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Applications of Mathematics, Vol. 47 (2002), No. 5, 397–410

Persistent URL: http://dml.cz/dmlcz/134504

A NOTE ON CONTACT SHAPE OPTIMIZATION WITH SEMICOERCIVE STATE PROBLEMS*

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(Received March 16, 2000)

Abstract. This note deals with contact shape optimization for problems involving "floating" structures. The boundedness of solutions to state problems with respect to admissible domains, which is the basic step in the existence analysis, is a consequence of Korn's inequality in coercive cases. In semicoercive cases (meaning that floating bodies are admitted), the Korn inequality cannot be directly applied and one has to proceed in another way: to use a decomposition of kinematically admissible functions and a Korn type inequality on appropriate subspaces. In addition, one has to show that the constant appearing in this inequality is independent with respect to a family of admissible domains.

Keywords: shape optimization, semicoercive problems

MSC 2000: 49A29, 73K25, 65K10

1. Introduction

Shape optimization is a special branch of the optimal control theory in which control variables are related to the geometry of systems (the shape, the thickness, e.g.). In particular, contact shape optimization deals with optimization of a structure composed of several individual deformable bodies, being in a mutual contact. It is well-known that contact shape optimization leads to a non-smooth problem, in general, due to the fact that the state of the system is described by a variational inequality (see [2], [3], [11]). A mathematical analysis including approximations of such problems is presented in detail in [2], [3]. All problems studied there are supposed to be *coercive*. The coercivness is guaranteed by appropriate Dirichlet boundary conditions, eliminating rigid body deformations, i.e. translations and rotations. On

* This work was supported by grants No. 101/01/0538 of the Grant Agency of the Czech Republic and No. A1075005 of the Academy of Sciences of the Czech Republic.

the other hand, in practice we meet a lot of situations when at least to one of the bodies of the structure is allowed to "float". In this way we arrive at *a semicoercive* case. It is well-known that semicoercive cases deserve special attention: solutions need not exist for any applied forces and if a solution exists then it is not necessarily unique.

The basic step of the existence analysis in shape optimization consists in proving that solutions to state problems depend *continuously* on shape variations (in an appropriate way). To this end, we first show that the solutions (defined on different domains) are bounded in the respective Sobolev spaces. In the coercive case this follows immediately from Korn's inequality, whose constant can be chosen independently of domains satisfying the so-called *uniform ε-cone property.* In semicoercive cases, Korn's inequality cannot be directly applied and one has to proceed in a different way. In this note it will be shown how the boundedness of solutions can be obtained by using the approach from [4]. Having established the boundedness, the existence analysis in contact shape optimization proceeds *exactly* step by step as in [2], [3].

The paper is organized as follows: in Section 2 we prove that the constants appearing in equivalent norms are independent with respect to a class of domains, provided that a "basic" norm possesses the same property. This is an extension of results from [5], [6]. In Section 3 we use a Korn type inequality on appropriate subspaces, whose constant can be chosen independently of the admissible domains. This result will be used in contact shape optimization.

2. Equivalent norms with constants uniform with respect to a class of domains

We start this section by notation, definitions and auxiliary results which will be useful in what follows.

Let $h > 0$, $\varepsilon \in (0, \pi/2)$ be given numbers and ξ a unit vector in $\mathbb{R}^n$. The set

$$C(\xi, \varepsilon, h) = \{x \in \mathbb{R}^n \mid (x, \xi) > \|x\| \cos \varepsilon, \|x\| < h\}$$

is a cone of angle ε and height h.

Definition 2.1. Let $\varepsilon \in (0, \pi/2)$, $h > 0$, $r > 0$ $(2r \leqslant h)$ be given. We say that a domain $\Omega \subset \mathbb{R}^n$ satisfies the ε-cone property if for every $x \in \partial\Omega$ there exists a vector $\xi_x \in \mathbb{R}^n$, $\|\xi_x\| = 1$ and a cone $C_x := C_x(\xi_x, \varepsilon, h)$ such that $\forall y \in B(x, r) \cap \Omega$ the set $y + C_x$ belongs to Ω, where $B(x, r)$ denotes the ball of radius r and centered at x.

Definition 2.2. Let h, ε, r be as above and let D be a closed bounded domain in $\mathbb{R}^n$. The system of all domains $\Omega \subset \mathbb{R}^n$, $\Omega \subset D$ satisfying the ε-cone property will be denoted by $\mathcal{O}_\varepsilon$.

It is well-known that $\mathcal{O}_\varepsilon$ possesses some useful properties, which are listed in

Theorem 2.1 (the uniform extension property). *Let h, r, ε and $\mathcal{O}_\varepsilon$ be the same as above. Then for all $\Omega \in \mathcal{O}_\varepsilon$ there exists an extension mapping $p_\Omega \in \mathcal{L}(H^m(\Omega), H^m(\mathbb{R}^n))$, $m \in \mathbb{N}$, such that its operator norm $\|p_\Omega\|$ depends solely on h, r, ε and m but not on the particular choice of $\Omega \in \mathcal{O}_\varepsilon$.*

P r o o f. For the proof see [1]. □

Consequence 2.1. *Since the numbers h, r, ε are the same for all $\Omega \in \mathcal{O}_\varepsilon$, the norm p_Ω can be estimated independently of $\Omega \in \mathcal{O}_\varepsilon$.*

The compactness of $\mathcal{O}_\varepsilon$ with respect to the Hausdorff metric and the convergence of the characteristic functions in the $L^2(D)$-norm is another important property of this class of domains. Recall that the Hausdorff distance $d(A, B)$ of two sets A, B in $\mathbb{R}^n$ is defined by

$$d(A,B) = \max\Big(\sup_{x\in A} \varrho(x,B), \sup_{x\in B} \varrho(x,A)\Big),$$

where $\varrho(x, A)$ is the distance of a point x from A.

We say that $\{A_k\}_{k=1}^{\infty}$ tends to A in the Hausdorff sense (and we write $A_k \xrightarrow{h} A$) iff $d(A_k, A) \to 0$, $k \to \infty$. Compactness properties of $\mathcal{O}_\varepsilon$ are summarized in

Theorem 2.2. *From every sequence $\{\Omega_k\}$, $\Omega_k \in \mathcal{O}_\varepsilon$ one can select a subsequence $\{\Omega_{k_j}\}$ and a domain $\Omega \in \mathcal{O}_\varepsilon$ such that*

$$\Omega_{k_j} \xrightarrow{h} \Omega, \quad j \to \infty, \tag{2.1}$$

$$\partial\Omega_{k_j} \xrightarrow{h} \partial\Omega, \quad j \to \infty, \tag{2.2}$$

$$\chi(\Omega_{k_j}) \to \chi(\Omega) \quad \text{in the } L^2(D)\text{-norm}, \tag{2.3}$$

where $\chi(\bullet)$ denotes the characteristic function of the set $\bullet$.

P r o o f. For the proof of (2.1), (2.3) we refer to [1], [10], (2.2) can be found in [8]. □

R e m a r k 2.1. From the definition of the Hausdorff metric it easily follows that if $\partial\Omega_k \xrightarrow{h} \partial\Omega$, then for any $\eta > 0$ there exists $k_0 := k_0(\eta) \in \mathbb{N}$ such that for any $k \geqslant k_0$ the boundary $\partial\Omega_k$ belongs to the η-neighbourhood of $\partial\Omega$.

On every $\Omega \in \mathcal{O}_\varepsilon$ we shall consider the Sobolev space $H^1(\Omega, \mathbb{R}^d)$, $d \in \mathbb{N}$ and its closed subspace $W(\Omega)$. The classical norm in $H^1(\Omega, \mathbb{R}^d)$ will be denoted by $\|\cdot\|_{1,\Omega}$, while the symbol $|\cdot|_{1,\Omega}$ stands for a seminorm. Further, let $a_\Omega\colon W(\Omega) \times W(\Omega) \to \mathbb{R}^1$ be a bilinear form satisfying

(A1) $m|v|^2_{1,\Omega} \leqslant a_\Omega(v,v) \leqslant M|v|^2_{1,\Omega}$ $\forall v \in W(\Omega)$, $\forall \Omega \in \mathcal{O}_\varepsilon$, where m, M are positive constants which do not depend on $\Omega \in \mathcal{O}_\varepsilon$;

(A2) for any decomposition of $\Omega \in \mathcal{O}_\varepsilon$ into two disjoint Lipschitz subdomains Ω_1, Ω_2 (i.e. $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, $\Omega_1 \cap \Omega_2 = \emptyset$) we have

$$a_\Omega(v,v) = a_{\Omega_1}(v,v) + a_{\Omega_2}(v,v) \ \ \forall v \in W(\Omega);$$

(A3) if $\Omega_k \xrightarrow{h} \Omega$, where $\Omega_k, \Omega \in \mathcal{O}_\varepsilon$ are such that $\Omega_k \subset \Omega$ $\forall k \in \mathbb{N}$ then

$$a_{\Omega_k}(v,v) \to a_\Omega(v,v) \ \ \forall v \in W(\Omega);$$

(A4) $\exists \beta = \text{const} > 0$ such that

$$a_\Omega(v,v) + \|v\|^2_{0,\Omega} \geqslant \beta \|v\|^2_{1,\Omega} \ \ \forall v \in W(\Omega), \ \ \forall \Omega \in \mathcal{O}_\varepsilon,$$

where $\|\cdot\|_{0,\Omega}$ denotes the $L^2(\Omega, \mathbb{R}^d)$-norm.

Remark 2.2. (A4) is the basic assumption in the subsequent analysis. It says that the expression $(a_\Omega(v,v) + \|v\|^2_{0,\Omega})^{1/2}$ defines a norm in $W(\Omega)$ which is equivalent to $\|\cdot\|_{1,\Omega}$ and in addition, the constant β is *independent* of $\Omega \in \mathcal{O}_\varepsilon$.

Let $\mathcal{W}(\Omega) = \ker a_\Omega$, i.e.

$$\mathcal{W}(\Omega) = \{v \in W(\Omega) \mid a_\Omega(v,v) = 0\} = \{v \in W(\Omega) \mid |v|_{1,\Omega} = 0\}.$$

Next we shall suppose that

(A5) $\dim \mathcal{W}(\Omega) = \kappa \in \mathbb{N}$, where κ does not depend on $\Omega \in \mathcal{O}_\varepsilon$.

If it is so, one can find a system $\{l^{(i)}_\Omega\}_{i=1}^\kappa$ of linear continuous functionals on $H^1(\Omega, R^d)$ such that

$$v \in \mathcal{W}(\Omega) \ \ \& \ \ l^{(i)}_\Omega(v) = 0, \ \ i = 1, \ldots, \kappa \Leftrightarrow v = 0 \ \text{ in } \ \Omega.$$

Next we shall suppose that the set D from Definition 2.2 is such that int D has Lipschitz boundary.

Convention. The mapping p_Ω from Theorem 2.1 will be now understood as the extension from Ω onto D (by taking the restriction of the original p_Ω to D).

Let the system $\{l_\Omega^{(i)}\}_{i=1}^\kappa$ be continuous in the following sense:

$$\text{(A6)}\quad \left.\begin{array}{r}\Omega_k \xrightarrow{h} \Omega,\ \ \Omega_k, \Omega \in \mathcal{O}_\varepsilon\\ p_{\Omega_k} v_k \rightharpoonup v \ \text{ in }\ H^1(D, R^d),\\ v_k \in H^1(\Omega_k, R^d)\end{array}\right\} \Longrightarrow l_{\Omega_k}^{(i)}(v_k) \to l_\Omega^{(i)}(v),\ \ k \to \infty,\ \ i = 1, \ldots, \kappa.$$

Finally we shall suppose that the following restriction property holds:

$$\text{(A7)}\quad \left.\begin{array}{l}\Omega_k \xrightarrow{h} \Omega,\ \ \Omega_k,\ \ \Omega \in \mathcal{O}_\varepsilon\\ p_{\Omega_k} v_k \rightharpoonup v \ \text{ in }\ H^1(D, R^d),\ \ v_k \in W(\Omega_k)\end{array}\right\} \Longrightarrow v_{|\Omega} \in W(\Omega).$$

It is known (see [7]) that

$$[|v|]_{1,\Omega} := \left\{a_\Omega(v, v) + \sum_{i=1}^{\kappa} [l_\Omega^{(i)}(v)]^2\right\}^{1/2}$$

defines a norm on $W(\Omega)$ which is equivalent to $\|\cdot\|_{1,\Omega}$:

$$\exists \tilde\beta = \text{const} > 0 \ \text{ such that }\ [|v|]_{1,\Omega} \geqslant \tilde\beta \|v\|_{1,\Omega}$$

holds for every $v \in W(\Omega)$. The constant $\tilde\beta$ can possibly depend on a particular choice of $\Omega \in \mathcal{O}_\varepsilon$. In what follows we shall prove that under the previous assumptions, $\tilde\beta$ can be chosen independently of $\Omega \in \mathcal{O}_\varepsilon$.

We start with an auxiliary result.

Lemma 2.1. *Let* (A1)–(A3) *and* (A7) *be satisfied. Further let* $\partial\Omega_k \xrightarrow{h} \partial\Omega$ *and* $p_{\Omega_k} v_k \rightharpoonup v$ *in* $H^1(D, R^d)$, *where* $\Omega_k, \Omega \in \mathcal{O}_\varepsilon$ *and* $v_k \in W(\Omega_k)$. *Then*

$$\liminf_{k\to\infty} a_{\Omega_k}(v_k, v_k) \geqslant a_\Omega(v, v).$$

Proof. Let $\Omega(s)$, $s \in \mathbb{N}$ denote the following subset of Ω:

$$\Omega(s) = \{x \in \Omega \mid \varrho(x, \partial\Omega) > 1/s\}.$$

Then $\Omega(s) \xrightarrow{h} \Omega$, $\partial\Omega(s) \xrightarrow{h} \partial\Omega$ as $s \to \infty$.

Let $s \in \mathbb{N}$ be fixed. There exists $k_0 := k_0(s) \in \mathbb{N}$ such that (see Remark 2.2) $\Omega_k \supset \Omega(s)$ $\forall k \geqslant k_0$. From (A1) and (A2) it follows that

$$a_{\Omega_k}(v_k, v_k) = a_{\Omega_{(s)}}(v_k, v_k) + a_{\Omega_k \setminus \overline{\Omega}_{(s)}}(v_k, v_k) \geqslant a_{\Omega_{(s)}}(v_k, v_k)$$

and consequently

$$\liminf_{k\to\infty} a_{\Omega_k}(v_k, v_k) \geqslant \liminf_{k\to\infty} a_{\Omega_{(s)}}(v_k, v_k) \geqslant a_{\Omega_{(s)}}(v, v), \tag{2.4}$$

by virtue of the lower semicontinuity of the mapping $a_{\Omega_{(s)}}\colon v \to a_{\Omega(s)}(v, v)$, $v \in W(\Omega(s))$. Letting $s \to \infty$ in (2.4) and using (A3) we arrive at the assertion of the theorem. □

The main result of this section is

Theorem 2.3. *Let* (A1)–(A7) *be satisfied. Then there exists a constant* $\tilde{\beta} > 0$ *independent of* $\Omega \in \mathcal{O}_\varepsilon$ *such that*

$$a_\Omega(v,v) + \sum_{i=1}^{\kappa} [l_\Omega^{(i)}(v)]^2 \geqslant \tilde{\beta} \|v\|_{1,\Omega}^2 \ \ \forall v \in W(\Omega), \ \ \forall \Omega \in \mathcal{O}_\varepsilon. \tag{2.5}$$

P r o o f (by contradiction). Let us suppose that (2.5) is not true. Then for any $k \in \mathbb{N}$ there exists $\Omega_k \in \mathcal{O}_\varepsilon$ and $v_k \in W(\Omega_k)$ such that

$$a_{\Omega_k}(v_k, v_k) + \sum_{i=1}^{\kappa} [l_{\Omega_k}^{(i)}(v_k)]^2 \leqslant 1/k \|v_k\|_{1,\Omega_k}^2. \tag{2.6}$$

We may suppose that $\|v_k\|_{1,\Omega_k} = 1 \ \forall k \in \mathbb{N}$. In view of Theorems 2.1 and 2.2 we may also suppose that

$$\Omega_k \xrightarrow{h} \Omega \in \mathcal{O}_\varepsilon, \quad \partial\Omega_k \xrightarrow{h} \partial\Omega, \quad k \to \infty \tag{2.7}$$

and

$$p_{\Omega_k}(v_k) \rightharpoonup v \ \text{ in } \ H^1(D, R^d), \ \ v_{|\Omega} \in W(\Omega) \tag{2.8}$$

(otherwise we pass to appropriate subsequences). From (2.6) we have that

$$a_{\Omega_k}(v_k, v_k) \to 0, \quad l_{\Omega_k}^{(i)}(v_k) \to 0 \ \ \forall i = 1, \ldots, \kappa. \tag{2.9}$$

Lemma 2.1, together with (2.7), (2.8) and (A6) yield

$$a_\Omega(v,v) = 0 \Leftrightarrow v_{|\Omega} \in \mathcal{W}(\Omega)$$

and $l_\Omega^{(i)}(v) = 0 \ \ \forall i = 1, \ldots, \kappa$ so that $v = 0$ in Ω. On the other hand,

$$a_{\Omega_k}(v_k, v_k) + \|v_k\|_{0,\Omega_k}^2 \geqslant \beta \ \ \forall k \in \mathbb{N}$$

making use of (A4) so that

$$\|v_k\|_{0,\Omega_k}^2 \geqslant \beta/2$$

for k sufficiently large as follows from (2.9). From (2.3) and compactness of the embedding of $H^1(\Omega, \mathbb{R}^d)$ into $L^2(\Omega, \mathbb{R}^d)$ we easily obtain that

$$\|v\|_{0,\Omega}^2 = \lim_{k\to\infty} \|v_k\|_{0,\Omega_k}^2 \geqslant \beta/2,$$

which leads to a contradiction with $v = 0$ in Ω. $\square$

Consequence 2.2. *Let all the assumptions of Theorem* 2.3 *be satisfied and let*

$$V(\Omega) = \{v \in W(\Omega) \mid l_\Omega^{(i)}(v) = 0 \ \ \forall i = 1, \ldots, \kappa\}.$$

Then (2.5) *implies that*

$$a_\Omega(v, v) \geqslant \tilde{\beta}\|v\|_{1,\Omega}^2 \ \ \forall v \in V(\Omega), \ \ \forall \Omega \in \mathcal{O}_\varepsilon,$$

with a constant $\tilde{\beta} > 0$ *independent of* $\Omega \in \mathcal{O}_\varepsilon$.

Next we present several applications of Theorem 2.3 and of Consequence 2.2. Let C_0, C_1 be two positive constants and let

$$\mathcal{U}_{\rm ad} = \{\alpha \in C^{0,1}([a,b]) \mid 0 \leqslant \alpha(x_1) \leqslant C_0 \ \ \forall x_1 \in [a,b], \ \ |\alpha'(x_1)| \leqslant C_1 \ \text{ a.e. in } (a,b)\}$$

be the set of *uniformly bounded* and *uniformly Lipschitz* functions defined in an interval $[a,b]$. With any $\alpha \in \mathcal{U}_{\rm ad}$ the domain $\Omega(\alpha) \subset \mathbb{R}^2$ will be associated:

$$\Omega(\alpha) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (a,b), \alpha(x_1) < x_2 < \overline{\gamma}\},$$

where $\overline{\gamma} > 0$ is a given number (see Fig. 2.1); let $\Gamma(\alpha)$ be the graph of the function $\alpha \in \mathcal{U}_{\rm ad}$.

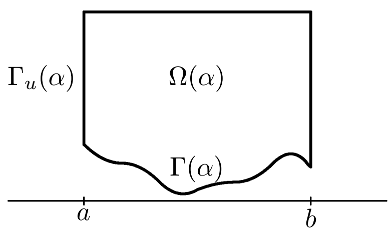

Figure 2.1.

The system $\mathcal{O}_\varepsilon = \{\Omega(\alpha) \mid \alpha \in \mathcal{U}_{\rm ad}\}$ satisfies the ε-cone property in the sense of Definition 2.1 for some $\varepsilon := \varepsilon(C_0, C_1) > 0$.

E x a m p l e 2.1. Let $W(\alpha) = H^1(\Omega(\alpha)) := H^1(\Omega(\alpha), \mathbb{R}^1)$, $\Omega(\alpha) \in \mathcal{O}_\varepsilon$ (as above) and $a_\alpha(v,v) = |v|_{1,\Omega(\alpha)}^2$ be the classical seminorm in $H^1(\Omega(\alpha))$. The convergence $\Omega(\alpha_k) \xrightarrow{h} \Omega(\alpha)$ in $\mathcal{O}_\varepsilon$ reduces to uniform convergence of $\{\alpha_k\}$ to α in $[a,b]$. Further,

$$\mathcal{W}(\alpha) = \ker a_\alpha = \mathbb{R}^1$$

and

$$\dim \mathcal{W}(\alpha) = 1 \quad \text{for any } \ \alpha \in \mathcal{U}_{\rm ad}.$$

Let $\Gamma_u(\alpha) = \{(x_1, x_2) \mid x_1 = a;\ \alpha(a) < x_2 < \overline{\gamma}\}$ be the left vertical side of $\partial\Omega(\alpha)$ and define

$$l_\alpha^{(1)}(v) := \int_{\Gamma_u(\alpha)} v \,\mathrm{d}x_2 = \int_{\alpha(a)}^{\bar{\gamma}} v(a, x_2) \,\mathrm{d}x_2, \quad v \in W(\alpha). \tag{2.10}$$

It is easy to verify that all the assumptions (A1)–(A7) are satisfied in this case so that the assertion of Theorem 2.3 reads as follows:

$$\exists \tilde{\beta} = \text{const} > 0\colon\ |v|^2_{1,\Omega(\alpha)} + \left(\int_{\Gamma_u(\alpha)} v \,\mathrm{d}x_2\right)^2 \geqslant \tilde{\beta}\|v\|^2_{1,\Omega(\alpha)} \tag{2.11}$$

holds for any $v \in H^1(\Omega(\alpha))$ and any $\alpha \in \mathcal{U}_{\mathrm{ad}}$. In particular, let

$$V(\alpha) = \{v \in H^1(\Omega(\alpha)) \mid v = 0 \text{ on } \Gamma_u(\alpha)\}.$$

Then

$$|v|^2_{1,\Omega(\alpha)} \geqslant \tilde{\beta}\|v\|^2_{1,\Omega(\alpha)} \quad \forall v \in V(\alpha),\ \forall \alpha \in \mathcal{U}_{\mathrm{ad}} \tag{2.12}$$

is the generalized *Friedrichs inequality* uniform with respect to $\alpha \in \mathcal{U}_{\mathrm{ad}}$. Another choice of $l_\alpha^{(1)}$ satisfying the previous assumptions is:

$$l_\alpha^{(1)}(v) = \int_{\Omega(\alpha)} v \,\mathrm{d}x_1 \,\mathrm{d}x_2$$

leading to the well-known *Poincaré inequality* uniform with respect to $\alpha \in \mathcal{U}_{\mathrm{ad}}$.

Example 2.2. Let $\mathcal{U}_{\mathrm{ad}}$, $\Omega(\alpha)$, $\Gamma_u(\alpha)$ be the same as in the previous example and

$$W(\alpha) = \{v = (v_1, v_2) \in H^1(\Omega(\alpha); \mathbb{R}^2) \mid v_1 = 0 \text{ on } \Gamma_u(\alpha)\},$$
$$a_\alpha(u, v) = \int_{\Omega(\alpha)} \varepsilon_{ij}(u)\varepsilon_{ij}(v) \,\mathrm{d}x_1 \,\mathrm{d}x_2,\ \ v \in W(\alpha),$$

where $\varepsilon_{ij}(v) = \frac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$, $i, j = 1, 2$ is the linearized strain tenzor. Then

$$\mathcal{W}(\alpha) = \ker a_\alpha = \{v\colon\ \Omega(\alpha) \to \mathbb{R}^2 \mid v = (0, a),\ \ a \in \mathbb{R}^1\}.$$

A non-trivial result says (see [9]) that the so-called second Korn inequality

$$a_\alpha(v, v) + \|v\|^2_{0,\Omega(\alpha)} \geqslant \beta\|v\|^2_{1,\Omega(\alpha)} \ \ \forall v \in W(\alpha)$$

is satisfied with a constant $\beta > 0$ which *does not depend* on $\alpha \in \mathcal{U}_{\rm ad}$. A possible choice of $l_\alpha^{(1)}$ recognizing the zero element of $\mathcal{W}(\alpha)$ is

$$l_\alpha^{(1)}(v) = \int_{\Gamma_u(\alpha)} v_2 \, \mathrm{d}x_2. \tag{2.13}$$

From Theorem 2.3 it follows that

$$a_\alpha(v,v) + \Big(\int_{\Gamma_u(\alpha)} v_2 \, \mathrm{d}x_2\Big)^2 \geqslant \tilde\beta \|v\|^2_{1,\Omega(\alpha)} \ \ \forall v \in W(\alpha), \ \ \forall \alpha \in \mathcal{U}_{\rm ad},$$

and $\tilde\beta > 0$ is independent of $\alpha \in \mathcal{U}_{\rm ad}$. Let

$$V(\alpha) = \{v \in W(\alpha) \mid l_\alpha^{(1)}(v) = 0\}. \tag{2.14}$$

Then

$$a_\alpha(v,v) \geqslant \tilde\beta \|v\|^2_{1,\Omega(\alpha)} \ \ \forall v \in V(\alpha), \ \ \forall \alpha \in \mathcal{U}_{\rm ad}. \tag{2.15}$$

An alternative choice of $l_\alpha^{(1)}$ satisfying especially (A6) is

$$l_\alpha^{(1)}(v) = \int_a^b v_2(x_1, \alpha(x_1)) \, \mathrm{d}x_1, \quad v \in W(\alpha) \tag{2.16}$$

(for the proof see [2], [3]) so that (2.15) holds true with $l_\alpha^{(1)}$ defined by (2.16). This result will be used in the next section.

3. Contact shape optimization with semicoercive state problems

In this section we will study an optimization problem for a system of elastic bodies whose shapes are as in Fig. 2.1. In particular the symbols $\mathcal{U}_{\rm ad}$, $\Omega(\alpha)$, $\Gamma_u(\alpha)$ and $\Gamma(\alpha)$ have the same meaning as at the end of Section 2. We suppose that the bodies are unilaterally supported along $\Gamma(\alpha)$ by a rigid, frictionless foundation $\mathbb{R}^2_- = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leqslant 0\}$, i.e. the following unilateral conditions are prescribed on $\Gamma(\alpha)$:

$$\begin{cases} u_2(x_1, \alpha(x_1)) \geqslant -\alpha(x_1) \ \ \forall x_1 \in (a,b), \\ T_2(u) := \sigma_{2j}(u)\nu_j \geqslant 0, \quad T_2(u)(u_2 + \alpha) = 0, \\ T_1(u) := \sigma_{1j}(u)\nu_j = 0, \end{cases} \tag{3.1}$$

where $\nu = (\nu_1, \nu_2)$ denotes the unit outward normal vector to $\partial\Omega$, the matrix $\sigma(u) = \{\sigma_{ij}(u)\}_{i,j=1}^2$ stands for the stress tensor which is related to the strain tensor $\varepsilon(u) = \{\varepsilon_{ij}(u)\}_{i,j=1}^2$ by means of a linear Hooke law

$$\sigma_{ij}(u) = c_{ijkl}\varepsilon_{kl}(u). \tag{3.2}$$

Elasticity coefficients $c_{ijkl} \in L^\infty(\hat{\Omega})$ satisfy the classical symmetry and ellipticity conditions

$$\begin{cases} c_{ijkl} = c_{jikl} = c_{klij} & \text{a.e. in } D \\ c_{ijkl}\xi_{ij}\xi_{kl} \geqslant m\xi_{ij}\xi_{ij} & \forall \xi_{ij} = \xi_{ji} \in \mathbb{R}^1 \\ & \text{for a.a. } x \in D, \end{cases} \tag{3.3}$$

where $m > 0$ is a positive constant and $D = (a, b) \times (0, \overline{\gamma})$. On $\Gamma_u(\alpha)$ (the left vertical side of $\partial\Omega(\alpha)$) the zero displacement in the x_1-direction will be prescribed:

$$u_1 = 0 \quad \text{on} \quad \Gamma_u(\alpha). \tag{3.4}$$

On the remaining part $\Gamma_P(\alpha) \subset \partial\Omega(\alpha)$, surface tractions $P \in L^2(\partial D; \mathbb{R}^2)$ are applied (notice that $\Gamma_P(\alpha) \subset \partial D \ \forall \alpha \in \mathcal{U}_{\mathrm{ad}}$):

$$\sigma_{ij}(u)\nu_j = P_i, \quad i = 1, 2 \text{ on } \Gamma_P(\alpha). \tag{3.5}$$

Finally, the body is subject to body forces $F \in L^2(D, \mathbb{R}^2)$.

By the *classical solution* of the Signorini problem we mean a displacement field $u = (u_1, u_2)$ satisfying (3.1), (3.2), (3.4), (3.5) and the equilibrium equations

$$\frac{\partial \sigma_{ij}(u)}{\partial x_j} + F_i = 0, \quad i = 1, 2 \text{ in } \Omega(\alpha). \tag{3.6}$$

In order to give the variational formulation, we introduce the following set of kinematically admissible displacements

$$K(\alpha) = \{v = (v_1, v_2) \in W(\alpha) \mid v_2(x_1, \alpha(x_1)) \geqslant -\alpha(x_1) \text{ for a.a. } x_1 \in (a, b)\},$$

where $W(\alpha)$ is the space from Example 2.2 on $\Omega := \Omega(\alpha)$.

The variational formulation of our problem reads as follows:

$$(\mathcal{P}(\alpha)) \qquad \text{Find } u := u(\alpha) \in K(\alpha)\colon\ J_\alpha(u) \leqslant J_\alpha(v) \ \forall v \in K(\alpha)$$

or equivalently:

$$(\mathcal{P}(\alpha))' \quad \text{Find } u := u(\alpha) \in K(\alpha)\colon\ a_\alpha(u, v - u) \geqslant L_\alpha(v - u) \ \forall v \in K(\alpha),$$

where

$$J_\alpha(v) = \frac{1}{2} a_\alpha(v,v) - L_\alpha(v),$$
$$a_\alpha(v,v) = \int_{\Omega(\alpha)} \sigma_{ij}(v)\varepsilon_{ij}(v)\,\mathrm{d}x_1\,\mathrm{d}x_2,$$
$$L_\alpha(v) = \int_{\Omega(\alpha)} F_i v_i\,\mathrm{d}x_1\,\mathrm{d}x_2 + \int_{\Gamma_P(\alpha)} P_i v_i\,\mathrm{d}s.$$

Since the bilinear form a_α is only *semicoercive* on $W(\alpha)$ (see Example 2.2), the existence of solutions to $(\mathcal{P}(\alpha))$ will be guaranteed under an additional assumption on the resultant of applied forces.

Theorem 3.1. *Let $\alpha \in \mathcal{U}_{\rm ad}$ be fixed and suppose that*

$$r_\alpha := \int_{\Omega(\alpha)} F_2\,\mathrm{d}x_1\,\mathrm{d}x_2 + \int_{\Gamma_P(\alpha)} P_2\,\mathrm{d}s < 0. \tag{3.7}$$

Then $(\mathcal{P}(\alpha))$ (or $(\mathcal{P}(\alpha))'$) has a unique solution.

Proof. For the proof we refer to [4]. □

Next we introduce a cost functional $I\colon (\alpha, y) \to \mathbb{R}^1$, $\alpha \in \mathcal{U}_{\rm ad}$, $y \in W(\alpha)$, and define an optimal shape design problem:

$$(\mathbb{P}) \qquad \text{Find } \alpha^* \in \mathcal{U}_{\rm ad} \text{ such that } I(\alpha^*, u(\alpha^*)) \leqslant I(\alpha, u(\alpha)) \;\; \forall \alpha \in \mathcal{U}_{\rm ad},$$

with $u(\alpha) \in K(\alpha)$ being a solution to the semicoercive $(\mathcal{P}(\alpha))$.

The basic step in the existence analysis for $(\mathbb{P})$ consists in showing that solutions of $(\mathcal{P}(\alpha))$ depend continuously on shape variations. In our particular case, the continuous dependence reads as follows: if $\alpha_k \rightrightarrows \alpha$ (uniformly) in $[a,b]$, $\alpha_k, \alpha \in \mathcal{U}_{\rm ad}$ and $u_k := u(\alpha_k)$ are solutions to $(\mathcal{P}(\alpha_k))$ then there exists $\tilde{u} \in H^1(D, \mathbb{R}^2)$ such that

$$p_{\alpha_k} u_k \rightharpoonup \tilde{u} \text{ in } H^1(D, \mathbb{R}^2), \quad k \to \infty \tag{3.8}$$

and $u := \tilde{u}_{|\Omega(\alpha)}$ solves $(\mathcal{P}(\alpha))$. The symbol p_{α_k} stands for the uniform extension of functions from $\Omega(\alpha_k)$ onto D, introduced in Section 2.

The first step for proving (3.8) is to show that the sequence $\{u_k\}$ is bounded:

$$\exists\, C := \text{const} > 0\colon \; \|u_k\|_{1,\Omega(\alpha_k)} \leqslant C \;\; \forall k \in \mathbb{N}. \tag{3.9}$$

In coercive cases, (3.9) is a consequence of Korn's inequality which is valid in the space of virtual displacements. Unfortunately, Korn's inequality is no longer true in $W(\alpha)$. To prove (3.9) we use the approach from [4].

We start with

Lemma 3.1. *Let there exist a negative* $\overline{r}$ *such that*

$$r_\alpha \leqslant \overline{r} \ \ \forall \alpha \in \mathcal{U}_{\text{ad}}, \tag{3.10}$$

where r_α *is defined by* (3.7). *Further, let* $v_k \in K(\alpha_k)$, $\alpha_k \in \mathcal{U}_{\text{ad}}$ *be such that* $\|v_k\|_{1,\Omega(\alpha_k)} \to \infty$, $k \to \infty$. *Then*

$$\lim_{k\to\infty} J_{\alpha_k}(v_k) \to \infty, \quad k \to \infty. \tag{3.11}$$

P r o o f. Recall that

$$\begin{aligned} W(\alpha) &= \{v \in H^1(\Omega(\alpha); \mathbb{R}^2) \mid v_1 = 0 \text{ on } \Gamma_u(\alpha)\};\\ \mathcal{W}(\alpha) &= \{v\colon \Omega(\alpha) \to \mathbb{R}^2 \mid v = (0,a), a \in \mathbb{R}^1\}\end{aligned}$$

and define $l_\alpha^{(1)}$ by means of (2.16). As we already know there exists a constant $\tilde{\beta} > 0$ independent of $\alpha \in \mathcal{U}_{\text{ad}}$ such that

$$a_\alpha(v,v) \geqslant \tilde{\beta}\|v\|^2_{1,\Omega(\alpha)} \ \ \forall v \in V(\alpha), \ \forall \alpha \in \mathcal{U}_{\text{ad}}, \tag{3.12}$$

where

$$V(\alpha) = \left\{v \in W(\alpha) \;\middle|\; \int_a^b v_2(x_1, \alpha(x_1))\,\mathrm{d}x_2 = 0\right\}.$$

Any function $v \in W(\alpha)$ can be decomposed and written as the sum

$$v = P_\alpha v + y_\alpha,$$

where $P_\alpha v \in V(\alpha)$ and $y_\alpha \in \mathcal{W}(\alpha)$. Indeed, taking $y_\alpha = (0, c_\alpha) \in \mathcal{W}(\alpha)$, where $c_\alpha = l_\alpha^{(1)}(v)(b-a)^{-1}$, we see that $P_\alpha v := v - y_\alpha$ belongs to $V(\alpha)$. Applying this decomposition to $\{v_k\}$, $v_k \in V(\alpha_k)$ and using (3.12) we have ($y_k := y_{\alpha_k}$)

$$\begin{aligned} J_{\alpha_k}(v_k) = J_{\alpha_k}(P_{\alpha_k}v_k + y_k) &= \tfrac{1}{2}\, a_{\alpha_k}(P_{\alpha_k}v_k, P_{\alpha_k}v_k) - L_{\alpha_k}(P_{\alpha_k}v_k) - L_{\alpha_k}(y_k)\\ &\geqslant \tilde{\beta}/2\|P_{\alpha_k}v_k\|^2_{1,\Omega(\alpha_k)} - c\|P_{\alpha_k}v_k\|_{1,\Omega(\alpha_k)} - L_{\alpha_k}(y_k),\end{aligned} \tag{3.13}$$

where c is a positive constant which also does not depend on $\alpha_k \in \mathcal{U}_{\text{ad}}$. Since $L_{\alpha_k}(y_k) = c_{\alpha_k} r_{\alpha_k}$, we have

$$J_{\alpha_k}(v_k) \geqslant \tilde{\beta}/2\|P_{\alpha_k}v_k\|^2_{1,\Omega(\alpha_k)} - c\|P_{\alpha_k}v_k\|_{1,\Omega(\alpha_k)} - c_{\alpha_k} r_{\alpha_k}. \tag{3.14}$$

Let $\|v_k\|_{1,\Omega(\alpha_k)} \to \infty$. Then either $\|P_{\alpha_k} v_k\|_{1,\Omega(\alpha_k)} \to \infty$ or $\|y_k\|_{1,\Omega(\alpha_k)} \to \infty$. A direct calculation shows that

$$\|y_k\|_{1,\Omega(\alpha_k)} = |c_{\alpha_k}| \, (\operatorname{meas} \Omega(\alpha_k))^{1/2} = \frac{|l^{(1)}_{\alpha_k}(v_k)|}{b-a} \, (\operatorname{meas} \Omega(\alpha_k))^{1/2}. \tag{3.15}$$

Considering $v_k \in K(\alpha_k)$ we see that

$$l^{(1)}_{\alpha_k}(v_k) = \int_a^b v_k(x_1, \alpha_k(x_1)) \, \mathrm{d}x_1 \geqslant - \int_a^b \alpha_k(x_1) \, \mathrm{d}x_1 \geqslant -(b-a)\overline{\gamma}, \tag{3.16}$$

i.e. $\{l^{(1)}_{\alpha_k}(v_k)\}_{k=1}^{\infty}$ is bounded from below.

Let $\|y_k\|_{1,\Omega(\alpha_k)} \to \infty$. Then from (3.15) and (3.16) it follows that $c_{\alpha_k} \to \infty$, $k \to \infty$. From this, (3.10) and (3.14) we conclude that $J_{\alpha_k}(v_k) \to \infty$ as $k \to \infty$. The same property holds if $\|P_k v_k\|_{1,\Omega(\alpha_k)} \to \infty$ and $\{\|y_k\|_{1,\Omega(\alpha_k)}\}$ is bounded. This is a trivial consequence of (3.14). □

The main result of this section is

Theorem 3.2. *Let* (3.10) *be satisfied. Then there exists a constant* $c > 0$ *which does not depend on* $\alpha \in \mathcal{U}_{\mathrm{ad}}$ *and such that*

$$\|u(\alpha)\|_{1,\Omega(\alpha)} \leqslant c \;\; \forall \alpha \in \mathcal{U}_{\mathrm{ad}}, \tag{3.17}$$

where $u(\alpha) \in K(\alpha)$ *is the solution of* $(\mathcal{P}(\alpha))$.

P r o o f. The definition of $u(\alpha)$ yields

$$J_\alpha(u(\alpha)) \leqslant J_\alpha(\theta) = 0 \;\; \forall \alpha \in \mathcal{U}_{\mathrm{ad}},$$

where $\theta = (0,0)$ is the zero element of $W(\alpha)$, which belongs to $K(\alpha)$ for every $\alpha \in \mathcal{U}_{\mathrm{ad}}$. Therefore $\{J_\alpha(u(\alpha))\}_{\alpha \in \mathcal{U}_{\mathrm{ad}}}$ is bounded from above. From this and Lemma 3.1, the assertion of the theorem follows. □

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