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NUMERICAL SOLUTION OF SEVERAL MODELS OF INTERNAL TRANSONIC FLOW*

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Abstract. The paper deals with numerical solution of internal flow problems. It mentions a long tradition of mathematical modeling of internal flow, especially transonic flow at our department. Several models of flow based on potential equation, Euler equations, Navier-Stokes and Reynolds averaged Navier-Stokes equations with proper closure are considered. Some mathematical and numerical properties of the model are mentioned and numerical results achieved by in-house developed methods are presented.

Keywords: transonic flow, mathematical models, numerical solution

MSC 2000: 76H05, 65M99, 35Q30, 35J25

1. Introduction

A fast progress in numerical solution of transonic flows in external and internal aerodynamics started by the first works published in 1971. Transonic flow (2D cascades) was solved in the Czechoslovak Republic firstly by the model of small disturbance potential equation [1]. Several extensions of this method were developed for the model of full potential equation. These were the first methods in Czechoslovakia which were used in practice for aerodynamical design of large axial compressors and steam turbines. Some flow regimes (eg. choked flow in turbine cascades with supersonic outlet flow) were successfully solved several years later by methods based on the model of Euler and Navier-Stokes equations.

Development of a proper scheme for discretization of convective terms is the main problem of numerical solution of laminar as well as turbulent viscous flows. We consider the same problem for all models, the flow through the plane cascade SE1050.

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It is a part of a large steam turbine of Škoda Pilsen, Energo and, together with experimental data of IT CAS, a test case of Qnet network.

2. Potential flow

The stationary isentropic irrotational flow of inviscid perfect gas can be described by one partial differential equation of the second order for a velocity potential Φ. It can be used also for transonic flow problems provided we consider only weak shock waves. The governing equation can be written in both the nonconservative

$$(1) \quad (a^2 - \Phi_x^2)\Phi_{xx} + 2\Phi_x\Phi_y\Phi_{xy} + (a^2 - \Phi_y^2)\Phi_{yy} = 0 \quad a^2 = a_0^2 - \frac{\kappa - 1}{2}(\Phi_x^2 + \Phi_y^2)$$

and the conservative form

$$(2) \qquad (\varrho\Phi_x)_x + (\varrho\Phi_y)_y = 0, \quad \varrho = \varrho_0\left[1 - \frac{\kappa - 1}{2a_0^2}(\Phi_x^2 + \Phi_y^2)\right]^{\frac{1}{\kappa-1}}.$$

We denote by Φ the velocity potential ($\vec{v} = (v_1, v_2) = \operatorname{grad}\Phi$), by ϱ the density, by a the local speed of sound, by κ the ratio of specific heats (Poisson's constant) and by v_1, v_2 the Cartesian components of velocity $\vec{v}$. The potential equation is the second order partial differential equation of mixed type. It is elliptic in the region of the subsonic flow ($v < a$), hyperbolic in the region of the supersonic flow ($v > a$) and parabolic along the sonic line ($v = a$). If subsonic inlet and outlet flows are considered, the problem of flow through a plane cascade can be formulated as a boundary value problem in the domain of one pitch of cascade. We assume a uniform inlet $\vec{v} = \vec{v}_{\text{in}}$ and outlet flow $\vec{v} = \vec{v}_{\text{out}}$ (and consequently, the position of these boundaries sufficiently far from the cascade profiles) and denote by γ the circulation of velocity around the profile P (γ is unknown in advance)

$$(3) \qquad \gamma = -\oint_{\partial P} v_1\,\mathrm{d}x + v_2\,\mathrm{d}y.$$

Its value characterizes the lift force in the case of isolated airfoil or the work output in the case of a (turbine) cascade of profiles. We split the boundary of the domain of solution Ω into an inlet part Γ_{in}, periodical boundaries Γ_{p_1}, Γ_{p_2}, walls of profiles Γ_w and an outlet part Γ_{out}. A non-homogeneous Dirichlet's boundary condition is prescribed on Γ_i:

$$(4) \qquad \Phi(x, y) = v_{2\text{in}}y \quad \forall\,[x, y] \in \Gamma_{\text{in}}.$$

The periodicity condition has the form

$$(5) \qquad \Phi(x, y + b) = \Phi(x, y) + K(v_{\text{in}}, b, \gamma) \quad \forall\,[x, y] \in \Gamma_{p_1}.$$

Then $[x, y+b]$ is a point of Γ_{p_2}, K is a constant and b denotes the distance of profiles in the cascade. A non-permeability condition

$$(6) \qquad (\operatorname{grad}\Phi, \vec{n}) = \frac{\partial\Phi}{\partial n} = 0 \quad \forall\,[x, y] \in \Gamma_w$$

is given on the walls of the profile with normal vector $\vec{n}$. A non-homogeneous Neumann condition

$$(7) \qquad (\operatorname{grad}\Phi, \vec{n}) = (\vec{v}_{\text{out}}, \vec{n}) \quad \forall\,[x, y] \in \Gamma_{\text{out}}$$

is prescribed on the outlet boundary Γ_{out} ($\vec{n}$ denotes again the normal vector). The constants in boundary conditions are not independent, only two of them can be prescribed [6], the other constants depend on the prescribed ones and the value of γ.

There are no complete results of existence or unicity of solution for transonic potential flow problems. This is connected with the mixed type of the governing differential equation and the possible existence of discontinuities in the velocity field. An analysis of this general problem was done e.g. in [8], [9], unfortunately some a posteriori considerations have to be taken into account.

The situation becomes much simpler, if only a subsonic flow is considered. Then the potential equation is elliptical in the whole domain of solution. Several cases of steady flow described by full or small perturbance potential equations in 2D and 3D can be formulated as weak boundary value problems with V-elliptic operators. The theory of monotone operators can be applied to prove existence and uniqueness of a solution of these problems [6]. We show it for our problem of flow through a plane cascade.

The weak problem is formulated in a suitable subset of the Sobolev space $W_1^2(\Omega)$. We define the space of periodical test functions V:

$$(8) \qquad V = \overline{M}^{W_2^1},$$

$$M = \{f \in C_\infty(\Omega),\ \ f(x, y) = f(x, y+b)\ \ \forall\,[x, y] \in \Gamma_{p_1},\ \ \operatorname{supp} f \cap \Gamma_{\text{in}} = \emptyset\}$$

with the norm

$$(9) \qquad \|u\|_V = \left(\iint_\Omega |\operatorname{grad} u|^2 \,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}$$

equivalent to the norm of W_2^1. Then we define a function $u_0 \in C^2(\Omega)$:

$$(10) \qquad \begin{aligned} u_0(x, y) &= v_{\text{in}} y && \forall\,[x, y] \in \Gamma_{\text{in}},\\ u_0(x, y) &= u_0(x, y+b) + K(v_{\text{in}}, b, \gamma) && \forall\,[x, y] \in \Gamma_{p_1},\\ \operatorname{grad} u_0(x, y) &= \operatorname{grad} u_0(x, y+b) && \forall\,[x, y] \in \Gamma_{p_1},\\ (\operatorname{grad} u_0(x, y), \vec{n}) &= 0 && \forall\,[x, y] \in \Gamma_w. \end{aligned}$$

Multiplying (2) by an arbitrary test function $f \in V$, integrating over Ω and using integration by parts we introduce

Definition. Let a proper set of constants $\vec{v}_{\rm in}$, $\vec{v}_{\rm out}$, γ be given. Let (10) hold for a function $u_0 \in C^2(\Omega)$. Let operators $\mathcal{T}\colon V \to V^*$ and $\mathcal{F} \in V^*$ be given by

$$\begin{aligned}\langle \mathcal{T}u, f\rangle &= \iint_\Omega \varrho(|\mathrm{grad}(u+u_0)|^2)\,\mathrm{grad}(u+u_0)\,\mathrm{grad}\, f\,\mathrm{d}\Omega,\\ \langle \mathcal{F}, f\rangle &= \int_{\Gamma_{\rm out}} \varrho(|\vec{v}_{\rm out}|^2)\vec{n}f\,\mathrm{d}s.\end{aligned} \tag{11}$$

Then $u \in V$ is a weak solution of the problem if

$$\langle \tilde{\mathcal{T}}u, f\rangle = \langle \mathcal{T}u - \mathcal{F}, f\rangle = 0, \tag{12}$$

holds for all $f \in V$.

Theorem. *There exists a unique solution of problem* (12) *formulated in the above definition.*

To prove this theorem, we show that the operator $\mathcal{T}$ and consequently $\tilde{\mathcal{T}}$ have such properties that Browder's fix point theorem can be applied for (12).

We formally prolongate the density function $\varrho \in C^2(\mathbb{R})$. (2) holds for subsonic velocities and for supersonic speeds $|\vec{v}| > a$ we define ϱ equal to a suitable constant.

Lemma 1.

- *$\varrho = \varrho(z)$, $\mathrm{d}\varrho(z)/\mathrm{d}z$, $w\mathrm{d}\varrho(w^2)/\mathrm{d}w$, where $z = w^2 = |\vec{v}|^2$ are monotone functions;*
- *$z\,\mathrm{d}\varrho(z)/\mathrm{d}z$, $2z\,\mathrm{d}\varrho/\mathrm{d}z + \varrho(z)$, where $z = |\vec{v}|^2$ are monotone and bounded functions;*
- *the Gâteaux derivative of $\mathcal{T}$ exists and*

$$\begin{aligned}D\mathcal{T}(u)(v,w) = \iint_\Omega \Big(& 2\frac{\mathrm{d}\varrho(z)}{\mathrm{d}z}\mathrm{grad}^2(u+u_0)\,\mathrm{grad}\, w\,\mathrm{grad}\, v\\ &+ \varrho(z)\,\mathrm{grad}\, w\,\mathrm{grad}\, v\Big)\,\mathrm{d}x\,\mathrm{d}y,\end{aligned} \tag{13}$$

where $z = |\mathrm{grad}(u+u_0)|^2$ holds for all $u, v, w \in V$.

Lemma 2. *The operator $\mathcal{T}$ is strongly monotone, i.e. there exists an increasing and continuous function $c\colon \langle 0,\infty) \to \langle 0,\infty)$, $c(0) = 0$, $\lim\limits_{r\to\infty} c(r) = \infty$ such that*

$$|\langle \mathcal{T}u - \mathcal{T}v, u - v\rangle \geqslant c(\|u-v\|)\|u-v\| \ \ \forall\, u, v \in V. \tag{14}$$

P r o o f. Let $u, v \in V$, let us denote $h = u - v$, $z = \operatorname{grad}(u + \Theta h + U_0)$. The mean value theorem applied to the left-hand side of (14) gives

$$
\begin{aligned}
(15)\qquad |\langle \mathcal{T}u - \mathcal{T}v, u - v\rangle| &= \iint_\Omega 2\varrho'(|z|^2)(z \operatorname{grad} h)^2 + \varrho(|z|^2)\operatorname{grad}^2 h \\
&\geqslant \iint_\Omega |2\varrho'(|z|^2)z^2 + \varrho(z)|\operatorname{grad}^2 h \\
&\geqslant C_1\|h\|_V^2 \geqslant C_2\|h\|^2_{W_2^1}.
\end{aligned}
$$

The inequalities hold due to the properties of arguments listed in Lemma 1. The function $c(r)$ in (14) can be defined as $c(r) = C_2 r$.

Lemma 3. *The operator $\mathcal{T}$ is continuous on any finite dimensional subspace $V_k \subset V$, i.e. $\forall \{u_n\}_{n=1}^\infty$, $u_n \to u$, $u, u_n \in V_k$ the relation*

$$
(16)\qquad \langle \mathcal{T}u_n, v\rangle \to \langle \mathcal{T}u, v\rangle
$$

holds $\forall v \in V$.

P r o o f. Let $u, v \in V$. We denote $h = u - v$, $w = v + \Theta(u - v)$, $z = \operatorname{grad}(w + u_0)$. The mean value theorem gives

$$
\begin{aligned}
(17)\quad \langle \mathcal{T}u - \mathcal{T}v, f\rangle &= D\mathcal{T}(u + \Theta(u - v))(f, v - u) \\
&= \iint_\Omega 2\frac{\mathrm{d}\varrho}{\mathrm{d}z}(|z|^2)(\operatorname{grad}(w + u_0)\operatorname{grad} h)(\operatorname{grad} f \operatorname{grad}(w + u_0))\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \iint_\Omega \varrho(|z|^2)\operatorname{grad} h \operatorname{grad} f\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant C_3\|f\|_V\|h\|_V \leqslant C_4\|f\|_{W_2^1}\|h\|_{W_2^1}.
\end{aligned}
$$

We have used again the properties listed in Lemma 1. Let now V_k be an arbitrary finite dimensional subset of V, $\{u_n\}_{n=1}^\infty$, $u_n \to u$, $u, u_n \in V_k$. Then the inequality (17) yields

$$
(18)\quad |\langle \mathcal{T}u_n, f\rangle - \langle \mathcal{T}u, f\rangle| = \langle \mathcal{T}u_n - \mathcal{T}u, f\rangle| \leqslant K\|u - u_n\|_{W_2^1}\|f\|_{W_2^1} \to 0 \ \ \forall f \in V.
$$

The operator $\mathcal{T}$ is strongly monotone in V and continuous in any finite dimensional subspace of V. The existence and unicity of the weak solution is then a consequence of the following version of Browder's fixed point theorem [7]:

Browder's theorem. *Let B be a reflexive Banach space, let $\mathcal{T}\colon B \to B^*$ be strongly monotone and continuous in each finite dimensional subspace of B. Then there exists a unique $u \in B$ such that*

$$
(19)\qquad \langle Tu, v\rangle = 0 \quad \forall v \in B.
$$

The problem of the potential transonic flow through the cascade was solved numerically by a finite difference method on a curvilinear body fitted grid.

The type of equation is tested at each grid point and the so called Jameson's rotated scheme, which combines upwind and central difference approximations of derivatives in dependence on the type of the equation and the mutual position of the streamline and grid coordinates is implemented. The system of equations is solved by the block relaxation method SLOR.

Convergence of the iterative method is very slow and was later accelerated by multi-grid methods (FAS or CS scheme).

We present a comparison of the computed result with the experiments of Institute of Thermomechanics, CAS for SE1050 cascade on Fig. 1.

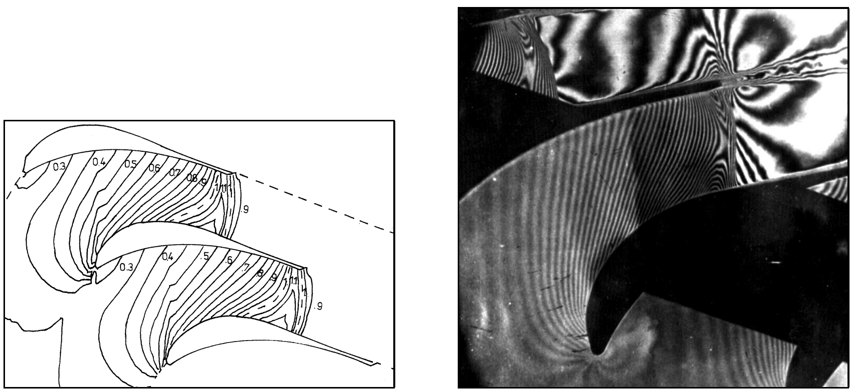

Figure 1. Turbine cascade SE1050. Comparison of computed results (potential model) with experiment IT CAS.

3. Euler equations

The 2D system of Euler equations can be written in the conservative vector form:

$$\mathbf{W}_t + \mathbf{F}_x + \mathbf{G}_y = 0, \tag{20}$$

where $\mathbf{W} = [\varrho, \varrho v_1, \varrho v_2, e]^T$ are conservative variables and $\mathbf{F} = [\varrho v_1, \varrho v_1^2 + p, \varrho v_1 v_2, v_1(e+p)]^T$, $\mathbf{G} = [\varrho v_2, \varrho v_1 v_2, \varrho v_2^2 + p, v_2(e+p)]^T$ are physical fluxes (convective). The system (20) is a hyperbolic system of equations independently of the ratio of the velocity and the speed of sound.

The weak solution from a proper functional set satisfies the integral form of (20).

A subsonic Mach number of normal velocity component is considered on both the inlet and outlet boundaries. Then we prescribe three conditions on the inlet and one