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ON THE MOTION OF RIGID BODIES IN A VISCOUS FLUID*

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Abstract. We consider the problem of motion of several rigid bodies in a viscous fluid. Both compressible and incompressible fluids are studied. In both cases, the existence of globally defined weak solutions is established regardless possible collisions of two or more rigid objects.

Keywords: rigid body, compressible fluid, incompressible fluid, global existence

MSC 2000: 35Q30, 35Q35

1. Classical formulation

1.1 Rigid bodies and motion.

A *rigid body* is a connected compact subset $\overline{S}$ of the Euclidean space $\mathbb{R}^N$. A *motion* is a mapping $\eta\colon\ I \times \mathbb{R}^N \mapsto \mathbb{R}^N$ such that

$$\eta(t,\cdot)\colon\ \mathbb{R}^N \mapsto \mathbb{R}^N \ \text{ is an isometry}$$

for any t belonging to a *time interval* $I \subset \mathbb{R}$. We will consider only motions which are absolutely continuous with respect to time, i.e., their time derivative exists for a.a. $t \in I$.

The mass distribution in the body is characterized by the *mass density* ϱ^S, its *total mass* is given as

$$m = \int_{\overline{S}} \varrho^S \,\mathrm{d}\boldsymbol{x}.$$

To describe the motion, we adopt the Eulerian (spatial) description where the coordinate system is attached to the region of the Euclidean space currently occupied by the body. The *place* $\boldsymbol{x}$ and the *time* t are taken as independent variables.

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1.2. Kinematics, equations of motion.

The mappings $\eta(t,\cdot)\colon \mathbb{R}^N \mapsto \mathbb{R}^N$ are isometries and, consequently, can be written in the form

$$\eta(t,\boldsymbol{x}) = \boldsymbol{X}_g(t) + \mathbb{O}(t)(\boldsymbol{x} - \boldsymbol{X}_g(0))$$

where $\boldsymbol{X}_g(t)$ is the position of the center of mass at a time $t \in I$ and $\mathbb{O}(t)$ is a matrix such that $\mathbb{O}^T(t)\mathbb{O}(t) = \mathbb{I}$. Since the motion is absolutely continuous, we can define

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}_g(t) = \boldsymbol{U}_g(t)\text{—the } \textit{translation velocity},$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{O}(t)\mathbb{O}^T(t) = \mathbb{Q}(t)\text{—the } \textit{angular velocity}$$

of the body $\overline{S}$.

The solid velocity in the Eulerian description reads

$$\boldsymbol{u}^S(t,\boldsymbol{x}) = \frac{\partial\eta}{\partial t}(t,\eta^{-1}(t,\boldsymbol{x})) = \boldsymbol{U}_g(t) + \mathbb{Q}(t)(\boldsymbol{x} - \boldsymbol{X}_g(t)),\ \ \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}_g(t) = \boldsymbol{U}_g(t).$$

The total force $\boldsymbol{F}^S$ acting on $\overline{S}$ consists of the body force and the contact force, specifically,

$$\boldsymbol{F}^S(t) = \int_{\partial\overline{S}(t)} \mathbb{T}\boldsymbol{n}\,\mathrm{d}\sigma + \int_{\overline{S}(t)} \varrho^S \boldsymbol{g}^S\,\mathrm{d}\boldsymbol{x}$$

where $\mathbb{T}$ denotes the Cauchy stress tensor, $\boldsymbol{n}$ is the unit outward normal, $\boldsymbol{g}^S$ denotes the specific body force, and

$$\overline{S}(t) = \eta(t,\overline{S}).$$

In accordance with Newton's second law, we get

$$m\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{U}_g(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\overline{S}(t)} \varrho^S\boldsymbol{u}^S\,\mathrm{d}\boldsymbol{x} = \int_{\partial\overline{S}(t)} \mathbb{T}\boldsymbol{n}\,\mathrm{d}\sigma + \int_{\overline{S}(t)} \varrho^S\boldsymbol{g}^S\,\mathrm{d}\boldsymbol{x}. \tag{1.1}$$

As the angular velocity $\mathbb{Q}$ is skew symmetric, there exists a vector ω such that

$$\mathbb{Q}(t)(\boldsymbol{x} - \boldsymbol{X}_g) = \omega(t) \times (\boldsymbol{x} - \boldsymbol{X}_g).$$

Accordingly, the balance of moment of momentum reads

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{J}\omega) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\overline{S}(t)} \varrho^S(\boldsymbol{x} - \boldsymbol{X}_g) \times \boldsymbol{u}^S\,\mathrm{d}\boldsymbol{x} \\ &= \int_{\partial\overline{S}(t)} (\boldsymbol{x} - \boldsymbol{X}_g) \times \mathbb{T}\boldsymbol{n}\,\mathrm{d}\sigma + \int_{\overline{S}(t)} \varrho^S(\boldsymbol{x} - \boldsymbol{X}_g) \times \boldsymbol{g}^S\,\mathrm{d}\boldsymbol{x},\end{aligned} \tag{1.2}$$

where $\mathbb{J}$ is the *inertial tensor* defined through

$$\mathbb{J}\boldsymbol{a}\cdot\boldsymbol{b} = \int_{\overline{S}(t)} \varrho^S (\boldsymbol{a}\times(\boldsymbol{x}-\boldsymbol{X}_g))\cdot(\boldsymbol{b}\times(\boldsymbol{x}-\boldsymbol{X}_g))\,\mathrm{d}\boldsymbol{x}.$$

The equations (1.1), (1.2) determine completely the motion of the rigid body $\overline{S}$.

1.3. The fluid motion.

The state of the fluid will be determined by the *density* ϱ^f and the *velocity* $\boldsymbol{u}^f$ satisfying the standard Navier-Stokes system

$$\partial_t \varrho^f + \operatorname{div}(\varrho^f \boldsymbol{u}^f) = 0, \tag{1.3}$$

$$\partial_t(\varrho^f \boldsymbol{u}^f) + \operatorname{div}(\varrho^f \boldsymbol{u}^f \otimes \boldsymbol{u}^f) + \nabla p = \operatorname{div}\mathbb{S} + \varrho^f \boldsymbol{g}^f, \tag{1.4}$$

where p is the *pressure*, $\boldsymbol{g}^f$ is the specific body force, and $\mathbb{S}$ is the viscous stress tensor. The total stress $\mathbb{T}$ is determined through Stokes' law

$$\mathbb{T} = \mathbb{S} - p\mathbb{I}.$$

The *compressible* fluids considered in this paper will be in a barotropic regime where the pressure p is uniquely determined by the density ϱ^f,

$$p = p(\varrho^f).$$

More specifically, we restrict ourselves to the isentropic constitutive relation where

$$p = a(\varrho^f)^\gamma, \ \ a > 0, \ \ \gamma > 1. \tag{1.5}$$

1.4. The boundary conditions.

As we focus on *viscous* fluids, we adopt the hypothesis of complete adherence of the fluid to the boundaries of rigid objects (the no-slip boundary conditions). This means

$$\boldsymbol{u}^f(t) = \boldsymbol{u}^S(t) \ \text{ on } \partial\overline{S}(t), \ \ t \in I, \ \ \boldsymbol{u}^f = 0 \ \text{ on } \partial\Omega \tag{1.6}$$

provided the fluid is contained in a fixed spatial domain $\Omega \subset \mathbb{R}^N$.

Under the hypothesis of continuity of the stresses, the equations (1.1)–(1.4) together with (1.6) form a coupled system governing the motion of a rigid body (which will be extended to the case of several bodies in the next section) in a viscous fluid contained in a fixed spatial domain $\Omega \subset \mathbb{R}^N$.

1.5. Constitutive laws.

It remains to establish a relation between the stress tensor $\mathbb{S}$ and the motion of the fluid. The most common class of fluids are linearly viscous or *Newtonian fluids* where $\mathbb{S}$ is a linear function of the symmetric velocity gradient:

$$\mathbb{S} = 2\mu\mathbb{D}(\boldsymbol{u}) + \lambda\mathbb{I}\,\mathrm{trace}(\mathbb{D}(\boldsymbol{u})) \ \text{ where } \mathbb{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T). \tag{1.7}$$

Here μ and λ are viscosity coefficients satisfying

$$\mu > 0, \quad \lambda + \mu \geqslant 0.$$

2. Variational formulation

2.1. The motion of rigid bodies.

The family of rigid bodies will be represented by sets $\overline{S}^i$, $i = 1, \ldots, m$,

$$\left\{ \begin{array}{c} \overline{S}^i \ \text{ compact, connected subsets of } \mathbb{R}^N, \\ S^i = \mathrm{int}(\overline{S}^i) \neq \emptyset, \\ |\overline{S}^i \setminus S^i| = |\partial \overline{S}^i| = 0. \end{array} \right\} \tag{2.1}$$

The motions $\eta^i(t,\cdot)\colon \mathbb{R}^N \mapsto \mathbb{R}^N$ are (affine) isometries for any t and absolutely continuous as functions of $t \in I$. We define $\overline{S}^i(t) = \eta^i(t, \overline{S}^i)$ and the region occupied by the solids

$$\overline{Q}^s = \left\{ (t, \boldsymbol{x}) \mid t \in I, \ \boldsymbol{x} \in \bigcup_{i=1}^m \overline{S}^i(t) \right\}.$$

2.2. The balance of mass.

The solid densities ϱ^{S^i}, $i = 1, \ldots, m$ as well as the fluid density ϱ^f can be considered as functions defined on the whole space $\mathbb{R}^3$ extended to be zero outside $\overline{S}^i(t)$ and the fluid region

$$Q^f = (I \times \Omega) \setminus \overline{Q}^s,$$

respectively. We set $\varrho = \varrho^f + \sum_{i=1}^m \varrho^{S^i}$.

Similarly, we introduce the velocity

$$\boldsymbol{u}(t, \boldsymbol{x}) = \begin{cases} \boldsymbol{u}^{S^i} & \text{for } t \in I, \ \boldsymbol{x} \in \overline{S}^i(t), \\ \boldsymbol{u}^f & \text{for } t \in I, \ \boldsymbol{x} \in \Omega \setminus \bigcup_{i=1}^m \overline{S}^i(t), \\ 0 & \text{for } t \in I, \ \boldsymbol{x} \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Since the normal component of the velocity is continuous on the solid-liquid boundary, the functions ϱ and $\boldsymbol{u}$ satisfy the continuity equation

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0 \text{ in } \mathcal{D}'(I \times \mathbb{R}^N). \tag{2.2}$$

2.3. The equations of motion.

In addition to the moving bodies $\overline{S}^i$, $i = 1, \ldots, m$, it seems convenient to define a "body at rest"

$$\overline{S}^0 = \mathbb{R}^N \setminus \Omega \quad \text{with } \eta^0(t, \boldsymbol{x}) = \boldsymbol{x} \text{ for all } t \in I, \ \boldsymbol{x} \in \mathbb{R}^N.$$

Given the motions η^j, $j = 0, \ldots, m$, the set of admissible velocities at a time $t \in I$ is defined as

$$\begin{aligned} \mathcal{R}_\delta(t) = \{\Phi \in C^1(\mathbb{R}^N) \mid \Phi = 0 \text{ on the } \delta\text{-neighbourhood of } \overline{S}^0, \\ \mathbb{D}(\Phi) = 0 \text{ on the } \delta\text{-neighbourhood of } \overline{S}^i(t), \ i = 1, \ldots, m\}, \ \delta > 0. \end{aligned}$$

The equations (1.1), (1.2), and (1.4) admit a very elegant variational formulation requiring the integral identity

$$\int_I \int_{\mathbb{R}^N} \varrho \boldsymbol{u} \cdot \partial_t \varphi + (\varrho \boldsymbol{u} \otimes \boldsymbol{u}) \colon \mathbb{D}(\varphi) + p \operatorname{div} \varphi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t = \int_I \int_{\mathbb{R}^N} \mathbb{S} \colon \mathbb{D}(\varphi) - \varrho \boldsymbol{g} \cdot \varphi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \tag{2.3}$$

to hold for any test function $\varphi \in \mathcal{D}(I \times \mathbb{R}^N)$,

$$\varphi(t) \in \mathcal{R}_\delta(t), \ t \in I \text{ for a certain } \delta > 0.$$

The fact that the space of admissible test functions depends on the motion of the rigid objects, i.e., on the velocity $\boldsymbol{u}$, is considered to be the main shortcoming of the present approach. Another possibility is to attach the reference frame to one of the bodies similarly as in Gunzburger et al. [10]. Note, however, that this formulation is more convenient only in the case there is only one rigid object contained in $\Omega = \mathbb{R}^N$.

Remark 2.1. It might seem that the choice of the space $\mathcal{R}_\delta$ restricts considerably the family of admissible test functions. Gunzburger et al. [10] use a "larger" space

$$\begin{aligned} \mathcal{R}(t) = \{\Phi \mid \Phi \text{ Lipschitz continuous on } \mathbb{R}^N, \ \Phi = 0 \text{ on } \overline{S}^0, \\ \mathbb{D}(\Phi) = 0 \text{ on } \overline{S}^i(t), \ i = 1, \ldots, m\}. \end{aligned}$$

However, the following result holds.

Lemma 2.1. *Let the sets* $\overline{S}^j$, $j = 0, \ldots, m$ *be regular in the sense that for any* $\boldsymbol{x} \in \partial\overline{S}^j$ *there exists a closed ball* $B \subset \overline{S}^j$ *such that* $\boldsymbol{x} \in B$.

Then the set $\bigcup_{\delta>0} \mathcal{R}_\delta(t)$ *is dense in* $\mathcal{R}(t)$ *with respect to the* $W^{1,q}(\mathbb{R}^N)$*-topology for any finite* $q \geqslant 1$.

Proof. (i) Clearly, it is enough to show that for any $\Phi \in \mathcal{R}(t)$ there is a sequence Φ_n of Lipschitz continuous functions such that

$$\Phi_n = 0 \text{ on a neighbourhood of } \overline{S}^0, \ \mathbb{D}(\Phi) = 0 \text{ on a neighbourhood of } \overline{S}^i,$$
$$i = 1, \ldots, m;$$
$$\Phi_n \to \Phi \text{ in } W^{1,q}(\mathbb{R}^N) \text{ as } n \to \infty.$$

(ii) To begin with, we observe the following property of functions in $\mathcal{R}(t)$: Assume $\Phi \in \mathcal{R}(t)$ and $\overline{S}^i(t) \cap \overline{S}^j(t) \neq \emptyset$. Then there exists a rigid velocity field $\boldsymbol{v}^{i,j}$ such that $\Phi = \boldsymbol{v}^{i,j}$ on $\overline{S}^i(t) \cup \overline{S}^j(t)$. Indeed, performing an affine transformation if necessary we can assume there are two closed balls $B_r([r,0])$, $B_r([-r,0])$ of radius r centered at $[r,0]$, $[-r,0]$ respectively such that

$$\Phi|_{B_r([r,0])} = \Phi|_{\overline{S}^i(t)} = \boldsymbol{P}^+ + \mathbb{Q}^+\boldsymbol{x}, \ \ \Phi|_{B_r([-r,0])} = \Phi|_{\overline{S}^j(t)} = \boldsymbol{P}^- + \mathbb{Q}^-\boldsymbol{x}$$

where we have denoted $\boldsymbol{x} = [y, \boldsymbol{z}]$. Since $B_r([-r,0]) \cap B_r([r,0]) = 0$ and Φ is continuous, we get $\boldsymbol{P}^+ = \boldsymbol{P}^-$.

On the other hand since Φ is Lipschitz continuous, one has

$$(\mathbb{Q}^+ - \mathbb{Q}^-)[0, \boldsymbol{z}] = o(|z|)$$

and, consequently,

$$\mathbb{Q}^+[0, \boldsymbol{z}] = \mathbb{Q}^-[0, \boldsymbol{z}] \text{ for all } \boldsymbol{z} \in \mathbb{R}^{N-1}.$$

Finally, as $\mathbb{Q}^+$, $\mathbb{Q}^-$ are skew symmetric, we have

$$(\mathbb{Q}^+ - \mathbb{Q}^-)^2 = 0,$$

which yields the desired conclusion $\mathbb{Q}^+ = \mathbb{Q}^-$. Consequently, it is enough to prove the conclusion of Lemma 2.1 in the situation when

$$\operatorname{dist}(\overline{S}^i(t), \overline{S}^j(t)) > 0 \quad \text{for all } i \neq j.$$

(iii) We define

$$d^j(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, \overline{S}^j(t)), \quad j = 0, \ldots, m$$

together with

$$D(\boldsymbol{x}) = \operatorname{dist}\Big(\boldsymbol{x}, \bigcup_{j=0}^{m} \overline{S}^j(t)\Big).$$

Moreover, let $h_n \in C^\infty(\mathbb{R})$ be a sequence of functions such that

$$h_n(z) = 0 \text{ for } z \leqslant \frac{1}{n}, \quad h_n(z) = 1 \text{ for } z \geqslant \frac{2}{n}, \quad |h_n'(z)| \leqslant 2n \text{ for all } z \in \mathbb{R}.$$

Now, let $\Phi \in \mathcal{R}(t)$, i.e., there exist rigid velocities $\boldsymbol{v}^{S^0} = 0$, $\boldsymbol{v}^{S^i}$, $i = 1, \dots, m$ such that

$$\Phi|_{\overline{S}^j(t)} = \boldsymbol{v}^{S^j} \text{ for } j = 0, \dots, m.$$

The approximate sequence Φ_n will be defined as

$$\Phi_n(\boldsymbol{x}) = h_n(D(\boldsymbol{x}))\Phi(\boldsymbol{x}) + \sum_{j=0}^{m}(1 - h_n(d^j(\boldsymbol{x}))\boldsymbol{v}^{S^j}(\boldsymbol{x}).$$

As there is no contact of two rigid bodies, one checks easily that $\Phi_n \in \mathcal{R}_{\delta_n}(t)$ for a suitably chosen $\delta_n > 0$ provided n is large enough. Moreover, it is not difficult to see that

$$\Phi_n \to \Phi \text{ in } W^{1,q}(\mathbb{R}^N)$$

provided we show

$$\chi_n = h_n'(D)\nabla D \cdot \Phi - \sum_{j=0}^{m} h_n'(d^j)\nabla d^j \cdot \boldsymbol{v}^{S^j} \to 0 \text{ in } L^q(\mathbb{R}^N) \text{ for } n \to \infty.$$

To this end, one observes that $\chi_n = 0$ on $\bigcup_{j=0}^{m} \overline{S}^j(t)$ for all n large enough. Next, for any compact $K \subset \mathbb{R}^N \setminus \bigcup_{j=1}^{m} \overline{S}^j$ there exists $n = n(K)$ such that $\chi_n|_K = 0$ for all $n \geqslant n(K)$.

Consequently, for n large enough and any $\boldsymbol{x} \in \mathbb{R}^N$ we have either $\chi_n(\boldsymbol{x}) = 0$ or

$$\chi_n(\boldsymbol{x}) = h_n'(d^j)\nabla d^j \cdot (\Phi - \boldsymbol{v}^{S^j}) \text{ for a certain } j.$$

Since Φ is Lipschitz continuous and $\Phi|_{\overline{S}^j(t)} = \boldsymbol{v}^{S^j}$, we get

$$|(\Phi - \boldsymbol{v}^{S^j})(\boldsymbol{x})| \leqslant L d^j(\boldsymbol{x}) \quad \text{for any } \boldsymbol{x} \in \mathbb{R}^N.$$

On the other hand $|\nabla d^j| \leqslant 1$ and $h_n'(d^j)d^j$ is bounded independently of n, which yields the desired conclusion $\chi_n \to 0$ in $L^q(\Omega)$ by means of the Lebesgue theorem.

□

2.4. Compatibility of the velocity with rigid motions.

To close the problem, we have to establish the relation between the velocity $\boldsymbol{u}$ and the motions η^i, $i = 1, \dots, m$.

The isometries are determined through

$$\eta^i(t, \boldsymbol{x}) = \boldsymbol{X}^i(t) + \mathbb{O}^i(t)\boldsymbol{x}, \quad i = 1, \dots, m,$$

where $\boldsymbol{X}^i$ and $\mathbb{O}^i$ are absolutely continuous functions of $t \in I$. We will say that $\boldsymbol{u}$, η^j, $j = 0, \dots, m$ are compatible if

$$\boldsymbol{u}(t, \boldsymbol{x}) = \boldsymbol{u}^{S^i}(t, \boldsymbol{x}) = \boldsymbol{U}^i(t) + \mathbb{Q}^i(t)(\boldsymbol{x} - \boldsymbol{X}^i(t)) \ \text{ for } \ \boldsymbol{x} \in \overline{S}^i(t), \ \ i = 1, \dots, m \tag{2.4}$$

for a.a. $t \in I$, where

$$\frac{\mathrm{d}}{\mathrm{dt}}\boldsymbol{X}^i = \boldsymbol{U}^i, \quad \left(\frac{\mathrm{d}}{\mathrm{dt}}\mathbb{O}^i\right)(\mathbb{O}^i)^T = \mathbb{Q}^i \ \text{ a.a. on } \ I.$$

The identity (2.4) is to be understood in the sense that there exists a sequence $\Phi_n \in \mathcal{D}(\mathbb{R}^N)$ such that

$$\Phi_n = \boldsymbol{u}^{S^i} \ \text{ on an open neighbourhood of } \ \overline{S}^i(t), \ \ \Phi_n \to \boldsymbol{u}(t) \ \text{ in } \ W^{1,2}(\mathbb{R}^N). \tag{2.5}$$

Similarly, we require

$$\boldsymbol{u}(t, \boldsymbol{x}) = \boldsymbol{u}^{S^0}(t, \boldsymbol{x}) = 0 \ \text{ for } \ \boldsymbol{x} \in \overline{S}^0 \tag{2.6}$$

in the sense of (2.5) for a.a. $t \in I$.

If Ω is a bounded domain, condition (2.6) is nothing else but $\boldsymbol{u}(t) \in W_0^{1,2}(\Omega)$ for a.a. $t \in I$.

2.5. Renormalized solutions.

DiPerna and Lions [3] developed a theory of variational solutions of the continuity equation (2.2) inspired by Kruzhkov's entropy solutions to nonlinear conservation laws. Multiplying (formally) (2.2) by $b'(\varrho)$ where b is a continuously differentiable function we obtain

$$\partial_t b(\varrho) + \operatorname{div}(b(\varrho)\boldsymbol{u}) + \big(b'(\varrho)\varrho - b(\varrho)\big) \operatorname{div} \boldsymbol{u} = 0. \tag{2.7}$$

The class of admissible functions $b\colon \ \mathbb{R} \to \mathbb{R}$ is determined by the integrability of ϱ. For practical purposes, it is enough to consider b such that

$$b \in C^1(\mathbb{R}), \ \ b'(\varrho) = 0 \ \text{ provided } \ \varrho \ \text{ is large enough, i.e., } \ \varrho > \varrho_b. \tag{2.8}$$

The equation (2.7) is to be satisfied in $\mathcal{D}'(I \times \mathbb{R}^N)$ for any b as in (2.8) and the function ϱ is termed a *renormalized solution*. A weak solution ϱ, $\boldsymbol{u}$ of (2.2) is not necessarily a renormalized solution and vice versa, there might be renormalized solutions of low integrability satisfying (2.7) but not (2.2). On the other hand, the information provided by (2.7) is quite useful to develop an existence theory for the variational (weak) solutions.

2.6. The energy inequality and a priori estimates.

Taking (formally) $\varphi = -\psi(t)\boldsymbol{u}$, $\psi \in \mathcal{D}(I)$ in (2.3) we deduce the *energy inequality*

$$E(t_2) + \int_{t_1}^{t_2} \int_\Omega \mathbb{S} : \mathbb{D}(\boldsymbol{u}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leqslant E(t_1) + \int_{t_1}^{t_2} \int_\Omega \varrho \boldsymbol{g} \cdot \boldsymbol{u} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \quad \text{for any } t_1 \leqslant t_2,\ t_1, t_2 \in I, \tag{2.9}$$

where

$$E(t) = \frac{1}{2} \int_\Omega \varrho |\boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} + \frac{a}{\gamma - 1} \int_\Omega \varrho^\gamma \,\mathrm{d}\boldsymbol{x}$$

provided the pressure p is related to the density ϱ through the isentropic constitutive law (1.5).

The second term on the left-hand side of (2.9) represents the energy dissipation which is the main and very often the only source of *a priori* estimates. For linearly viscous (Newtonian) fluids we deduce the *energy estimates*

$$\left\{ \begin{array}{c} \varrho \in L^\infty(I; L^\gamma(\mathbb{R}^N)),\ \ p(\varrho) \in L^\infty(I, L^1(\mathbb{R}^N)), \\ \boldsymbol{u} \ \in L^2(I; W^{1,2}(\mathbb{R}^N)), \\ \varrho |\boldsymbol{u}|^2 \in L^\infty(I; L^1(\mathbb{R}^N)) \end{array} \right\} \tag{2.10}$$

provided I is bounded. Of course, we have tacitly assumed boundedness of the external force density $\boldsymbol{g}$. If not specified otherwise, we will always assume that $\boldsymbol{g}$ is a bounded measurable function of $t \in I$, $\boldsymbol{x} \in \mathbb{R}^N$.

Moreover, integrating (formally) the continuity equation (2.2) we deduce that the total mass is a constant of motion, specifically

$$\int_{\mathbb{R}^N} \varrho(t) \,\mathrm{d}\boldsymbol{x} = \text{const}, \quad \varrho \geqslant 0. \tag{2.11}$$

2.7. Weak solutions.

Motivated by the preceding discussion, we introduce the concept of a *variational* (weak) solution to the problem of solids immersed in a compressible viscous fluid which will be referred to as Problem ($\mathrm{P^c}$) in what follows.

Problem (P^{c}). *We will say that ϱ, $\boldsymbol{u}$, and $\{\overline{S}^j, \eta^j\}_{j=0}^m$ is a variational solution of Problem* (P^{c}) *on a time interval I if the following conditions are satisfied:*

- *The function ϱ, $\boldsymbol{u}$ comply with the energy estimates, more precisely, they belong to the function spaces specified in* (2.10), (2.11).
- *The continuity equation* (2.2) *together with its renormalized form* (2.7) *holds.*
- *The variational form of the momentum equations* (2.3) *is satisfied for any admissible test function Φ.*
- *The mappings η^j, $j = 0, \ldots, m$ are affine isometries compatible with the velocity $\boldsymbol{u}$.*

3. Continuity in time and the Cauchy problem

3.1. The rigid motions.

In accordance with hypothesis (2.1), the sets $\overline{S}^i$ have a non-empty interior and, consequently, one can "differentiate" (2.4) with respect to $\boldsymbol{x}$ and integrate over $\mathbb{R}^N$ to obtain

$$\mathbb{Q}^i \in L^2(I) \text{ for all } i = 1, \ldots, m. \tag{3.1}$$

Similarly, we have

$$\boldsymbol{U}^i \in L^2(I) \text{ for all } i = 1, \ldots, m. \tag{3.2}$$

This means that the motion of the rigid bodies described through the isometries η^i is at least Hölder continuous in time. As a matter of fact, one can show the motion is even Lipschitz in time provided the total energy is bounded and the rigid densities strictly positive.

3.2. Time continuity of the density.

By definition, the density ϱ is bounded as a function of the time $t \in I$ with values in the Lebesgue space or $L^\gamma \cap L^1(\mathbb{R}^N)$ provided ϱ is extended to be zero outside Ω. Moreover, it follows from (2.10) that the *momentum* $\varrho\boldsymbol{u}$ is at least integrable, specifically,

$$\varrho\boldsymbol{u} \in L^\infty(I; L^{2\gamma/(\gamma+1)}(\mathbb{R}^N)) \text{ provided } \gamma > 1.$$

Thus we can deduce from the continuity equation (2.2) that

$$\varrho \in C(I; L^\gamma_{\mathrm{weak}}(\mathbb{R}^N))$$

for any variational solution of Problem ($\mathrm{P^c}$). Finally, since ϱ is also a renormalized solution of the continuity equation, one can use the regularizing procedure due to DiPerna and Lions [3] to deduce

$$\varrho \in C(I; L^1(\mathbb{R}^N)). \tag{3.3}$$

Observe that ϱ is constant in time outside Ω. More specifically, we report the following result (see [5], Lemma 3.2).

Lemma 3.1. *Let ϱ, $\boldsymbol{u}$ satisfy the continuity equation* (2.2) *in $\mathcal{D}'(I \times \mathbb{R}^N)$,*

$$\varrho \in L^\infty(I; L^\gamma \cap L^1(\mathbb{R}^N)), \ \ \boldsymbol{u} \in L^2(I; W^{1,2}(\mathbb{R}^N)), \ \ \gamma > 1.$$

Let, moreover, $\boldsymbol{u}$ be compatible with a system $\{\overline{S}^i, \eta^j\}_{j=0}^m$ where $\eta^i(t,\cdot)\colon \mathbb{R}^N \to \mathbb{R}^N$ are isometries for any $t \in I$, $i = 1, \dots, m$, and

$$\overline{S}^0 \ \text{ is an exterior of a bounded domain, } \eta^0(t, \boldsymbol{x}) = \boldsymbol{x}.$$

Then

$$\varrho(t_2, \eta^j(t_2, \boldsymbol{x})) = \varrho(t_1, \eta^j(t_1, \boldsymbol{x}))$$

for a.a. $\boldsymbol{x} \in \operatorname{int}(\overline{S}^j)$ and any $t_1, t_2 \in I$, $j = 0, \dots, m$.

What Lemma 3.1 says is nothing else but that the density is constant along characteristics on the region of space-time occupied by the rigid bodies, where the velocity is regular.

3.3. The momentum.

The time continuity of the momentum $\varrho\boldsymbol{u}$ is a more subtle issue. Using the variational formulation of the momentum equations (2.3) one deduces easily that the mapping

$$t \mapsto \int_{\mathbb{R}^N} (\varrho \boldsymbol{u})(t) \cdot \Phi \,\mathrm{d}\boldsymbol{x} \tag{3.4}$$

is continuous on an interval $J \cap \overline{I}$ containing τ for any $\Phi \in \mathcal{R}_\delta(\tau)$ where $\tau \in \overline{I}$ and $J = J(\Phi)$ is an open interval the length of which depends on Φ.

Accordingly one can define the instantaneous value of the momentum $\varrho\boldsymbol{u}$ via (3.4). More specifically, we have the following result.

Lemma 3.2. *Let ϱ, $\boldsymbol{u}$ be a variational solution of Problem* ($\mathrm{P^c}$) *on a bounded time interval I, and let $\tau \in \overline{I}$.*

(i) *Then there exists a function* $\boldsymbol{q} \in L^{2\gamma/(\gamma+1)}(\mathbb{R}^N)$ *such that*

$$\lim_{t\to\tau}\int_{\mathbb{R}^N}(\varrho\boldsymbol{u})(t)\cdot\Phi\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^N}\boldsymbol{q}\cdot\Phi\,\mathrm{d}x \ \text{ for any } \ \Phi\in\bigcup_{\delta>0}\mathcal{R}_\delta(\tau).$$

(ii) *Moreover, if*

$$m^i = \int_{\overline{S}^i}\varrho^{S^i}\,\mathrm{d}\boldsymbol{x} > 0 \ \text{ for all } \ i=1,\dots,m, \tag{3.5}$$

then there is a unique function $\boldsymbol{q}^f \in L^{2\gamma/(\gamma+1)}\Big(\mathbb{R}^N \setminus \bigcup\limits_{j=0}^{m}\overline{S}^j(\tau)\Big)$ *and rigid velocities* $\boldsymbol{v}^{S^i}$ *such that*

$$\boldsymbol{q} = \begin{cases} \boldsymbol{q}^f & \text{on } \mathbb{R}^N\setminus\bigcup\limits_{i=1}^{m}\overline{S}^i(\tau),\\ \varrho^{S^i}\boldsymbol{v}^{S^i} & \text{on } \overline{S}^i(\tau),\\ 0 & \text{on } \overline{S}^0. \end{cases} \tag{3.6}$$

(iii) *Finally, if*

$$\operatorname{dist}(\overline{S}^i(\tau),\overline{S}^j(\tau)) > 0 \ \text{ for all } \ i\neq j, \tag{3.7}$$

then the functions $\overline{v}^{S^i}$ *in* (3.6) *are uniquely determined. In particular, one can take* $\boldsymbol{v}^{S^i}(\tau) = \boldsymbol{u}^{S^i}(\tau)$ *for a.a.* $\tau\in J$ *provided* (3.7) *holds for any* $\tau \in J$.

P r o o f. The existence of $\boldsymbol{q}$ follows from the Hahn-Banach theorem. Moreover, if (3.5) holds, the set $\bigcup\limits_{\delta>0}\mathcal{R}_\delta(\tau)$ can be considered as a subspace of the Banach space

$$Y = \{\boldsymbol{q}\in L^{2\gamma/(\gamma+1)}(\mathbb{R}^N) \mid \boldsymbol{q} = 0 \text{ on } \overline{S}^0,\ \ \boldsymbol{q} = \boldsymbol{v}^{S^i} \text{ on } \overline{S}^i(\tau),\ \ i=1,\dots,m\}$$

endowed with a norm

$$\|\boldsymbol{q}\|_Y = \|\boldsymbol{q}\|_{L^{2\gamma/(\gamma+1)}\left(\mathbb{R}^N\setminus\bigcup\limits_{j=0}^{m}\overline{S}^j(\tau)\right)} + \sum_{i=1}^{m}\left(\int_{\overline{S}^i(\tau)}\varrho^{S^i}|\boldsymbol{v}^{S^i}|^2\,\mathrm{d}\boldsymbol{x}\right)^{1/2}.$$

Thus (3.6) follows from the Riesz representation theorem.

Finally, it is easy to check that the set $\bigcup\limits_{\delta>0}\mathcal{R}_\delta(\tau)$ is dense in Y provided (3.7) holds, which completes the proof. $\square$

In accordance with Lemma 3.2, the instantaneous value of the momentum coincides with $\varrho\boldsymbol{u}$ a.e. on I provided there is no collision of two rigid objects, i.e., if (3.7) holds. In the case of a collision, the rigid velocities $\boldsymbol{v}^{S^i}$ are not uniquely determined, and the momentum $\varrho\boldsymbol{u}$ need not be continuous with respect to time not even in a weak sense. This is related to several possibilities how to continue the solution after the time of contact. Such an ambiguity is one of the main obstacles to constructing physically meaningful global-in-time variational solutions.

3.4. The initial value problem.

To conclude, consider a time interval $I = (0, T)$. In accordance with the previous discussion, it makes sense to supplement Problem ($\mathrm{P^c}$) by the following set of initial conditions.

We can prescribe the initial position of the rigid objects,

$$\overline{S}^i(0) = \eta^i(0, \overline{S}^i) = \overline{S}^i_0, \quad i = 1, \dots, m, \tag{3.8}$$

the initial density distribution

$$\varrho(0, \boldsymbol{x}) = \varrho_0(\boldsymbol{x}) \geqslant 0, \quad \boldsymbol{x} \in \Omega, \tag{3.9}$$

while the initial value $\boldsymbol{q}$ of the momentum is attained in the sense that

$$\lim_{t\to 0+} \int_{\mathbb{R}^N} \varrho\boldsymbol{u} \cdot \Phi \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^N} \boldsymbol{q} \cdot \Phi \,\mathrm{d}\boldsymbol{x} \ \text{ for any } \ \Phi \in \bigcup_{\delta>0} \mathcal{R}_\delta(0). \tag{3.10}$$

It is convenient to normalize $\overline{S}^i(0) = \overline{S}^i$, i.e., $\eta^i(0, \boldsymbol{x}) = \boldsymbol{x}$, $i = 1, \dots, m$.

4. Remarks, comments, discussion

Because of numerous applications, the two constituent solid-liquid flows have been the subject of many theoretical as well as numerical studies. One of the most classical contributions is the study of Stokes [16], and there has been an enormous amount of recent literature dealing mostly with the movement of one or several solid bodies in a linearly viscous incompressible fluid. We mention the monographs of Giovangigli [9], Rajagopal and Tao [13], or the paper by Galdi [8] to name only a few.

The variational formulation analogous to that introduced in Section 2 was introduced by Serre [15], and recently used by Desjardins and Esteban [1] to show local existence of strong solutions of the initial-boundary value problem in the incompressible case. The same authors discuss also the problem of global existence "up to collision" for both incompressible and compressible isentropic case ([2]).

Probably the weakest point of the present formulation related to the static frame attached to Ω is the fact that the space of admissible test functions for the momentum equation depends on the velocity component $\boldsymbol{u}$. In fact, the class of test functions used in literature varies in a considerable way. Several authors—Desjardins and Esteban [2], San Martin, Starovoitov and Tucsnak [14]—use the class of test functions

$$\mathcal{V} = \{\varphi \in W^{1,2}((0,T)\times\Omega) \mid \varphi|_{\partial\Omega} = 0,\ \ \varrho^{S^i}\mathbb{D}(\varphi) = 0,\ \ i = 1,\dots,m\}.$$

Here, one should observe that given the maximal expected regularity of the solutions given by the energy estimates presented in Section 2.6, this class can be used only in the incompressible case restricted to two space dimensions. Indeed, neither the convective term $\varrho\boldsymbol{u}\otimes\boldsymbol{u}$ nor the pressure p in the compressible case are known to be square integrable—a necessary condition for (2.3) to make sense.

The principle of compatibility of the velocity $\boldsymbol{u}$ with the motion of the rigid objects characterized by $\{\overline{S}^j, \eta^j\}_{j=0}^m$ introduced in Section 2 coincides basically with the definition of the weak solutions introduced by Gunzburger, Lee and Seregin [10] even though they use the reference frame attached to (one) moving body rather than to the spatial domain Ω. There is another possible approach employed in [2], [14] and others, namely, it is assumed that

$$\boldsymbol{u} \in L^2(0,T; V^s \cap W_0^{1,2}(\Omega))$$

where the sets $V^s = V^s(t)$ are defined through

$$V^s = \{\boldsymbol{v} \in W^{1,2}(\Omega) \mid \mathbb{D}(\boldsymbol{v})\varrho^{S^i}(t) = 0 \ \text{ for } \ i = 1,\dots,m\}.$$

Note, however, that this definition combined only with the continuity equation (2.2) for the densities ϱ^{S^i} does not prevent the rigid bodies from splitting into a finite number of parts, i.e., the sets $\overline{S}^i(t)$ need not be connected for $t > 0$. This is the main shortcoming of using the continuity equation as the only description of the motion of rigid bodies.

Another feature of the problem which should be captured by the concept of variational (weak) solutions are collisions of two or more rigid bodies. One should be aware of the fact that our definition of weak solutions as well as others introduced in the above mentioned references do not seem to cope with this task. In fact the variational solutions are too general and seem to include all possible contacts allowed by relevant physical principles. This is due to the fact that the velocity gradient $\nabla\boldsymbol{u}$ need not be bounded, and, in view of Lemma 3.2, the instantaneous value of the momentum at the collision time is not well defined. Indeed, if a collision occurs, the weak solutions do not comply with the principle of "formal interpretation". This means that even if a weak solution is regular, it need not be a classical solution of the problem. Consider for instance the situation when Ω is a unit ball in $\mathbb{R}^3$ containing only one solid ball $\overline{S}$ of metal of high constant density ϱ^S touching the boundary $\partial\Omega$ at the point $(1,0,0)$. The only external force considered is the gravity $\boldsymbol{g} = (-g,0,0)$ acting in the vertical direction. It is easy to see that there is a time-independent weak solution in the sense of Section 2, namely,

$$\boldsymbol{u} \equiv 0, \;\; \varrho = \varrho^f + \varrho^S, \;\; \{\overline{S}, \mathbb{I}\}$$

where

$$\varrho^S(t,\boldsymbol{x}) = \varrho^S \;\; 1_{\overline{S}}(\boldsymbol{x}),$$

and ϱ^f is a static density distribution, i.e., ϱ^f satisfies

$$\nabla p(\varrho^f) = \varrho^f \boldsymbol{g}. \tag{4.1}$$

Note that the static problem (4.1) admits a unique solution with a given total mass $m^f = \int_\Omega \varrho^f \, \mathrm{d}\boldsymbol{x}$. To stress even more the paradoxical character of this situation, the static solution ϱ^f contains a vacuum zone, i.e., vanishes identically in a neighbourhood of $\overline{S}$ provided m^f is chosen small enough.

This seems to be another pathological feature of the problem: The physically admissible solutions cannot be smooth while there are smooth variational (weak) solutions which are obviously not acceptable by the common sense.

The final question we want to address now is the principle of *impermeability*. It is satisfied even in the class of weak solutions as shown in the following assertion (see [5], Lemma 3.1).

Lemma 4.1. *Let $\{\overline{S}^i, \eta^i\}$, $i = 1,2$ be two families of compact connected sets where η^i are isometries on $\mathbb{R}^N$ compatible with the same velocity $\boldsymbol{u}$. Denote $S^i \equiv \operatorname{int}(\overline{S}^i)$, $i = 1,2$.*

Then either $S^1(t) \cap S^2(t) = \emptyset$ for all $t \in [0,T]$ or $S^1(\tau) \cap S^2(\tau) \neq \emptyset$ for a certain $\tau \in [0,T]$ in which case $\eta^1(t) = \eta^2(t)$ for all $t \in [0,T]$.

Thus the solid bodies are allowed to touch—$\overline{S}^i \cap \overline{S}^j \neq \emptyset$—but not penetrate one another unless they did so at the initial time. Moreover, one observes easily the following corollary.

Corollary 4.1. *Let a family $\{\overline{S}, \eta\}$ be compatible with a velocity $\boldsymbol{u}$ such that*

$$\boldsymbol{u} = 0 \;\; \text{a.e. on the set} \;\; \mathbb{R}^N \setminus \Omega$$

where $\Omega \subset \mathbb{R}^N$ is an open set.

Then either $S(t) \subset \Omega$ *for all* $t \in [0,T]$ *or* $\eta \equiv \mathrm{Id}$, *i.e.*, $\overline{S}(s) = \overline{S}(t)$ *for any* $0 \leqslant s \leqslant t \leqslant T$.

5. A global existence theorem in the compressible case

Now, we consider the initial value problem for the case of a compressible isentropic fluid discussed in Sections 2 and 3. To be more specific, let

$$\varrho_0 \geqslant 0, \ \ \varrho \in L^\gamma \cap L^1(\mathbb{R}^N) \tag{5.1}$$

be the initial distribution of the density; the sets

$$\overline{S}_0^i, \ \ i = 1, \ldots, m$$

characterize the initial position of the rigid bodies; and $\boldsymbol{q}$ is the initial momentum satisfying the compatibility condition

$$\boldsymbol{q} = 0 \ \text{ a.e. on the set } \{\varrho_0 = 0\}. \tag{5.2}$$

We suppose that the initial kinetic energy is finite, i.e.,

$$\frac{|\boldsymbol{q}|^2}{\varrho_0} \in L^1(\mathbb{R}^N). \tag{5.3}$$

For simplicity, the underlying (fixed) spatial domain Ω will be bounded.

We report the following result (see [5], Theorem 4.1).

Theorem 5.1. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain. Let the initial data* ϱ_0, $\boldsymbol{q}$, $\overline{S}_0^i$ *be given satisfying hypotheses* (5.1)–(5.3) *together with* (2.1)*. Let the external force density* $\boldsymbol{g}$ *be a bounded measurable function of* $t \in (0,T)$ *and* $\boldsymbol{x} \in \mathbb{R}^N$*. Finally, let the pressure* p *be given by the isentropic constitutive relation*

$$p(\varrho) = a\varrho^\gamma, \ \ a > 0 \ \textit{ with } \gamma > \frac{N}{2}.$$

Then Problem (P^{c}) *admits a variational solution* ϱ, $\boldsymbol{u}$, $\{\overline{S}^i, \eta^i\}_{i=1}^m$ *on* $(0,T)$ *satisfying the initial conditions*

$$\varrho(0) = \varrho_0, \ \ (\varrho\boldsymbol{u})(0) = \boldsymbol{q}, \ \ \overline{S}^i(0) = \overline{S}_0^i$$

where the second equality is to be understood in the sense of (3.10). *Moreover, the solution satisfies the energy inequality*

$$(5.4) \qquad E(\tau) + \int_0^\tau \int_{\mathbb{R}^N} \mu|\nabla \boldsymbol{u}|^2 + (\lambda+\mu)|\operatorname{div} \boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$\leqslant \int_{\mathbb{R}^N} \frac{1}{2}\frac{|\boldsymbol{q}|^2}{\varrho_0} + \frac{a}{\gamma-1}\varrho_0^\gamma \,\mathrm{d}\boldsymbol{x} + \int_0^\tau \varrho \boldsymbol{g}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \ \textit{for a.a. } \tau \in (0,T).$$

Unlike most of the recent results of similar type (see e.g. Desjardins and Esteban [2]), we do not assume any bound on the existence time $T > 0$. Possible collisions of two bodies or a body with the boundary $\partial\Omega$ are allowed. Another advantage of the present result are very mild hypotheses concerning the regularity of the boundary of solids.

The complete proof of Theorem 5.1 is done in [5]. Adopting the recent existence theory for the isentropic Navier-Stokes equations based on the work of Lions [12] (see also [7]) one immediately encounters the two major stumbling blocks inherent to the present problem: One has to show compactness of the density component, specifically,

$$(5.5) \qquad \varrho_n \to \varrho \ \text{ in } \ C([0,T]; L^1(\mathbb{R}^N))$$

as well as of the convective term

$$(5.6) \qquad \varrho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n \to \varrho \boldsymbol{u} \times \boldsymbol{u} \ \text{ weakly in } \ L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^N)$$

where ϱ_n, $\boldsymbol{u}_n$ is a sequence of suitable approximate solutions. The construction of the approximate solutions for this type of problem is an independent issue.

While (5.6) can be proved with help of the method developed in [4], the convergence in (5.6) is more delicate in both the compressible and incompressible cases. This is the main difference between the "classical" problem with no rigid objects, where (5.6) follows easily from the Lions-Aubin lemma, and the present situation. In most theoretical studies, the authors try to prove (5.6) on the whole space-time domain while a short inspection of (2.3) shows that it is needed only on the part occupied by the fluid.

This simple observation is one of the main ingredients of the proof. Indeed, from this point of view, the compressible case is "easier" than the incompressible one. The reason is that there are local pressure estimates in the fluid region

$$Q^f = \left\{ (t,\boldsymbol{x}) \mid t \in (0,T),\ \ \boldsymbol{x} \in \mathbb{R}^N \setminus \bigcup_{j=0}^m \overline{S}^j(t) \right\}$$

independent of the presence of the rigid objects. More specifically, in addition to the energy estimates presented in Section 2.6, we have an a priori estimate for the pressure

$$\int_{K^f} \varrho^{\gamma+\theta} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t < \text{const.}$$

for a certain $\theta > 0$ and for any compact $K^f \subset Q^f$ (cf. [5], Lemma 8.1).

The approximate solutions are constructed by means of a method which is a combination of the penalization technique of San Martin et al. [14] with the approximation scheme developed in [7]. More specifically, we consider the problem

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = d\Delta\varrho, \;\; d > 0$$

in $(0, T) \times \Omega$ supplemented by the Neumann boundary conditions

$$\nabla\varrho \cdot \boldsymbol{n}|_{\partial\Omega} = 0;$$

$$\partial_t(\varrho\boldsymbol{u}) + \operatorname{div}(\varrho\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p(\varrho) + d\nabla\boldsymbol{u}\nabla\varrho$$
$$= \operatorname{div}\big(2\mu(\chi)\mathbb{D}(\boldsymbol{u}) + \lambda(\chi)\mathbb{I} \operatorname{div}\boldsymbol{u}\big) + \varrho\boldsymbol{g}$$

in $(0, T) \times \Omega$ with

$$\boldsymbol{u}|_{\partial\Omega} = 0$$

and

$$p(\varrho) = a\varrho^\gamma + b\varrho^\beta, \;\; a, b > 0.$$

The extra term $d\Delta\varrho$ in the continuity equation represents a standard artificial viscosity approximation. Observe that the momentum equation has been modified accordingly to preserve the energy inequality. Moreover, an artificial pressure term $b\varrho^\beta$, $\beta > 1$ large enough, is added for technical reasons explained in detail in [7].

In comparison with [7], the main new ingredient is that the viscosity coefficients μ and λ are no longer constant but depend on a function χ determined by the velocity $\boldsymbol{u}$ in the following way. Similarly as in [14], we define a regularized velocity $\boldsymbol{u}_\delta = \boldsymbol{u} * \vartheta_\delta$ where ϑ_δ is a sequence of regularizing kernels. Now, we can define the corresponding characteristic curves

$$\frac{\partial}{\partial t}\eta(t, \boldsymbol{x}) = \boldsymbol{u}_\delta(t, \eta(t, \boldsymbol{x})), \quad \eta(0, \boldsymbol{x}) = \boldsymbol{x}.$$

Finally, let $O \subset \mathbb{R}^N$ be a bounded open set characterizing, roughly speaking, the initial position of the rigid bodies. We set

$$O(t) = \eta(t, O) \text{ and } \chi(t, \boldsymbol{x}) = \mathbf{db}_{O(t)}(\boldsymbol{x}) \text{ for any } t \in [0, T], \;\; \boldsymbol{x} \in \mathbb{R}^N,$$

where $\mathbf{db}_O$ is the boundary distance defined through

$$\mathbf{db}_O(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, \overline{\mathbb{R}^N \setminus O}) - \operatorname{dist}(\boldsymbol{x}, \overline{O}).$$

Now, we take the viscosity coefficients of the form

$$\mu = \mu_\varepsilon = \mu + \frac{1}{\varepsilon} H(\chi + \delta), \quad \lambda = \lambda_\varepsilon = \lambda + \frac{1}{\varepsilon} H(\chi + \delta)$$

where H is a smooth convex function, $H(z) = 0$ for $z \leqslant 0$, $H(z) > 0$ otherwise, and μ and λ are the fluid viscosity coefficients satisfying

$$\mu > 0, \quad \lambda + \mu \geqslant 0.$$

The above system can be solved by means of an approximation scheme of the Faedo-Galerkin type (see [5], Section 6). The next step is to let $\varepsilon \to 0$. Intuitively, introducing the viscosity coefficients depending on χ we have replaced the rigid bodies by a fluid of very high viscosity. Letting $\varepsilon \to 0$, the solids will "appear" as occupying a specific region of the space time. The final two steps consist in letting first $d \to 0$ and then $b \to 0$ to obtain a solution of the original problem (see [5] for the complete proof).

6. The incompressible fluids

To conclude, we shall briefly comment on the results which can be obtained in the case when the fluid is incompressible. As already mentioned above, the pressure term makes this problem more difficult than in the compressible case measured in the complexity of the path from the standard existence theory for the sole fluid with no objects inside to the present setting.

Obviously, the weak formulation presented in Section 2 needs some modification. Now, the term $a/(\gamma - 1)\varrho^\gamma$ will disappear from the energy inequality (2.9) while the space of admissible test function $\mathcal{R}_\delta(t)$ must be supplemented by the requirement that the functions are divergence free.

Denoting the resulting problem as Problem ($\mathrm{P^i}$) we report the following result (see [6], Theorem 1.1).

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Let $T > 0$ be given. Assume that the initial position of the rigid bodies is given through a family of sets $\overline{S}_0^i$ satisfying* (2.1) *and such that*

$$\overline{S}_0^i \cap \overline{S}_0^j = \emptyset \ \text{ for } i \neq j, \ \ \overline{S}_0^i \cap (\mathbb{R}^3 \setminus \Omega) = \emptyset, \ \ i = 1, \ldots, m.$$

Let the initial distribution of the density be given by a measurable function ϱ_0 *such that*

$$0 < \underline{\varrho} \leqslant \varrho_0(\boldsymbol{x}) \leqslant \overline{\varrho} \ \textit{ for a.a } \boldsymbol{x} \in \Omega.$$

Finally, let $\boldsymbol{u}_0 \in L^2_\sigma(\Omega)$ *and* $\boldsymbol{g} \in L^\infty((0,T) \times \Omega)$ *be given.*

Then Problem $(\mathrm{P}^{\mathrm{i}})$ *supplemented with the initial conditions*

$$\varrho(0) = \varrho_0, \ \ \boldsymbol{u}(0) = \boldsymbol{u}_0, \ \ \overline{S}^i(0) = \overline{S}^i_0, \ \ i = 1, \dots, m,$$

admits a variational solution ϱ, $\boldsymbol{u}$, $\{\overline{S}^i, \eta^i\}_{i=1}^m$ *satisfying the energy inequality*

$$\frac{1}{2}\int_\Omega \varrho(\tau)|\boldsymbol{u}(\tau)|^2 \,\mathrm{d}\boldsymbol{x} + \mu \int_0^\tau \int_\Omega |\nabla \boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leqslant \frac{1}{2}\int_\Omega \varrho_0 |\boldsymbol{u}_0|^2 \,\mathrm{d}\boldsymbol{x} + \int_0^\tau \int_\Omega \varrho \boldsymbol{g} \cdot \boldsymbol{u} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

for a.a. $\tau \in (0,T)$.

Here, the symbol $L^2_\sigma(\Omega)$ stands for the closure of the set of all divergence free functions in $\mathcal{D}(\Omega)$ with respect to the L^2-norm.

The proof of Theorem 6.1 is rather technical but the main idea is very simple. As discussed above, there are several results of "local" type asserting existence up to the first collision. The only truly global result seem to be that of Hoffmann and Starovoitov [11] where there is only one rigid body—a ball—contained in a fixed domain Ω—another ball. The second global-in-time existence result was proved by San Martin et al. [14] in two space dimensions. In both cases, the essential feature of the problem is that at the contact time the translation velocities of the two colliding objects are identical. This is, however, not the case in general. Accordingly, the space of all variational solutions seems to be too large to ensure compactness of the velocity in the Lebesgue space L^2. The way out is, of course, to restrict the class of admissible solutions after the contact time. One of many possibilities is to assume that once two rigid objects collide they will stay together forever. This is a very naive realization of the least energy principle.

In accordance with the previous discussion, the global solution, the existence of which is claimed in Theorem 6.1, will be constructed with help of the following (local) result (see [6], Proposition 3.1).

Proposition 6.1. *Under the hypotheses of Theorem* 6.1, Problem $(\mathrm{P}^{\mathrm{i}})$ *admits a variational solution defined on a maximal time interval* $(0, T_{\max})$, $T_{\max} > 0$. *More-*

over, if $T_{\max} < \infty$, we have either

$$\overline{S}^i(T_{\max}) \cap \overline{S}^j(T_{\max}) \neq \emptyset \ \text{ for certain } \ i \neq j$$

or

$$\overline{S}^i(T_{\max}) \cap (\mathbb{R}^3 \setminus \Omega) \neq \emptyset \ \text{ for a certain } \ i.$$

Proposition 6.1 is nothing else but an "up to the first collision" weaker form of Theorem 6.1. From this point of view, its proof might seem to be a relatively easy modification of the techniques available in literature. However, this is not the case for the following reason. Neither the boundary of the rigid objects nor that of Ω are supposed to be smooth. The standard proofs for smooth objects are based on some constructions based on solving several elliptic-like problems very sensitive to rough boundary changes. In particular, the boundary should be at least Lipschitz, which is in general not the case in the "after collision" situation when two colliding objects are considered as one after the collision time.

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