $\mathbf{L}$ has the form

$$L_{ijkl} = \lambda\delta_{ij}\delta_{kl} + 2\mu I_{ijkl}, \tag{2.3}$$

where λ and μ are Lame's constants. Instead of μ, the shear modulus G is sometimes introduced.

Comparing the two equations (2.1), we get

$$\lambda_{ij} = -L_{ijkl}\mu_{kl}, \quad \mu_{ij} = -M_{ijkl}\lambda_{kl}. \tag{2.4}$$

Kinematic equations may be written as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2.5}$$

Note that displacements $\mathbf{u} \in [H^1(\Omega)]^3$ and $\mathbf{u} = \bar{\mathbf{u}} \in [H^{\frac{1}{2}}(\Gamma_u)]^3$ are prescribed. They are said to be kinematically admissible if the relation (2.5) holds.

Eventually, static equations or equations of equilibrium yield

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \tag{2.6}$$

provided no volume weight forces are taken into account. The last relation has to be taken in the sense of distributions.

Note that one says that the stress tensor is statically admissible or its components are statically admissible, if $[\sigma_{ij}] \in H^{\mathrm{sym}}_{\mathrm{div}}(\Omega)$, statistical boundary conditions on Γ_p are prescribed and (2.6) is fulfilled.

Substituting the kinematical equations into the equations of equilibrium leads to Lame's equations for the unknown displacement vector $\mathbf{u} \equiv \{u_1, u_2, u_3\} \in H^1(\Omega)$, which are written in the sense of distributions:

$$\frac{\partial}{\partial x_j}\left[L_{ijkl}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} - 2\mu_{kl}\right)\right] = 0 \quad \text{in } \Omega, \tag{2.7}$$

or alternatively

$$\frac{\partial}{\partial x_j}\left[L_{ijkl}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) + 2\lambda_{ij}\right] = 0 \quad \text{in } \Omega, \tag{2.8}$$

for a given field $\boldsymbol{\mu}$, or $\boldsymbol{\lambda}$, both in $[H^{\mathrm{sym}}_{\mathrm{div}}(\Omega)]^3$.

Recall that on the part Γ_u of the boundary Γ the displacement vector $\bar{\mathbf{u}} \in [H^{\frac{1}{2}}(\Gamma_u)]^3$ is prescribed, and the traction field $\bar{\mathbf{p}}$ is given on $\Gamma_p \in [L^2(\Gamma_p)]^3$. Assuming smooth enough fields $\mathbf{u} \in \Omega$, we can formulate a variational principle which is equivalent to the equation (2.7) or (2.8):

Lagrange's primary principle: For given tractions $\mathbf{p} = \bar{\mathbf{p}}$ on Γ_p find the minimum value of the functional $\Pi_u(\mathbf{u}) = \Pi_i^u(\mathbf{u}) + \Pi_e^u(\mathbf{u})$ on the set of kinematically admissible displacements ($\mathbf{u} \equiv \{u_1, u_2, u_3\}$) on Γ_u, i.e., $\mathbf{u} = \bar{\mathbf{u}} \in [H^{\frac{1}{2}}(\Gamma_u)]^3$ and $\bar{\mathbf{u}}_3$ is prescribed, where

$$(2.9)\quad \Pi_i^u(\mathbf{u}) = \frac{1}{2}\int_\Omega W\,\mathrm{d}\Omega = \frac{1}{2}\int_\Omega [\boldsymbol{\varepsilon}(\mathbf{u}(\mathbf{x})) - \boldsymbol{\mu}(\mathbf{x})] : \mathbf{L}(\mathbf{x}) : [\boldsymbol{\varepsilon}(\mathbf{u}(\mathbf{x})) - \boldsymbol{\mu}(\mathbf{x})]\,\mathrm{d}\Omega$$
$$= \frac{1}{2}\int_\Omega L_{ijkl}(\mathbf{x})[\varepsilon_{ij}(\mathbf{u}(\mathbf{x})) - \mu_{ij}(\mathbf{x})][\varepsilon_{kl}(\mathbf{u}(\mathbf{x})) - \mu_{kl}(\mathbf{x})]\,\mathrm{d}\Omega$$
$$= \frac{1}{2}\int_\Omega \boldsymbol{\sigma}(\mathbf{x}) : \mathbf{M}(\mathbf{x}) : \boldsymbol{\sigma}(\mathbf{x})\,\mathrm{d}\Omega$$
$$= \frac{1}{2}\int_\Omega M_{ijkl}(\mathbf{x})\sigma_{ij}(\mathbf{x})\sigma_{kl}(\mathbf{x})\,\mathrm{d}\Omega,$$

$$(2.10)\quad \Pi_e^u(\mathbf{u}) = -\int_{\Gamma_p} \bar{\mathbf{p}}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) = -\int_{\Gamma_p} \bar{p}_i(\mathbf{x})u_i(\mathbf{x})\,\mathrm{d}\Gamma.$$

Here Π_i^u is the *the energy of internal forces*, potential energy, whereas Π_e^u is the *energy of external forces*. W is the density of internal energy.

Assuming the validity of (2.5), the principle is equivalent to (2.1), or, if (2.1) and the boundary condition on Γ_u are fulfilled, then the variational principle is equivalent to (2.5).

The dual, or Castigliano's principle can be formulated for the stress tensor $\boldsymbol{\sigma}$ in (2.7), or (2.8):

Castigliano's principle: For given boundary displacements $\mathbf{u} = \bar{\mathbf{u}}$ on Γ_u find the maximum of the functional $\Pi_p(\boldsymbol{\sigma}) = \Pi_i^p(\boldsymbol{\sigma}) + \Pi_e^p(\boldsymbol{\sigma})$ on the set of statically admissible stress fields on the boundary Γ_p, i.e., $\boldsymbol{\sigma}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) = \bar{\mathbf{p}}(\mathbf{x}) \in [L^2(\Gamma_p)]^3$, where $\mathbf{n}$ is the unit outward normal to the boundary Γ_p and $\bar{\mathbf{p}}$ are prescribed tractions:

$$(2.11)\quad \Pi_i^p(\boldsymbol{\sigma})$$
$$= \frac{1}{2}\int_\Omega W^*\,\mathrm{d}\Omega(\mathbf{x}) = \frac{1}{2}\int_\Omega [\boldsymbol{\sigma}(\mathbf{x} : \mathbf{M}(\mathbf{x}) : \boldsymbol{\sigma}(\mathbf{x}) + 2\boldsymbol{\sigma}(\mathbf{x}) : \boldsymbol{\mu}(\mathbf{x})]\,\mathrm{d}\Omega$$
$$= \frac{1}{2}\int_\Omega [M_{ijkl}(\mathbf{x})\sigma_{ij}(\mathbf{x})\sigma_{kl}(\mathbf{x}) + 2\sigma_{ij}(\mathbf{x})\mu_{ij}(\mathbf{x})]\,\mathrm{d}\Omega$$
$$= \frac{1}{2}\int_\Omega \{[\boldsymbol{\sigma}(\mathbf{x}) - \boldsymbol{\lambda}(\mathbf{x})] : \mathbf{M}(\mathbf{x}) : [\boldsymbol{\sigma}(\mathbf{x}) - \boldsymbol{\lambda}(\mathbf{x})] - \boldsymbol{\lambda}(\mathbf{x}) : \mathbf{M}(\mathbf{x}) : \boldsymbol{\lambda}(\mathbf{x})\}\,\mathrm{d}\Omega$$
$$= \frac{1}{2}\int_\Omega \{M_{ijkl}(\mathbf{x})[\sigma_{ij}(\mathbf{x}) - \lambda_{ij}(\mathbf{x})][\sigma_{kl}(\mathbf{x}) - \lambda_{kl}(\mathbf{x})]$$
$$- M_{ijkl}(\mathbf{x})\lambda_{ij}(\mathbf{x})\lambda_{kl}(\mathbf{x})\}\,\mathrm{d}\Omega,$$

$$(2.12)\quad \Pi_e^p(\boldsymbol{\sigma})$$
$$= -\int_{\Gamma_u} \boldsymbol{\sigma}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x})\cdot\bar{\mathbf{u}}(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) = -\int_{\Gamma_u} \sigma_{ij}(\mathbf{x})\cdot n(\mathbf{x})_j\bar{u}_i(\mathbf{x})\,\mathrm{d}\Gamma,$$

Π_i^p is the *complementary energy of internal forces*, whereas Π_e^p is the *external energy*. W^* is the density of the complementary energy of internal forces.

In the sense of *Legendre's transformation* we have, see Fig. 2.1:

$$\Pi_i^u + \Pi_i^p = \int_\Omega \sigma(\mathbf{x}) : \varepsilon(\mathbf{x})\, d\Omega. \tag{2.13}$$

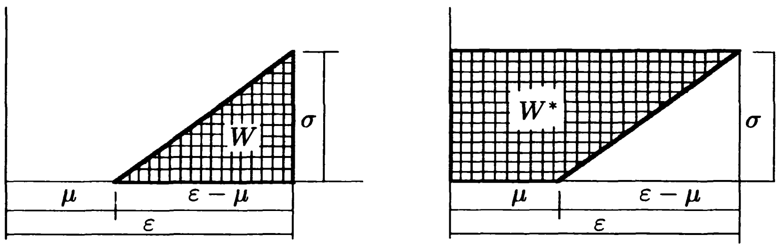

Figure 2.1. Internal energies for primary and dual principles.

3. Extended Hashin-Shtrikman variational principles

In this section we extend the Hashin-Shtrikman variational principle [15], by introducing both the eigenstrain and eigenstress fields into the formulation. For the sake of simplicity assume that no body forces are present.

3.1. Preliminary considerations

The idea of Hashin and Shtrikman consists in introducing new variables τ_{ij} or γ_{ij} (components of polarization tensors) to get another free variables which may be used for "the best" estimation of bounds on overall material properties of nonhomogeneous and anisotropic media.

Let us consider a bounded domain Ω with bounded Lipschitz's boundary Γ and with subdomains Ω_i, $i = 1, \ldots, n$, describing local inhomogeneities, see Fig. 3.1.

Following the Hashin and Shtrikman idea, let us split the procedure into two steps. First, let ε_{ij}^0 and σ_{ij}^0 be the strain field and the stress field, respectively. The stresses σ_{ij}^0 and the small strains ε_{ij}^0 are related by linear homogeneous isotropic Hooke's law:

$$\sigma_{ij}^0 = L_{ijkl}^0 \varepsilon_{kl}^0 \quad \text{in } \Omega, \tag{3.1}$$

or

$$\varepsilon_{ij}^0 = M_{ijkl}^0 \sigma_{kl}^0 \quad \text{in } \Omega, \tag{3.2}$$

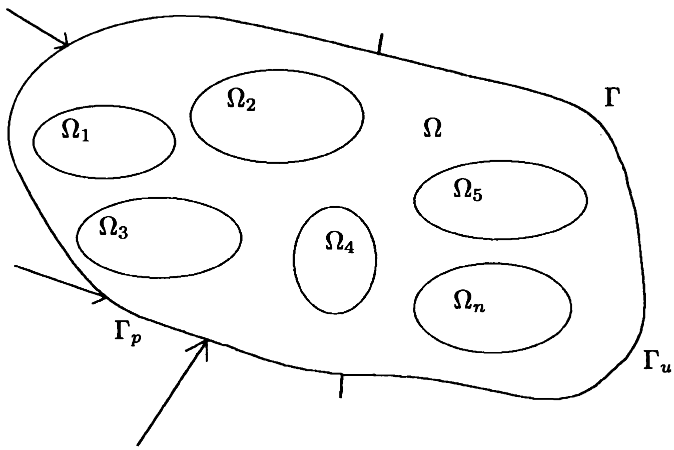

Figure 3.1. Geometry and denotation of the trial body.

where L^0_{ijkl} and M^0_{ijkl} are constant components of material stiffnesses and compliances, respectively. Subscripts in (3.1) and (3.2) run from 1 to 3. It is worth noting that the stresses σ^0_{ij} are in Ω statically admissible, since linear elasticity is considered in the above comparison media of the trial body (the quantities in which are denoted by 0). Similarly, kinematic equations $\left(\varepsilon^0_{ij} = \partial u^0_i/\partial x_j + \partial u^0_j/\partial x_i\right)$ are valid to get the proper relation between the components of the strain tensor and the displacement vector. These conditions will be necessary in what follows. In this sense, the quantities with 0 are considered to be given.

In the second step a geometrically identical body is considered, which is anisotropic and nonhomogeneous. Displacements u_i, strains ε_{ij} and stresses σ_{ij} are unknown and the generalized Hooke's law including the eigenstresses λ_{ij} can be written as

$$\sigma_{ij} = L_{ijkl}\varepsilon_{kl} + \lambda_{ij}, \quad \lambda_{ij} = -L_{ijkl}\mu_{kl}, \quad \text{in } \Omega, \tag{3.3}$$

where μ_{kl} are the eigenstrains. The inverse Hooke's law holds in the form

$$\varepsilon_{ij} = M_{ijkl}\sigma_{kl} + \mu_{ij}, \quad \mu_{ij} = -M_{ijkl}\lambda_{kl}, \quad \text{in } \Omega. \tag{3.4}$$

Similarly to the classical Hashin-Shtrikman principles, define the symmetric stress polarization tensor τ_{ij} and the symmetric strain polarization tensor γ_{ij} by

$$\sigma_{ij} = L^0_{ijkl}\varepsilon_{kl} + \tau_{ij}, \tag{3.5}$$

$$\varepsilon_{ij} = M^0_{ijkl}\sigma_{kl} + \gamma_{ij}. \tag{3.6}$$

The definition of polarization tensors follows from a comparison of (3.3) and (3.5) (for $\boldsymbol{\tau}$) and from a comparison of (3.4) and (3.6) (for $\boldsymbol{\gamma}$):

$$\tau_{ij} = [L_{ijkl}]\varepsilon_{kl} + \lambda_{ij} \tag{3.7}$$

and

$$\gamma_{ij} = [M_{ijkl}]\sigma_{kl} + \mu_{ij}, \tag{3.8}$$

where

$$[L_{ijkl}] = L_{ijkl} - L^0_{ijkl}, \quad [M_{ijkl}] = M_{ijkl} - M^0_{ijkl}.$$

Define also

$$u'_i = u_i - u^0_i, \quad \sigma'_{ij} = \sigma_{ij} - \sigma^0_{ij} \quad \text{in } \Omega, \tag{3.9}$$

and the kinematic equations

$$\varepsilon'_{ij} = \varepsilon_{ij} - \varepsilon^0_{ij} = \frac{\partial u_i}{\partial u_j} - \frac{\partial u_j}{\partial u_i} - \left(\frac{\partial u^0_i}{\partial u_j} - \frac{\partial u^0_j}{\partial u_i}\right). \tag{3.9a}$$

Let us introduce two assumptions:

Assumption A: the surface displacements $\bar{u}_i \in [H^{\frac{1}{2}}(\Gamma_u)]^3$ are prescribed along the entire boundary $\Gamma \equiv \Gamma_u$, and $u^0_i(\boldsymbol{\xi}) \equiv \bar{u}_i(\boldsymbol{\xi})$, $\boldsymbol{\xi} \in \Gamma$, hence $u'_i(\boldsymbol{\xi}) = \bar{u}_i(\boldsymbol{\xi}) - u^0_i(\boldsymbol{\xi}) = 0$, $\boldsymbol{\xi} \in \Gamma$.

Assumption B: the tractions $\bar{p}_i \in [L^2(\Gamma_p)]^3$ are given along the entire boundary $\Gamma \equiv \Gamma_p$, and $p^0_i(\boldsymbol{\xi}) \equiv \bar{p}_i(\boldsymbol{\xi})$, $\boldsymbol{\xi} \in \Gamma$, hence $p'_i(\boldsymbol{\xi}) = \sigma'_{ij}(\boldsymbol{\xi})n_j(\boldsymbol{\xi}) = \bar{p}_i(\boldsymbol{\xi}) - p^0_i(\boldsymbol{\xi}) = 0$, $\boldsymbol{\xi} \in \Gamma$.

Lemma 1. *In the case of Assumption* A *together with the equilibrium* (2.6) *and the kinematic equations, for the stress fields* σ_{ij}, σ^0_{ij}, σ'_{ij} *we have*

$$\int_\Omega \sigma_{ij}\varepsilon'_{ij}\,\mathrm{d}\Omega = \int_\Omega \sigma^0_{ij}\varepsilon'_{ij}\,\mathrm{d}\Omega = \int_\Omega \sigma'_{ij}\varepsilon'_{ij}\,\mathrm{d}\Omega = 0.$$

In the case of Assumption B *and assuming the validity of kinematic equations, for the stress fields* σ_{ij}, σ^0_{ij}, σ'_{ij} *obeying* (2.6) *we have*

$$\int_\Omega \sigma'_{ij}\varepsilon_{ij}\,\mathrm{d}\Omega = \int_\Omega \sigma'_{ij}\varepsilon^0_{ij}\,\mathrm{d}\Omega = \int_\Omega \sigma'_{ij}\varepsilon'_{ij}\,\mathrm{d}\Omega = 0.$$

P r o o f. For example,

$$\int_\Omega \sigma'_{ij}\varepsilon'_{ij}\,\mathrm{d}\Omega = \frac{1}{2}\int_\Omega \sigma'_{ij}\left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}\right)\mathrm{d}\Omega = \{\text{from Green's theorem}\} = 0$$

as either $u'_i = 0$, or $\sigma'_{ij}n_j = 0$ on Γ and σ'_{ij} fulfils (2.6). In the same manner one gets the other expressions in Lemma 1.

Lemma 2. *Denoting the first variation (Gateau's derivative in the direction of a small change of the vector by which we differentiate) by δ, we have*

$$\int_\Omega (\varepsilon'_{kl}\delta\tau_{kl} - \tau_{kl}\delta\varepsilon'_{kl})\,\mathrm{d}\Omega = 0.$$

P r o o f.

$$\begin{aligned}
&\int_\Omega (\varepsilon'_{kl}\delta\tau_{kl} - \tau_{kl}\delta\varepsilon'_{kl})\,\mathrm{d}\Omega \\
&= \int_\Omega (\varepsilon'_{kl}\delta\tau_{kl} - \tau_{kl}\delta\varepsilon'_{kl} + L^0_{ijkl}\varepsilon'_{ij}\delta\varepsilon'_{kl} - L^0_{ijkl}\varepsilon'_{ij}\delta\varepsilon'_{kl})\,\mathrm{d}\Omega \\
&\{\text{because of the symmetry } L^0_{ijkl} = L^0_{klij},\ (3.5) \text{ and } (3.1)\} \\
&= \int_\Omega (\varepsilon'_{kl}\delta\sigma'_{kl} - \sigma'_{kl}\delta\varepsilon'_{kl})\,\mathrm{d}\Omega.
\end{aligned}$$

□

The last integral is equal to zero by virtue of the next lemma.

Lemma 3. *For the second polarization tensor one has*

$$\int_\Omega (\sigma'_{ij}\delta\gamma_{ij} - \gamma_{ij}\delta\sigma'_{ij})\,\mathrm{d}\Omega = 0.$$

P r o o f.

$$\begin{aligned}
&\int_\Omega (\sigma'_{ij}\delta\gamma_{ij} - \gamma_{ij}\delta\sigma'_{ij})\,\mathrm{d}\Omega \\
&= \int_\Omega (\sigma'_{ij}\delta\gamma_{ij} - \gamma_{ij}\delta\sigma'_{ij} + M^0_{ijkl}\sigma'_{ij}\delta\sigma'_{kl} - M^0_{ijkl}\sigma'_{ij}\delta\sigma'_{kl})\,\mathrm{d}\Omega \\
&= \int_\Omega (\sigma'_{ij}\delta\varepsilon'_{ij} - \varepsilon'_{ij}\delta\sigma'_{ij})\,\mathrm{d}\Omega.
\end{aligned}$$

The last integral vanishes because of result of Lemma 1. □

3.2. Extended primary Hashin-Shtrikman variational principle

Let Assumption A be fulfilled. Subtracting (3.1) from (3.5) yields

$$\sigma'_{ij} = L^0_{ijkl}\varepsilon'_{kl} + \tau_{ij}. \tag{3.10}$$

Since both σ_{ij} and σ^0_{ij} are statically admissible, and u_i and u^0_i are kinematically admissible, taking into consideration (3.10) the following equations have to be satisfied in the sense of distributions:

$$\frac{\partial\sigma'_{ij}}{\partial x_j} = \frac{\partial(L^0_{ijkl}\varepsilon'_{kl} + \tau_{ij})}{\partial x_j} = 0 \quad \text{in } \Omega, \tag{3.11}$$

$$\tau_{ij} - [L_{ijkl}]\varepsilon_{kl} - \lambda_{ij} = 0 \quad \text{in } \Omega, \tag{3.12}$$

$$u'_i = 0 \quad \text{on } \Gamma, \tag{3.13}$$

where

$$[L_{ijkl}] = L_{ijkl} - L^0_{ijkl},$$

and (3.13) follows from Assumption A.

Formula (3.12) can be recast as

$$C_{ijkl}(\tau_{kl} - \lambda_{kl}) - \varepsilon_{ij} = 0, \tag{3.14}$$

where $[L_{ijrs}]C_{rskl} = I_{ijkl}$.

Theorem 3.1. *If (3.11) to (3.13) and (3.9a) are fulfilled, the following variational principle can be formulated: find the stationary point of the extended functional U defined as*

$$U(\tau_{ij},\varepsilon'_{ij}) = U^0 - \frac{1}{2}\int_\Omega \{C_{ijkl}(\tau_{ij} - \lambda_{ij})(\tau_{kl} - \lambda_{kl}) - 2\tau_{ij}\varepsilon^0_{ij} - \varepsilon'_{ij}\,\tau_{ij} - M_{ijkl}\lambda_{ij}\lambda_{kl}\}\,d\Omega. \tag{3.15}$$

In (3.15) we have denoted

$$U^0 = \frac{1}{2}\int_\Omega \sigma^0_{ij}\varepsilon^0_{ij}\,d\Omega = \frac{1}{2}\int_\Omega L^0_{ijkl}\varepsilon^0_{ij}\varepsilon^0_{kl}\,d\Omega.$$

Proof. The first variation of (3.15) with respect to the two independent fields τ_{ij} and ε'_{ij} yields

$$\begin{aligned}\delta U &= -\int_\Omega \left[C_{ijkl}(\tau_{ij} - \lambda_{ij})\delta\tau_{kl} - \varepsilon^0_{kl}\delta\tau_{kl} - \frac{1}{2}\tau_{kl}\delta\varepsilon'_{kl} - \frac{1}{2}\varepsilon'_{kl}\delta\tau_{kl}\right] d\Omega \\ &= -\int_\Omega \left\{[C_{ijkl}(\tau_{ij} - \lambda_{ij}) - \varepsilon_{kl}]\delta\tau_{kl} + \frac{1}{2}[\varepsilon'_{kl}\delta\tau_{kl} - \tau_{kl}\delta\varepsilon'_{kl}]\right\} d\Omega.\end{aligned}$$

From (3.14), the first term is zero and the second term vanishes because of Lemma 2. □

Theorem 3.2. *The functional U is equal to the actual potential strain energy stored in the anisotropic and heterogeneous body—see Fig. 2.1:*

$$U = \int_\Omega W \, \mathrm{d}\Omega,$$

where

$$W = \frac{1}{2} L_{ijkl}(\varepsilon_{ij} - \mu_{ij})(\varepsilon_{kl} - \mu_{kl}), \quad \mu_{ij} = -M_{ijkl}\, \lambda_{kl}.$$

Proof. Substituting λ_{ij} for μ_{ij}, and owing to (3.14), the integrand of (3.15) may be written as

$$\begin{gathered} \varepsilon^0_{ij} L^0_{ijkl} \varepsilon^0_{kl} - \varepsilon_{ij}(\tau_{ij} + L_{ijkl}\mu_{kl}) + 2\tau_{ij}\varepsilon^0_{ij} + \varepsilon'_{ij}\tau_{ij} + \mu_{ij} L_{ijkl}\mu_{kl} \\ \{\text{sum up all terms at } \boldsymbol{\tau}, \text{ and use the definition (3.12)}\} \\ = \varepsilon^0_{ij} L^0_{ijkl} \varepsilon^0_{kl} - \varepsilon_{ij} L_{ijkl}\mu_{kl} + [L_{ijkl}(\varepsilon_{ij} - \mu_{ij}) - L^0_{ijkl}\varepsilon^0_{ij} - L^0_{ijkl}\varepsilon'_{ij}]\varepsilon^0_{kl} + \mu_{ij} L_{ijkl}\mu_{kl}. \end{gathered}$$

On the other hand,

$$(\varepsilon_{ij} - \mu_{ij}) L_{ijkl}(\varepsilon_{kl} - \mu_{kl}) = \varepsilon_{ij} L_{ijkl}\varepsilon_{kl} - \varepsilon_{ij} L_{ijkl}\mu_{kl} - \mu_{ij} L_{ijkl}\varepsilon_{kl} + \mu_{ij} L_{ijkl}\mu_{kl}.$$

Comparing the right-hand sides of the last two relations, integrating the result, and taking into consideration Lemma 1, one arrives at

$$\int_\Omega \sigma_{ij}\varepsilon'_{ij} \, \mathrm{d}\Omega = \int_\Omega \sigma^0_{ij}\varepsilon'_{ij} \, \mathrm{d}\Omega = 0,$$

and one obtains the assertion of the theorem. □

Theorem 3.3. *Assuming the conditions of Theorem 3.1, the functional U in (3.15) attains its absolute maximum if $\mathbf{L}$ is positive definite, and it attains its absolute minimum if $\mathbf{C} + \mathbf{M}^0$ is negative semidefinite.*

Proof. The second variation of the functional U is

$$\delta^2 U = -\int_\Omega \{C_{ijkl}\delta\tau_{ij}\delta\tau_{kl} - \delta\varepsilon'_{ij}\delta\tau_{ij}\} \, \mathrm{d}\Omega. \tag{3.16}$$

Substituting from (3.10) to (3.16) for τ_{ij} only in the second term results in

$$\delta^2 U = -\int_\Omega \{C_{ijkl}\delta\tau_{ij}\delta\tau_{kl} - \delta\varepsilon'_{ij}(\delta\sigma'_{ij} - L^0_{ijkl}\delta\varepsilon'_{kl})\} \, \mathrm{d}\Omega.$$

The second term vanishes because of Lemma 1. Since L^0_{ijkl} is the tensor of elastic material constants, it has to be positive definite. Consequently, if the tensor C_{ijkl} is also positive definite, i.e. so is its inverse $[L_{ijkl}]$, the second variation of U is negative and in this case the maximum is attained (sufficient condition).

In order to prove the condition of minimum of the functional (3.15), let us consider the integral

$$I = \int_\Omega M^0_{ijkl}\delta\tau_{ij}\delta\tau_{kl}\,\mathrm{d}\Omega,$$

where M^0_{ijkl} is inverse to L^0_{ijkl}. Substituting from (3.10) for $\delta\tau_{ij}$ and $\delta\tau_{kl}$ yields

$$I = \int_\Omega \{M^0_{ijkl}\delta\sigma'_{ij}\delta\sigma'_{kl} + L^0_{ijkl}\delta\varepsilon'_{ij}\delta\varepsilon'_{kl} - 2\delta\varepsilon'_{ij}\delta\sigma'_{ij}\}\,\mathrm{d}\Omega.$$

The last term vanishes because of Lemma 1. Since both the tensors M^0_{ijkl} and L^0_{ijkl} are positive definite, we conclude

$$\int_\Omega M^0_{ijkl}\delta\tau_{ij}\delta\tau_{kl}\,\mathrm{d}\Omega = \int_\Omega \{M^0_{ijkl}\delta\sigma'_{ij}\delta\sigma'_{kl} + L^0_{ijkl}\delta\varepsilon'_{ij}\delta\varepsilon'_{kl}\}\,\mathrm{d}\Omega$$
$$\Rightarrow \int_\Omega M^0_{ijkl}\delta\tau_{ij}\delta\tau_{kl}\,\mathrm{d}\Omega > \int_\Omega L^0_{ijkl}\delta\varepsilon'_{ij}\delta\varepsilon'_{kl}\,\mathrm{d}\Omega.$$

Coming back to the second variation (3.16) we find out that a sufficient condition for the minimum of the functional U is: $\mathbf{C} + \mathbf{M}^0$ is negative semidefinite. □

3.3. Extended dual Hashin-Shtrikman variational principle

In this section we extend the dual Hashin-Shtrikman variational principle to a body with prescribed surface tractions $\bar{p}_i$ and obeying Assumption B. Assume again that no body forces are present.

Following the classical dual Hashin-Shtrikman theorem, define the symmetric strain polarization tensor γ_{ij} by (3.6). Further, let the primed system be defined by (3.9).

Subtracting (3.2) from (3.6), we obtain

$$\varepsilon'_{ij} = \varepsilon_{ij} - \varepsilon^0_{ij} = M^0_{ijkl}\sigma'_{kl} + \gamma_{ij}. \tag{3.17}$$

Since both the fields ε_{ij} and ε^0_{ij} are kinematically admissible, and the stresses σ_{ij} and σ^0_{ij} are statically admissible, (3.11) still holds in the sense of distributions:

$$\frac{\partial\sigma'_{ij}}{\partial x_j} = 0 \quad \text{in } \Omega, \tag{3.18}$$

$$\gamma_{ij} - [M_{ijkl}]\sigma_{kl} - \mu_{ij} = 0 \quad \text{in } \Omega, \tag{3.19}$$

$$\sigma'_{ij}n_j = 0 \quad \text{on } \Gamma, \tag{3.20}$$

where

$$[M_{ijkl}] = M_{ijkl} - M^0_{ijkl},$$

and (3.20) follows from Assumption B.

The definition of the polarization tensor (3.19) can be rewritten as

$$D_{ijkl}(\gamma_{kl} - \mu_{kl}) - \sigma_{ij} = 0, \tag{3.21}$$

where $[M_{ijrs}]D_{rskl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) = I_{ijkl}$ is the fourth-order unit tensor.

Theorem 3.4. *If* (3.17) *to* (3.20) *are fulfilled, the dual extended H-S variational principle can be formulated, namely, the variation of the functional*

$$U^*(\gamma_{ij}, \sigma'_{ij}) = U^{0*} - \frac{1}{2}\int_\Omega \{D_{ijkl}(\gamma_{ij} - \mu_{ij})(\gamma_{kl} - \mu_{kl}) - 2\gamma_{ij}\sigma^0_{ij} - \sigma'_{ij}\gamma_{ij}\}\,\mathrm{d}\Omega \tag{3.22}$$

attains its stationary value with respect to the fields γ_{ij} and σ'_{ij}. In (3.22) *we have denoted*

$$U^{0*} = \frac{1}{2}\int_\Omega \sigma^0_{ij}\varepsilon^0_{ij}\,\mathrm{d}\Omega = \frac{1}{2}\int_\Omega M^0_{ijkl}\sigma^0_{ij}\sigma^0_{kl}\,\mathrm{d}\Omega.$$

Proof. The first variation of (3.22) leads to the following expression:

$$\begin{aligned}\delta U^* &= -\int_\Omega \left[D_{ijkl}(\gamma_{ij} - \mu_{ij})\delta\gamma_{kl} - \sigma^0_{kl}\delta\gamma_{kl} - \frac{1}{2}\gamma_{ij}\delta\sigma'_{ij} - \frac{1}{2}\sigma'_{ij}\delta\gamma_{ij}\right]\mathrm{d}\Omega\\ &= -\int_\Omega \left\{[D_{ijkl}(\gamma_{ij} - \mu_{ij}) - \sigma_{ij}]\delta\gamma_{kl} + \frac{1}{2}[\sigma'_{ij}\delta\gamma_{ij} - \gamma_{ij}\delta\sigma'_{ij}]\right\}\mathrm{d}\Omega.\end{aligned}$$

The last integral is zero. The first part of it vanishes because of the validity of (3.21) and the second part is zero according to Lemma 3. □

Theorem 3.5. *The functional U^* is equal to the actual complementary energy stored in the anisotropic and heterogeneous body, see Fig.* 2.1*:*

$$U^{s*} = \int_\Omega W^*\,\mathrm{d}\Omega,$$

where

$$W^* = \frac{1}{2}(M_{ijkl}\sigma_{ij}\sigma_{kl} + 2\sigma_{ij}\mu_{ij}).$$

Proof. By virtue of (3.21) and (3.9) the integrand of (3.22) may be written as

$$M^0_{ijkl}\sigma^0_{ij}\sigma^0_{kl} + \sigma_{ij}\mu_{ij} + (\sigma'_{ij} + 2\sigma^0_{ij} - \sigma_{ij})\gamma_{ij}$$

{using (3.6) for removing γ_{ij} and by virtue of (3.9)}

$$= \sigma_{ij}\mu_{ij} + \varepsilon_{ij}(\sigma_{ij} - \sigma'_{ij}).$$

Integration over the domain Ω, Lemma 1, and (2.1_2) result in the assertion of the theorem. □

Theorem 3.6. *Assuming the conditions of Theorem 3.4, the functional U^* attains its absolute maximum if* [**M**] *is positive definite, and it attains its absolute minimum if* $\mathbf{D} + \mathbf{L}^0$ *is negative semidefinite.*

Proof. The second variation of U^* can be expressed as

$$\delta^2 U^* = -\int_\Omega \{D_{ijkl}\delta\gamma_{ij}\delta\gamma_{kl} - \delta\sigma'_{ij}\delta\gamma_{ij}\}\,\mathrm{d}\Gamma.$$

Substituting for γ_{ij} from (3.17) in the second term of the above integral yields

$$\delta^2 U = -\int_\Omega \{D_{ijkl}\delta\gamma_{ij}\delta\gamma_{kl} - \delta\sigma'_{ij}(\delta\varepsilon'_{ij} - M^0_{ijkl}\delta\sigma'_{kl})\}\,\mathrm{d}\Omega.$$

The second term disappears due to Lemma 1. Since M^0_{ijkl} is the tensor of elastic material constants, it has to be positive definite. Consequently, if the tensor C_{ijkl} is also positive definite, i.e. so is its inverse $[L_{ijkl}]$, the second variation of U is negative and in this case the maximum is attained.

In order to prove the condition of the minimum of the functional (3.22), consider the integral

$$I^* = \int_\Omega L^0_{ijkl}\delta\gamma_{ij}\delta\gamma_{kl}\,\mathrm{d}\Omega.$$

Substituting from (3.17) for $\delta\gamma_{ij}$ and $\delta\gamma_{kl}$, we get

$$I^* = \int_\Omega \{L^0_{ijkl}\delta\varepsilon'_{ij}\delta\varepsilon'_{kl} + M^0_{ijkl}\delta\sigma'_{ij}\delta\sigma'_{kl} - 2\delta\sigma'_{ij}\delta\varepsilon'_{ij}\}\,\mathrm{d}\Omega.$$

The last term disappears because of Lemma 1. Since both the tensors L^0_{ijkl} and M^0_{ijkl} are positively definite, we conclude

$$\begin{aligned}\int_\Omega L^0_{ijkl}\delta\gamma_{ij}\delta\gamma_{kl}\,\mathrm{d}\Omega &= \int_\Omega \{L^0_{ijkl}\delta\varepsilon'_{ij}\delta\varepsilon'_{kl} + M^0_{ijkl}\delta\sigma'_{ij}\delta\sigma'_{kl}\}\,\mathrm{d}\Omega \\ &\Rightarrow \int_\Omega L^0_{ijkl}\delta\gamma_{ij}\delta\gamma_{kl}\,\mathrm{d}\Omega > \int_\Omega M^0_{ijkl}\delta\sigma'_{ij}\delta\sigma'_{kl}\,\mathrm{d}\Omega.\end{aligned}$$

From the last inequality follows that a sufficient condition for minimum of the functional U^* is: $\mathbf{D} + \mathbf{L}^0$ is negatively definite. □

4. Applications

Mechanically nonlinear behavior can be introduced using eigenparameters (plastic strain, relaxation stresses), and using these parameters visco-elastic or visco-plastic material can be described. In this way, time dependent problems or hereditary problems can be involved in the eigenparameters. Shape optimization of prestressed fibers can start with the above principles. From the H-S principles a very weak integral formulation directly follows, and the BEM is applicable to nonlinear and time dependent problems [13]. The unpleasant term involving hypersingular integral, which has to be integrated in the sense of Hadamard, can be avoided by Eshelby's trick.

A typical application of the above established extended principles is an introduction of the change of temperature instead of the eigenstrains. The bounds obtained here can be derived in a similar manner as the classical H-S bounds on material constants [8]. But, to derive bounds on the overall temperature characteristics requires very extensive calculation.

5. Conclusion

In this paper, classical Hashin-Shtrikman has been extended by the eigenparameters (eigenstrains or eigenstresses). These internal parameters can stand for a large range of quantities, which are studied in mechanics of solid media. Basically, a similar process to that published in [8] on how to calculate bounds on nonlinear or time-dependent characteristics describing mechanical properties can be applied. On the other hand, each such a problem requires a specific treatment and the solution is not trivial.

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EXTENDED HASHIN-SHTRIKMAN VARIATIONAL PRINCIPLES*

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Abstract. Internal parameters, eigenstrains, or eigenstresses, arise in functionally graded materials, which are typically present in particulate, layered, or rock bodies. These parameters may be realized in different ways, e.g., by prestressing, temperature changes, effects of wetting, swelling, they may also represent inelastic strains, etc. In order to clarify the use of eigenparameters (eigenstrains or eigenstresses) in physical description, the classical formulation of elasticity is presented, and the two most important Lagrange's and Castigliano's variational principles are formulated in the sequel. Then the classical Hashin-Shtrikman principles are recalled and the involvement of eigenparameters is studied in more detail.

Keywords: extended Hashin-Shtrikman variational principle, eigenparameter, transformation field analysis

MSC 2000: 74E30, 74B10, 49S05

1. Introduction

Eigenstresses and eigenstrains play a very important role in many branches of applied mechanics, e.g., in composites, geotechnics, concrete structures, etc. In previous papers, [13], [14], the authors have formulated an effective approach to the analysis and optimization of nonhomogeneous bodies with prescribed boundary displacements or tractions and have used the transformation field analysis for relating the components of stress or strain tensors and of eigenstrains or eigenstresses. The transformation field analysis established by Dvorak in [2] has been applied to localization of stresses and strains in two-phase composites. The eigenstresses stood for relaxation stresses while eigenstrains represented plastic strains. This idea was extended in [3], [4], [15], where applications of a large scale of combinations of internal material situations together with prestress of composite structures were considered.

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In [3], thick-walled cylindrical structures were studied while in [4] and [15] submerged cylindrical laminates with different properties in combination with prestress were discussed.

R. Hill in [10] presented one of the first comprehensive approaches on how to solve elastic problems with sudden change of material parameters in terms of variational principles. An introduction of special material constants belongs also to Hill, who enabled researchers to split three-dimensional problems into pure shear and pure compression (tension) problems.

In [12] and [17] an interesting attempt at obtaining effective material properties of a nonlinear isotropic composite has been made. A new variational approach was proposed that provides the effective energy potentials of nonlinear composites in terms of the corresponding energy potentials for linear composites with the same microstructural distributions. When using the eigenparameters in the sense of [2] and generalize it to the macrostructure (localization) of composites, one can obtain procedures that involve a very wide scale of nonlinear problems (plasticity, viscoplasticity, damage, etc.). This is why we have been interested in such a variational formulation which is naturally valid for composites and allows us to extend the well-known variational principles using eigenparameters. To this end the most appropriate means are Hashin-Shtrikman variational principles [7], [9], which have been applied to estimation of material bounds in [8]. Using Eshelby's trick [6], an integral formulation can be stated [14], and the boundary element method is then applicable [1]. In comparison with the finite element method the boundary element method appears to be far more efficient in this case.

It is worth noting that the eigenparameters are an extension of, among other, the influence of change of temperature (eigenstrain); this has been discussed in the well-known paper by Levin [11].

Our approach is based on the idea of augmented Hashin-Shtrikman variational principles. This paper deals with extended primary and dual variational principles for nonhomogeneous bodies. By means of internal parameters, eigenstrains or eigenstresses, involved in H-S principles, it is possible to obtain new bounds on mechanical properties of the trial material, increase the bearing capacity of structures, and to minimize the stress excesses.

The paper deals with the deterministic solution of overall properties of composite materials. Randomly distributed phases (fibers) in connection with H-S principles have recently been studied by Willis [16], and Drugan and Willis [5].

1. Basic relations

We start with basic relations which are valid in mechanics of continuum and are appropriate for our next considerations.

Denote by $\Omega \in \mathbb{R}_3 \equiv 0x_1x_2x_3$ a bounded domain, $\Gamma = \Gamma_u \cup \Gamma_p$ $(\Gamma_u \cap \Gamma_p = 0)$ being its Lipschitz's boundary, both representing the trial body. On Γ_u the displacement vector $\bar{\mathbf{u}} \equiv \{\bar{u}_1, \bar{u}_2, \bar{u}_3\} \in [H^{\frac{1}{2}}(\Gamma_u)]^3$ is prescribed, and on Γ_p the vector of tractions $\bar{\mathbf{p}} \equiv \{\bar{p}_1, \bar{p}_2, \bar{p}_3\} \in [L^2(\Gamma_p)]^3$ is given. Recall the relation stresses-tractions on the boundary Γ_p: $p_i(\boldsymbol{\xi}) = \sigma_{ij}(\boldsymbol{\xi})n_j(\boldsymbol{\xi})$, where $\mathbf{n} \equiv \{n_1, n_2, n_3\}$ is the outward unit normal to the boundary Γ, $\boldsymbol{\xi} \equiv \{\xi_1, \xi_2, \xi_3\} \in \Gamma_p$.

Hooke's law for anisotropic and nonhomogeneous field is introduced in the form

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{L}(\mathbf{x}) \colon \boldsymbol{\varepsilon}(\mathbf{x}) + \boldsymbol{\lambda}(\mathbf{x}), \quad \boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{M}(\mathbf{x}) \colon \boldsymbol{\sigma}(\mathbf{x}) + \boldsymbol{\mu}(\mathbf{x}) \tag{2.1}$$

or

$$\sigma_{ij}(\mathbf{x}) = L_{ijkl}(\mathbf{x})\varepsilon_{kl}(\mathbf{x}) + \lambda_{ij}(\mathbf{x}), \quad \varepsilon_{ij}(\mathbf{x}) = M_{ijkl}(\mathbf{x})\sigma_{kl}(\mathbf{x}) + \mu_{ij}(\mathbf{x}),$$

where $\boldsymbol{\sigma} \equiv [\sigma_{ij}] \in H^{\mathrm{sym}}_{\mathrm{div}}(\Omega)$ is the stress tensor, $\boldsymbol{\varepsilon} \equiv [\varepsilon_{ij}] \in H^{\mathrm{sym}}_{\mathrm{div}}(\Omega)$ is the strain tensor, $\boldsymbol{\lambda} \equiv [\lambda_{ij}] \in H^{\mathrm{sym}}_{\mathrm{div}}(\Omega)$ is the eigenstress tensor, $\boldsymbol{\mu} \equiv [\mu_{ij}] \in H^{\mathrm{sym}}_{\mathrm{div}}(\Omega)$ is the eigenstrain tensor, $\mathbf{x} \equiv \{x_1, x_2, x_3\} \in \Omega$ is a position at which the material relations are studied, $\mathbf{L} \equiv L_{ijkl}$, $L_{ijkl} \in L^\infty(\Omega)$ is the material stiffness tensor and $\mathbf{M} \equiv M_{ijkl}$, $M_{ijkl} \in L^\infty(\Omega)$ is its compliance material tensor, both with the standard symmetry; the subscripts run the set $\{1, 2, 3\}$,

$$[\sigma_{ij}] \in H^{\mathrm{sym}}_{\mathrm{div}}(\Omega) \equiv \left((\sigma_{ij})^3_{i,j=1} \in L_2(\Omega),\ \frac{\partial \sigma_{ij}}{\partial x_j} \in L_2(\Omega),\ \sigma_{ij} = \sigma_{ji}\right).$$

Moreover, we have

$$L_{ijkl}M_{klmn} = I_{ijmn}, \quad I_{ijmn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}), \tag{2.2}$$

where $\mathbf{I} \equiv I_{ijkl}$ is the fourth-order unit tensor, δ_{ij} is the Kronecker delta.

Note that for a homogeneous and isotropic material the tensor $\mathbf{L}$ has the form

$$L_{ijkl} = \lambda\delta_{ij}\delta_{kl} + 2\mu I_{ijkl}, \tag{2.3}$$

where λ and μ are Lame's constants. Instead of μ, the shear modulus G is sometimes introduced.

Comparing the two equations (2.1), we get

$$\lambda_{ij} = -L_{ijkl}\mu_{kl}, \quad \mu_{ij} = -M_{ijkl}\lambda_{kl}. \tag{2.4}$$

Kinematic equations may be written as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2.5}$$

Note that displacements $\mathbf{u} \in [H^1(\Omega)]^3$ and $\mathbf{u} = \bar{\mathbf{u}} \in [H^{\frac{1}{2}}(\Gamma_u)]^3$ are prescribed. They are said to be kinematically admissible if the relation (2.5) holds.

Eventually, static equations or equations of equilibrium yield

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \tag{2.6}$$

provided no volume weight forces are taken into account. The last relation has to be taken in the sense of distributions.

Note that one says that the stress tensor is statically admissible or its components are statically admissible, if $[\sigma_{ij}] \in H^{\mathrm{sym}}_{\mathrm{div}}(\Omega)$, statistical boundary conditions on Γ_p are prescribed and (2.6) is fulfilled.

Substituting the kinematical equations into the equations of equilibrium leads to Lame's equations for the unknown displacement vector $\mathbf{u} \equiv \{u_1, u_2, u_3\} \in H^1(\Omega)$, which are written in the sense of distributions:

$$\frac{\partial}{\partial x_j}\left[L_{ijkl}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} - 2\mu_{kl}\right)\right] = 0 \quad \text{in } \Omega, \tag{2.7}$$

or alternatively

$$\frac{\partial}{\partial x_j}\left[L_{ijkl}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) + 2\lambda_{ij}\right] = 0 \quad \text{in } \Omega, \tag{2.8}$$

for a given field $\boldsymbol{\mu}$, or $\boldsymbol{\lambda}$, both in $[H^{\mathrm{sym}}_{\mathrm{div}}(\Omega)]^3$.

Recall that on the part Γ_u of the boundary Γ the displacement vector $\bar{\mathbf{u}} \in [H^{\frac{1}{2}}(\Gamma_u)]^3$ is prescribed, and the traction field $\bar{\mathbf{p}}$ is given on $\Gamma_p \in [L^2(\Gamma_p)]^3$. Assuming smooth enough fields $\mathbf{u} \in \Omega$, we can formulate a variational principle which is equivalent to the equation (2.7) or (2.8):

Lagrange's primary principle: For given tractions $\mathbf{p} = \bar{\mathbf{p}}$ on Γ_p find the minimum value of the functional $\Pi_u(\mathbf{u}) = \Pi_i^u(\mathbf{u}) + \Pi_e^u(\mathbf{u})$ on the set of kinematically admissible displacements ($\mathbf{u} \equiv \{u_1, u_2, u_3\}$) on Γ_u, i.e., $\mathbf{u} = \bar{\mathbf{u}} \in [H^{\frac{1}{2}}(\Gamma_u)]^3$ and $\bar{\mathbf{u}}_3$ is prescribed, where

$$
\begin{aligned}
(2.9)\qquad \Pi_i^u(\mathbf{u}) &= \frac{1}{2}\int_\Omega W\,\mathrm{d}\Omega = \frac{1}{2}\int_\Omega [\boldsymbol{\varepsilon}(\mathbf{u}(\mathbf{x})) - \boldsymbol{\mu}(\mathbf{x})] : \mathbf{L}(\mathbf{x}) : [\boldsymbol{\varepsilon}(\mathbf{u}(\mathbf{x})) - \boldsymbol{\mu}(\mathbf{x})]\,\mathrm{d}\Omega \\
&= \frac{1}{2}\int_\Omega L_{ijkl}(\mathbf{x})[\varepsilon_{ij}(\mathbf{u}(\mathbf{x})) - \mu_{ij}(\mathbf{x})][\varepsilon_{kl}(\mathbf{u}(\mathbf{x})) - \mu_{kl}(\mathbf{x})]\,\mathrm{d}\Omega \\
&= \frac{1}{2}\int_\Omega \boldsymbol{\sigma}(\mathbf{x}) : \mathbf{M}(\mathbf{x}) : \boldsymbol{\sigma}(\mathbf{x})\,\mathrm{d}\Omega \\
&= \frac{1}{2}\int_\Omega M_{ijkl}(\mathbf{x})\sigma_{ij}(\mathbf{x})\sigma_{kl}(\mathbf{x})\,\mathrm{d}\Omega, \\
(2.10)\qquad \Pi_e^u(\mathbf{u}) &= -\int_{\Gamma_p} \bar{\mathbf{p}}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) = -\int_{\Gamma_p} \bar{p}_i(\mathbf{x})u_i(\mathbf{x})\,\mathrm{d}\Gamma.
\end{aligned}
$$

Here Π_i^u is the *the energy of internal forces*, potential energy, whereas Π_e^u is the *energy of external forces*. W is the density of internal energy.

Assuming the validity of (2.5), the principle is equivalent to (2.1), or, if (2.1) and the boundary condition on Γ_u are fulfilled, then the variational principle is equivalent to (2.5).

The dual, or Castigliano's principle can be formulated for the stress tensor $\boldsymbol{\sigma}$ in (2.7), or (2.8):

Castigliano's principle: For given boundary displacements $\mathbf{u} = \bar{\mathbf{u}}$ on Γ_u find the maximum of the functional $\Pi_p(\boldsymbol{\sigma}) = \Pi_i^p(\boldsymbol{\sigma}) + \Pi_e^p(\boldsymbol{\sigma})$ on the set of statically admissible stress fields on the boundary Γ_p, i.e., $\boldsymbol{\sigma}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) = \bar{\mathbf{p}}(\mathbf{x}) \in [L^2(\Gamma_p)]^3$, where $\mathbf{n}$ is the unit outward normal to the boundary Γ_p and $\bar{\mathbf{p}}$ are prescribed tractions:

$$
\begin{aligned}
(2.11)\quad & \Pi_i^p(\boldsymbol{\sigma}) \\
&= \frac{1}{2}\int_\Omega W^*\,\mathrm{d}\Omega(\mathbf{x}) = \frac{1}{2}\int_\Omega [\boldsymbol{\sigma}(\mathbf{x} : \mathbf{M}(\mathbf{x}) : \boldsymbol{\sigma}(\mathbf{x}) + 2\boldsymbol{\sigma}(\mathbf{x}) : \boldsymbol{\mu}(\mathbf{x})]\,\mathrm{d}\Omega \\
&= \frac{1}{2}\int_\Omega [M_{ijkl}(\mathbf{x})\sigma_{ij}(\mathbf{x})\sigma_{kl}(\mathbf{x}) + 2\sigma_{ij}(\mathbf{x})\mu_{ij}(\mathbf{x})]\,\mathrm{d}\Omega \\
&= \frac{1}{2}\int_\Omega \{[\boldsymbol{\sigma}(\mathbf{x}) - \boldsymbol{\lambda}(\mathbf{x})] : \mathbf{M}(\mathbf{x}) : [\boldsymbol{\sigma}(\mathbf{x}) - \boldsymbol{\lambda}(\mathbf{x})] - \boldsymbol{\lambda}(\mathbf{x}) : \mathbf{M}(\mathbf{x}) : \boldsymbol{\lambda}(\mathbf{x})\}\,\mathrm{d}\Omega \\
&= \frac{1}{2}\int_\Omega \{M_{ijkl}(\mathbf{x})[\sigma_{ij}(\mathbf{x}) - \lambda_{ij}(\mathbf{x})][\sigma_{kl}(\mathbf{x}) - \lambda_{kl}(\mathbf{x})] \\
&\qquad\qquad - M_{ijkl}(\mathbf{x})\lambda_{ij}(\mathbf{x})\lambda_{kl}(\mathbf{x})\}\,\mathrm{d}\Omega, \\
(2.12)\quad & \Pi_e^p(\boldsymbol{\sigma}) \\
&= -\int_{\Gamma_u} \boldsymbol{\sigma}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x})\cdot\bar{\mathbf{u}}(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) = -\int_{\Gamma_u} \sigma_{ij}(\mathbf{x})\cdot n(\mathbf{x})_j\bar{u}_i(\mathbf{x})\,\mathrm{d}\Gamma,
\end{aligned}
$$

Π_i^p is the *complementary energy of internal forces*, whereas Π_e^p is the *external energy*. W^* is the density of the complementary energy of internal forces.

In the sense of *Legendre's transformation* we have, see Fig. 2.1:

$$\Pi_i^u + \Pi_i^p = \int_\Omega \boldsymbol{\sigma}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x})\,\mathrm{d}\Omega. \tag{2.13}$$

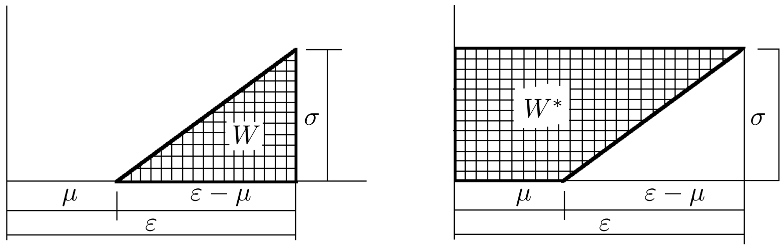

Figure 2.1. Internal energies for primary and dual principles.

3. Extended Hashin-Shtrikman variational principles

In this section we extend the Hashin-Shtrikman variational principle [15], by introducing both the eigenstrain and eigenstress fields into the formulation. For the sake of simplicity assume that no body forces are present.

3.1. Preliminary considerations

The idea of Hashin and Shtrikman consists in introducing new variables τ_{ij} or γ_{ij} (components of polarization tensors) to get another free variables which may be used for "the best" estimation of bounds on overall material properties of nonhomogeneous and anisotropic media.

Let us consider a bounded domain Ω with bounded Lipschitz's boundary Γ and with subdomains Ω_i, $i = 1, \ldots, n$, describing local inhomogeneities, see Fig. 3.1.

Following the Hashin and Shtrikman idea, let us split the procedure into two steps. First, let ε_{ij}^0 and σ_{ij}^0 be the strain field and the stress field, respectively. The stresses σ_{ij}^0 and the small strains ε_{ij}^0 are related by linear homogeneous isotropic Hooke's law:

$$\sigma_{ij}^0 = L_{ijkl}^0 \varepsilon_{kl}^0 \quad \text{in } \Omega, \tag{3.1}$$

or

$$\varepsilon_{ij}^0 = M_{ijkl}^0 \sigma_{kl}^0 \quad \text{in } \Omega, \tag{3.2}$$

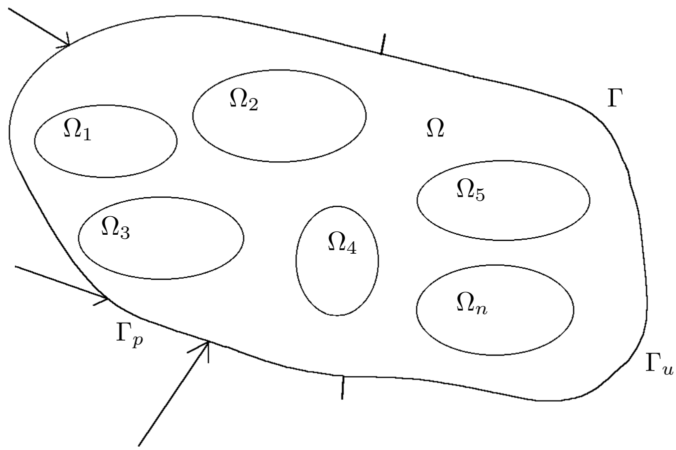

Figure 3.1. Geometry and denotation of the trial body.

where L^0_{ijkl} and M^0_{ijkl} are constant components of material stiffnesses and compliances, respectively. Subscripts in (3.1) and (3.2) run from 1 to 3. It is worth noting that the stresses σ^0_{ij} are in Ω statically admissible, since linear elasticity is considered in the above comparison media of the trial body (the quantities in which are denoted by 0). Similarly, kinematic equations $\left(\varepsilon^0_{ij} = \partial u^0_i/\partial x_j + \partial u^0_j/\partial x_i\right)$ are valid to get the proper relation between the components of the strain tensor and the displacement vector. These conditions will be necessary in what follows. In this sense, the quantities with 0 are considered to be given.

In the second step a geometrically identical body is considered, which is anisotropic and nonhomogeneous. Displacements u_i, strains ε_{ij} and stresses σ_{ij} are unknown and the generalized Hooke's law including the eigenstresses λ_{ij} can be written as

$$\sigma_{ij} = L_{ijkl}\varepsilon_{kl} + \lambda_{ij}, \quad \lambda_{ij} = -L_{ijkl}\mu_{kl}, \quad \text{in } \Omega, \tag{3.3}$$

where μ_{kl} are the eigenstrains. The inverse Hooke's law holds in the form

$$\varepsilon_{ij} = M_{ijkl}\sigma_{kl} + \mu_{ij}, \quad \mu_{ij} = -M_{ijkl}\lambda_{kl}, \quad \text{in } \Omega. \tag{3.4}$$

Similarly to the classical Hashin-Shtrikman principles, define the symmetric stress polarization tensor τ_{ij} and the symmetric strain polarization tensor γ_{ij} by

$$\sigma_{ij} = L^0_{ijkl}\varepsilon_{kl} + \tau_{ij}, \tag{3.5}$$

$$\varepsilon_{ij} = M^0_{ijkl}\sigma_{kl} + \gamma_{ij}. \tag{3.6}$$

The definition of polarization tensors follows from a comparison of (3.3) and (3.5) (for $\boldsymbol{\tau}$) and from a comparison of (3.4) and (3.6) (for γ):

$$\tau_{ij} = [L_{ijkl}]\varepsilon_{kl} + \lambda_{ij} \tag{3.7}$$

and

$$\gamma_{ij} = [M_{ijkl}]\sigma_{kl} + \mu_{ij}, \tag{3.8}$$

where

$$[L_{ijkl}] = L_{ijkl} - L^0_{ijkl}, \quad [M_{ijkl}] = M_{ijkl} - M^0_{ijkl}.$$

Define also

$$u'_i = u_i - u^0_i, \quad \sigma'_{ij} = \sigma_{ij} - \sigma^0_{ij} \quad \text{in } \Omega, \tag{3.9}$$

and the kinematic equations

$$\varepsilon'_{ij} = \varepsilon_{ij} - \varepsilon^0_{ij} = \frac{\partial u_i}{\partial u_j} - \frac{\partial u_j}{\partial u_i} - \left(\frac{\partial u^0_i}{\partial u_j} - \frac{\partial u^0_j}{\partial u_i}\right). \tag{3.9a}$$

Let us introduce two assumptions:

Assumption A: the surface displacements $\bar{u}_i \in [H^{\frac{1}{2}}(\Gamma_u)]^3$ are prescribed along the entire boundary $\Gamma \equiv \Gamma_u$, and $u^0_i(\boldsymbol{\xi}) \equiv \bar{u}_i(\boldsymbol{\xi})$, $\boldsymbol{\xi} \in \Gamma$, hence $u'_i(\boldsymbol{\xi}) = \bar{u}_i(\boldsymbol{\xi}) - u^0_i(\boldsymbol{\xi}) = 0$, $\boldsymbol{\xi} \in \Gamma$.

Assumption B: the tractions $\bar{p}_i \in [L^2(\Gamma_p)]^3$ are given along the entire boundary $\Gamma \equiv \Gamma_p$, and $p^0_i(\boldsymbol{\xi}) \equiv \bar{p}_i(\boldsymbol{\xi})$, $\boldsymbol{\xi} \in \Gamma$, hence $p'_i(\boldsymbol{\xi}) = \sigma'_{ij}(\boldsymbol{\xi})n_j(\boldsymbol{\xi}) = \bar{p}_i(\boldsymbol{\xi}) - p^0_i(\boldsymbol{\xi}) = 0$, $\boldsymbol{\xi} \in \Gamma$.

Lemma 1. *In the case of Assumption* A *together with the equilibrium* (2.6) *and the kinematic equations, for the stress fields* σ_{ij}, σ^0_{ij}, σ'_{ij} *we have*

$$\int_\Omega \sigma_{ij}\varepsilon'_{ij}\, \mathrm{d}\Omega = \int_\Omega \sigma^0_{ij}\varepsilon'_{ij}\, \mathrm{d}\Omega = \int_\Omega \sigma'_{ij}\varepsilon'_{ij}\, \mathrm{d}\Omega = 0.$$

In the case of Assumption B *and assuming the validity of kinematic equations, for the stress fields* σ_{ij}, σ^0_{ij}, σ'_{ij} *obeying* (2.6) *we have*

$$\int_\Omega \sigma'_{ij}\varepsilon_{ij}\, \mathrm{d}\Omega = \int_\Omega \sigma'_{ij}\varepsilon^0_{ij}\, \mathrm{d}\Omega = \int_\Omega \sigma'_{ij}\varepsilon'_{ij}\, \mathrm{d}\Omega = 0.$$

P r o o f. For example,

$$\int_\Omega \sigma'_{ij}\varepsilon'_{ij}\,\mathrm{d}\Omega = \frac{1}{2}\int_\Omega \sigma'_{ij}\left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}\right)\mathrm{d}\Omega = \{\text{from Green's theorem}\} = 0$$

as either $u'_i = 0$, or $\sigma'_{ij}n_j = 0$ on Γ and σ'_{ij} fulfils (2.6). In the same manner one gets the other expressions in Lemma 1.

Lemma 2. *Denoting the first variation (Gateau's derivative in the direction of a small change of the vector by which we differentiate) by δ, we have*

$$\int_\Omega (\varepsilon'_{kl}\delta\tau_{kl} - \tau_{kl}\delta\varepsilon'_{kl})\,\mathrm{d}\Omega = 0.$$

P r o o f.

$$\begin{aligned}
&\int_\Omega (\varepsilon'_{kl}\delta\tau_{kl} - \tau_{kl}\delta\varepsilon'_{kl})\,\mathrm{d}\Omega \\
&= \int_\Omega (\varepsilon'_{kl}\delta\tau_{kl} - \tau_{kl}\delta\varepsilon'_{kl} + L^0_{ijkl}\varepsilon'_{ij}\delta\varepsilon'_{kl} - L^0_{ijkl}\varepsilon'_{ij}\delta\varepsilon'_{kl})\,\mathrm{d}\Omega \\
&\{\text{because of the symmetry } L^0_{ijkl} = L^0_{klij},\ (3.5) \text{ and } (3.1)\} \\
&= \int_\Omega (\varepsilon'_{kl}\delta\sigma'_{kl} - \sigma'_{kl}\delta\varepsilon'_{kl})\,\mathrm{d}\Omega.
\end{aligned}$$

□

The last integral is equal to zero by virtue of the next lemma.

Lemma 3. *For the second polarization tensor one has*

$$\int_\Omega (\sigma'_{ij}\delta\gamma_{ij} - \gamma_{ij}\delta\sigma'_{ij})\,\mathrm{d}\Omega = 0.$$

P r o o f.

$$\begin{aligned}
&\int_\Omega (\sigma'_{ij}\delta\gamma_{ij} - \gamma_{ij}\delta\sigma'_{ij})\,\mathrm{d}\Omega \\
&\qquad = \int_\Omega (\sigma'_{ij}\delta\gamma_{ij} - \gamma_{ij}\delta\sigma'_{ij} + M^0_{ijkl}\sigma'_{ij}\delta\sigma'_{kl} - M^0_{ijkl}\sigma'_{ij}\delta\sigma'_{kl})\,\mathrm{d}\Omega \\
&\qquad = \int_\Omega (\sigma'_{ij}\delta\varepsilon'_{ij} - \varepsilon'_{ij}\delta\sigma'_{ij})\,\mathrm{d}\Omega.
\end{aligned}$$

The last integral vanishes because of result of Lemma 1. □

3.2. Extended primary Hashin-Shtrikman variational principle

Let Assumption A be fulfilled. Subtracting (3.1) from (3.5) yields

$$\sigma'_{ij} = L^0_{ijkl}\varepsilon'_{kl} + \tau_{ij}. \tag{3.10}$$

Since both σ_{ij} and σ^0_{ij} are statically admissible, and u_i and u^0_i are kinematically admissible, taking into consideration (3.10) the following equations have to be satisfied in the sense of distributions:

$$\frac{\partial \sigma'_{ij}}{\partial x_j} = \frac{\partial (L^0_{ijkl}\varepsilon'_{kl} + \tau_{ij})}{\partial x_j} = 0 \quad \text{in } \Omega, \tag{3.11}$$

$$\tau_{ij} - [L_{ijkl}]\varepsilon_{kl} - \lambda_{ij} = 0 \quad \text{in } \Omega, \tag{3.12}$$

$$u'_i = 0 \quad \text{on } \Gamma, \tag{3.13}$$

where

$$[L_{ijkl}] = L_{ijkl} - L^0_{ijkl},$$

and (3.13) follows from Assumption A.

Formula (3.12) can be recast as

$$C_{ijkl}(\tau_{kl} - \lambda_{kl}) - \varepsilon_{ij} = 0, \tag{3.14}$$

where $[L_{ijrs}]C_{rskl} = I_{ijkl}$.

Theorem 3.1. *If* (3.11) *to* (3.13) *and* (3.9a) *are fulfilled, the following variational principle can be formulated: find the stationary point of the extended functional U defined as*

$$U(\tau_{ij}, \varepsilon'_{ij}) = U^0 - \frac{1}{2}\int_\Omega \{C_{ijkl}(\tau_{ij} - \lambda_{ij})(\tau_{kl} - \lambda_{kl}) - 2\tau_{ij}\varepsilon^0_{ij} - \varepsilon'_{ij}\,\tau_{ij} - M_{ijkl}\lambda_{ij}\lambda_{kl}\}\,\mathrm{d}\Omega. \tag{3.15}$$

In (3.15) *we have denoted*

$$U^0 = \frac{1}{2}\int_\Omega \sigma^0_{ij}\varepsilon^0_{ij}\,\mathrm{d}\Omega = \frac{1}{2}\int_\Omega L^0_{ijkl}\varepsilon^0_{ij}\varepsilon^0_{kl}\,\mathrm{d}\Omega.$$

P r o o f. The first variation of (3.15) with respect to the two independent fields τ_{ij} and ε'_{ij} yields

$$\begin{aligned}\delta U &= -\int_\Omega \left[C_{ijkl}(\tau_{ij} - \lambda_{ij})\delta\tau_{kl} - \varepsilon^0_{kl}\delta\tau_{kl} - \frac{1}{2}\tau_{kl}\delta\varepsilon'_{kl} - \frac{1}{2}\varepsilon'_{kl}\delta\tau_{kl}\right]\mathrm{d}\Omega \\ &= -\int_\Omega \left\{[C_{ijkl}(\tau_{ij} - \lambda_{ij}) - \varepsilon_{kl}]\delta\tau_{kl} + \frac{1}{2}[\varepsilon'_{kl}\delta\tau_{kl} - \tau_{kl}\delta\varepsilon'_{kl}]\right\}\mathrm{d}\Omega.\end{aligned}$$

From (3.14), the first term is zero and the second term vanishes because of Lemma 2. □

Theorem 3.2. *The functional U is equal to the actual potential strain energy stored in the anisotropic and heterogeneous body—see Fig.* 2.1*:*

$$U = \int_\Omega W \, \mathrm{d}\Omega,$$

where

$$W = \frac{1}{2} L_{ijkl}(\varepsilon_{ij} - \mu_{ij})(\varepsilon_{kl} - \mu_{kl}), \quad \mu_{ij} = -M_{ijkl}\,\lambda_{kl}.$$

P r o o f. Substituting λ_{ij} for μ_{ij}, and owing to (3.14), the integrand of (3.15) may be written as

$$\begin{gathered} \varepsilon^0_{ij} L^0_{ijkl} \varepsilon^0_{kl} - \varepsilon_{ij}(\tau_{ij} + L_{ijkl}\mu_{kl}) + 2\tau_{ij}\varepsilon^0_{ij} + \varepsilon'_{ij}\tau_{ij} + \mu_{ij} L_{ijkl}\mu_{kl} \\ \{\text{sum up all terms at } \boldsymbol{\tau}, \text{ and use the definition (3.12)}\} \\ = \varepsilon^0_{ij} L^0_{ijkl} \varepsilon^0_{kl} - \varepsilon_{ij} L_{ijkl}\mu_{kl} + [L_{ijkl}(\varepsilon_{ij} - \mu_{ij}) - L^0_{ijkl}\varepsilon^0_{ij} - L^0_{ijkl}\varepsilon'_{ij}]\varepsilon^0_{kl} + \mu_{ij} L_{ijkl}\mu_{kl}. \end{gathered}$$

On the other hand,

$$(\varepsilon_{ij} - \mu_{ij}) L_{ijkl}(\varepsilon_{kl} - \mu_{kl}) = \varepsilon_{ij} L_{ijkl}\varepsilon_{kl} - \varepsilon_{ij} L_{ijkl}\mu_{kl} - \mu_{ij} L_{ijkl}\varepsilon_{kl} + \mu_{ij} L_{ijkl}\mu_{kl}.$$

Comparing the right-hand sides of the last two relations, integrating the result, and taking into consideration Lemma 1, one arrives at

$$\int_\Omega \sigma_{ij}\varepsilon'_{ij} \, \mathrm{d}\Omega = \int_\Omega \sigma^0_{ij}\varepsilon'_{ij} \, \mathrm{d}\Omega = 0,$$

and one obtains the assertion of the theorem. □

Theorem 3.3. *Assuming the conditions of Theorem* 3.1*, the functional U in* (3.15) *attains its absolute maximum if $\mathbf{L}$ is positive definite, and it attains its absolute minimum if $\mathbf{C} + \mathbf{M}^0$ is negative semidefinite.*

P r o o f. The second variation of the functional U is

$$\delta^2 U = -\int_\Omega \{C_{ijkl}\delta\tau_{ij}\delta\tau_{kl} - \delta\varepsilon'_{ij}\delta\tau_{ij}\} \, \mathrm{d}\Omega. \tag{3.16}$$

Substituting from (3.10) to (3.16) for τ_{ij} only in the second term results in

$$\delta^2 U = -\int_\Omega \{C_{ijkl}\delta\tau_{ij}\delta\tau_{kl} - \delta\varepsilon'_{ij}(\delta\sigma'_{ij} - L^0_{ijkl}\delta\varepsilon'_{kl})\} \, \mathrm{d}\Omega.$$

The second term vanishes because of Lemma 1. Since L^0_{ijkl} is the tensor of elastic material constants, it has to be positive definite. Consequently, if the tensor C_{ijkl} is also positive definite, i.e. so is its inverse $[L_{ijkl}]$, the second variation of U is negative and in this case the maximum is attained (sufficient condition).

In order to prove the condition of minimum of the functional (3.15), let us consider the integral

$$I = \int_\Omega M^0_{ijkl} \delta\tau_{ij} \delta\tau_{kl} \, \mathrm{d}\Omega,$$

where M^0_{ijkl} is inverse to L^0_{ijkl}. Substituting from (3.10) for $\delta\tau_{ij}$ and $\delta\tau_{kl}$ yields

$$I = \int_\Omega \{ M^0_{ijkl} \delta\sigma'_{ij} \delta\sigma'_{kl} + L^0_{ijkl} \delta\varepsilon'_{ij} \delta\varepsilon'_{kl} - 2\delta\varepsilon'_{ij} \delta\sigma'_{ij} \} \, \mathrm{d}\Omega.$$

The last term vanishes because of Lemma 1. Since both the tensors M^0_{ijkl} and L^0_{ijkl} are positive definite, we conclude

$$\int_\Omega M^0_{ijkl} \delta\tau_{ij} \delta\tau_{kl} \, \mathrm{d}\Omega = \int_\Omega \{ M^0_{ijkl} \delta\sigma'_{ij} \delta\sigma'_{kl} + L^0_{ijkl} \delta\varepsilon'_{ij} \delta\varepsilon'_{kl} \} \, \mathrm{d}\Omega$$
$$\Rightarrow \int_\Omega M^0_{ijkl} \delta\tau_{ij} \delta\tau_{kl} \, \mathrm{d}\Omega > \int_\Omega L^0_{ijkl} \delta\varepsilon'_{ij} \delta\varepsilon'_{kl} \, \mathrm{d}\Omega.$$

Coming back to the second variation (3.16) we find out that a sufficient condition for the minimum of the functional U is: $\mathbf{C} + \mathbf{M}^0$ is negative semidefinite. $\square$

3.3. Extended dual Hashin-Shtrikman variational principle

In this section we extend the dual Hashin-Shtrikman variational principle to a body with prescribed surface tractions $\bar{p}_i$ and obeying Assumption B. Assume again that no body forces are present.

Following the classical dual Hashin-Shtrikman theorem, define the symmetric strain polarization tensor γ_{ij} by (3.6). Further, let the primed system be defined by (3.9).

Subtracting (3.2) from (3.6), we obtain

$$\varepsilon'_{ij} = \varepsilon_{ij} - \varepsilon^0_{ij} = M^0_{ijkl} \sigma'_{kl} + \gamma_{ij}. \tag{3.17}$$

Since both the fields ε_{ij} and ε^0_{ij} are kinematically admissible, and the stresses σ_{ij} and σ^0_{ij} are statically admissible, (3.11) still holds in the sense of distributions:

$$\frac{\partial \sigma'_{ij}}{\partial x_j} = 0 \quad \text{in } \Omega, \tag{3.18}$$

$$\gamma_{ij} - [M_{ijkl}]\sigma_{kl} - \mu_{ij} = 0 \quad \text{in } \Omega, \tag{3.19}$$

$$\sigma'_{ij} n_j = 0 \quad \text{on } \Gamma, \tag{3.20}$$

where

$$[M_{ijkl}] = M_{ijkl} - M^0_{ijkl},$$

and (3.20) follows from Assumption B.

The definition of the polarization tensor (3.19) can be rewritten as

$$D_{ijkl}(\gamma_{kl} - \mu_{kl}) - \sigma_{ij} = 0, \tag{3.21}$$

where $[M_{ijrs}]D_{rskl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) = I_{ijkl}$ is the fourth-order unit tensor.

Theorem 3.4. *If* (3.17) *to* (3.20) *are fulfilled, the dual extended H-S variational principle can be formulated, namely, the variation of the functional*

$$U^*(\gamma_{ij}, \sigma'_{ij}) = U^{0*} - \frac{1}{2}\int_\Omega \{D_{ijkl}(\gamma_{ij} - \mu_{ij})(\gamma_{kl} - \mu_{kl}) - 2\gamma_{ij}\sigma^0_{ij} - \sigma'_{ij}\gamma_{ij}\}\,\mathrm{d}\Omega \tag{3.22}$$

attains its stationary value with respect to the fields γ_{ij} *and* σ'_{ij}. *In* (3.22) *we have denoted*

$$U^{0*} = \frac{1}{2}\int_\Omega \sigma^0_{ij}\varepsilon^0_{ij}\,\mathrm{d}\Omega = \frac{1}{2}\int_\Omega M^0_{ijkl}\sigma^0_{ij}\sigma^0_{kl}\,\mathrm{d}\Omega.$$

P r o o f. The first variation of (3.22) leads to the following expression:

$$\begin{aligned}
\delta U^* &= -\int_\Omega \left[D_{ijkl}(\gamma_{ij} - \mu_{ij})\delta\gamma_{kl} - \sigma^0_{kl}\delta\gamma_{kl} - \frac{1}{2}\gamma_{ij}\delta\sigma'_{ij} - \frac{1}{2}\sigma'_{ij}\delta\gamma_{ij}\right]\mathrm{d}\Omega \\
&= -\int_\Omega \left\{[D_{ijkl}(\gamma_{ij} - \mu_{ij}) - \sigma_{ij}]\delta\gamma_{kl} + \frac{1}{2}[\sigma'_{ij}\delta\gamma_{ij} - \gamma_{ij}\delta\sigma'_{ij}]\right\}\mathrm{d}\Omega.
\end{aligned}$$

The last integral is zero. The first part of it vanishes because of the validity of (3.21) and the second part is zero according to Lemma 3. □

Theorem 3.5. *The functional* U^* *is equal to the actual complementary energy stored in the anisotropic and heterogeneous body, see Fig.* 2.1*:*

$$U^{s*} = \int_\Omega W^*\,\mathrm{d}\Omega,$$

where

$$W^* = \frac{1}{2}(M_{ijkl}\sigma_{ij}\sigma_{kl} + 2\sigma_{ij}\mu_{ij}).$$

P r o o f. By virtue of (3.21) and (3.9) the integrand of (3.22) may be written as

$$\begin{gathered}
M^0_{ijkl}\sigma^0_{ij}\sigma^0_{kl} + \sigma_{ij}\mu_{ij} + (\sigma'_{ij} + 2\sigma^0_{ij} - \sigma_{ij})\gamma_{ij} \\
\{\text{using (3.6) for removing } \gamma_{ij} \text{ and by virtue of (3.9)}\} \\
= \sigma_{ij}\mu_{ij} + \varepsilon_{ij}(\sigma_{ij} - \sigma'_{ij}).
\end{gathered}$$

Integration over the domain Ω, Lemma 1, and (2.1_2) result in the assertion of the theorem. $\square$

Theorem 3.6. *Assuming the conditions of Theorem 3.4, the functional U^* attains its absolute maximum if $[\mathbf{M}]$ is positive definite, and it attains its absolute minimum if $\mathbf{D}+\mathbf{L}^0$ is negative semidefinite.*

Proof. The second variation of U^* can be expressed as

$$\delta^2 U^* = -\int_\Omega \{D_{ijkl}\delta\gamma_{ij}\delta\gamma_{kl} - \delta\sigma'_{ij}\delta\gamma_{ij}\}\,\mathrm{d}\Gamma.$$

Substituting for γ_{ij} from (3.17) in the second term of the above integral yields

$$\delta^2 U = -\int_\Omega \{D_{ijkl}\delta\gamma_{ij}\delta\gamma_{kl} - \delta\sigma'_{ij}(\delta\varepsilon'_{ij} - M^0_{ijkl}\delta\sigma'_{kl})\}\,\mathrm{d}\Omega.$$

The second term disappears due to Lemma 1. Since M^0_{ijkl} is the tensor of elastic material constants, it has to be positive definite. Consequently, if the tensor C_{ijkl} is also positive definite, i.e. so is its inverse $[L_{ijkl}]$, the second variation of U is negative and in this case the maximum is attained.

In order to prove the condition of the minimum of the functional (3.22), consider the integral

$$I^* = \int_\Omega L^0_{ijkl}\delta\gamma_{ij}\delta\gamma_{kl}\,\mathrm{d}\Omega.$$

Substituting from (3.17) for $\delta\gamma_{ij}$ and $\delta\gamma_{kl}$, we get

$$I^* = \int_\Omega \{L^0_{ijkl}\delta\varepsilon'_{ij}\delta\varepsilon'_{kl} + M^0_{ijkl}\delta\sigma'_{ij}\delta\sigma'_{kl} - 2\delta\sigma'_{ij}\delta\varepsilon'_{ij}\}\,\mathrm{d}\Omega.$$

The last term disappears because of Lemma 1. Since both the tensors L^0_{ijkl} and M^0_{ijkl} are positively definite, we conclude

$$\begin{aligned}\int_\Omega L^0_{ijkl}\delta\gamma_{ij}\delta\gamma_{kl}\,\mathrm{d}\Omega &= \int_\Omega \{L^0_{ijkl}\delta\varepsilon'_{ij}\delta\varepsilon'_{kl} + M^0_{ijkl}\delta\sigma'_{ij}\delta\sigma'_{kl}\}\,\mathrm{d}\Omega\\ &\Rightarrow \int_\Omega L^0_{ijkl}\delta\gamma_{ij}\delta\gamma_{kl}\,\mathrm{d}\Omega > \int_\Omega M^0_{ijkl}\delta\sigma'_{ij}\delta\sigma'_{kl}\,\mathrm{d}\Omega.\end{aligned}$$

From the last inequality follows that a sufficient condition for minimum of the functional U^* is: $\mathbf{D}+\mathbf{L}^0$ is negatively definite. $\square$

4. Applications

Mechanically nonlinear behavior can be introduced using eigenparameters (plastic strain, relaxation stresses), and using these parameters visco-elastic or visco-plastic material can be described. In this way, time dependent problems or hereditary problems can be involved in the eigenparameters. Shape optimization of prestressed fibers can start with the above principles. From the H-S principles a very weak integral formulation directly follows, and the BEM is applicable to nonlinear and time dependent problems [13]. The unpleasant term involving hypersingular integral, which has to be integrated in the sense of Hadamard, can be avoided by Eshelby's trick.

A typical application of the above established extended principles is an introduction of the change of temperature instead of the eigenstrains. The bounds obtained here can be derived in a similar manner as the classical H-S bounds on material constants [8]. But, to derive bounds on the overall temperature characteristics requires very extensive calculation.

5. Conclusion

In this paper, classical Hashin-Shtrikman has been extended by the eigenparameters (eigenstrains or eigenstresses). These internal parameters can stand for a large range of quantities, which are studied in mechanics of solid media. Basically, a similar process to that published in [8] on how to calculate bounds on nonlinear or time-dependent characteristics describing mechanical properties can be applied. On the other hand, each such a problem requires a specific treatment and the solution is not trivial.

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