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EIGENVALUES AND EIGENFUNCTIONS OF THE LAPLACE OPERATOR ON AN EQUILATERAL TRIANGLE FOR THE DISCRETE CASE*

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Abstract. A discretized boundary value problem for the Laplace equation with the Dirichlet and Neumann boundary conditions on an equilateral triangle with a triangular mesh is transformed into a problem of the same type on a rectangle. Explicit formulae for all eigenvalues and all eigenfunctions are given.

Keywords: discrete Laplace operator, discrete boundary value problem, eigenvalues, eigenfunctions

MSC 2000: 35J05, 65N25, 65N06

0. Introduction

In the previous paper [1], we have given formulae for eigenfunctions and eigenvalues of the Laplace operator on an equilateral triangle in the continuous case. In this paper we show that the eigenvectors for the discretization on a triangular mesh are given by the same formulae and the eigenvalues converge to the continuous ones when the mesh is refined. For details of some manipulations we refer the reader to [1].

Let T be a closed equilateral triangle with vertices $\big(\frac{-1}{\sqrt{3}},0\big),\big(\frac{1}{\sqrt{3}},0\big),(0,1)$. Its altitude is equal to one and its side is equal to $\frac{2}{\sqrt{3}}$.

For a given integer N, we define $h' = 1/N$, the meshsize $h = 2h'/\sqrt{3}$ and we introduce a triangular mesh T_h on T, i.e. the set of points $V_{ij} = (ih/2, jh')$, $j = 0,\dots,N$; $|i| \leqslant N-j$, $i+j$ of the same parity as N. The mesh of all interior points of T_h will be denoted by T_h°.

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Let R be the rectangle $[0,\sqrt{3}]\times[0,1]$. On this rectangle, we introduce a rectangular mesh R_h^r as the set of points $V_{ij}^r = (ih/2, jh')$, $j = 0,\dots,N$; $i = 0,\dots 3N$, and a triangular mesh R_h as the set of those points from R_h^r where $i+j$ is of the same parity as N. The mesh of all interior points of R_h will be denoted by R_h°.

Further, let $T_1 = T\cap R$ and $T_{1h} = T_h \cap R_h$. We divide the mesh T_{1h} into four parts: the mesh of the interior meshpoints T_{1h}°, the mesh of the interior part of the vertical boundary, i.e. the meshpoints (x,y) lying on the open segment $x = 0$, $y \in (0,1)$, denoted by T_{1h}^v, the meshpoints at the endpoints of the vertical boundary, i.e. the meshpoints coinciding with the points $(0,0)$ and $(0,1)$, denoted by T_{1h}^c, and the rest of the boundary T_{1h}^r.

In the following table ($p(N) = 2$ for N even and $p(N) = 1$ for N odd), the numbers of points of the meshes are summarized:

mesh	number of points
R_h^r	$3N^2+4N+1$
R_h	$(3N^2+4N+p(N))/2$
R_h°	$(3N^2-4N+p(N))/2$
T_h	$(N^2+3N+2)/2$
T_h°	$(N^2-3N+2)/2$
T_{1h}	$(N^2+4N+2+p(N))/4$
T_{1h}°	$(N^2-4N+2+p(N))/4$
T_{1h}^v	$(N-p(N))/2$
T_{1h}^c	$p(N)$
T_{1h}^r	$(3N-p(N))/2$

In what follows we will use the prolongation of the vector v defined on T_{1h} onto R_h so that we prolong it successively by symmetry or by skew symmetry with respect to the dotted lines (see Fig. 1).

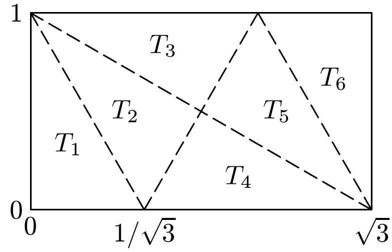

Figure 1. The triangle T_1 and its images T_i, $i = 2,\dots,6$, in R.

Thus we introduce transformations K_i of the triangle T_1 onto the triangles T_i by the equations

$$(1)\qquad \begin{aligned} x_1 &= \xi, & x_2 &= \tfrac12\bigl(-\xi-\sqrt3\eta+\sqrt3\bigr), & x_3 &= \tfrac12\bigl(\xi-\sqrt3\eta+\sqrt3\bigr),\\ y_1 &= \eta, & y_2 &= \tfrac12\bigl(-\sqrt3\xi+\eta+1\bigr), & y_3 &= \tfrac12\bigl(\sqrt3\xi+\eta+1\bigr), \end{aligned}$$

$$\begin{aligned} x_4 &= \tfrac12\bigl(-\xi+\sqrt3\eta+\sqrt3\bigr), & x_5 &= \tfrac12\bigl(\xi+\sqrt3\eta+\sqrt3\bigr), & x_6 &= \sqrt3-\xi,\\ y_4 &= \tfrac12\bigl(-\sqrt3\xi-\eta+1\bigr), & y_5 &= \tfrac12\bigl(\sqrt3\xi-\eta+1\bigr), & y_6 &= 1-\eta. \end{aligned}$$

The meshpoints of the triangular mesh are transformed by every K_i again to meshpoints. The corresponding mesh on the triangle T_i will be denoted by T_{ih}. We thus have $B_i = K_i B$ where $B_i = (x_i, y_i) \in T_{ih}$ for $B = (\xi, \eta) \in T_{1h}$.

The prolongation $\mathscr{P}v$ of a vector v from T_{1h} onto R_h is defined by

$$(2)\qquad \mathscr{P}v(B_i) = c_i v(B) \quad \text{on} \quad T_i,$$

where c_i (equal to $+1$ or -1) are appropriately chosen. The choice will be specified later in dependence on the type of the boundary conditions.

Let further u be a vector defined on T_{ih} or R_h. We denote by $\mathscr{H}u$ the boundary modification of u obtained by multiplication of the boundary values of u by coefficients as follows:

on the straight parts of the boundary by $\frac12$,

at the vertex of angle $\frac{\pi}{m}$ by $\frac{1}{2m}$.

R e m a r k. The multiplier is the ratio of the given angle to the angle of 2π.

Now, we define the transformation $\mathscr{F}$, which we call the folding, from R_h onto T_{1h} as follows: $\mathscr{F}v(B) = \sum_{i=1}^{6} c_i v(B_i)$, where $B = K_i^{-1} B_i$.

Lemma 1. *The equality*

$$\sum_{B\in T_{1h}} u(B)\mathscr{H}\mathscr{F}v(B) = \sum_{A\in R_h} \mathscr{P}u(A)\mathscr{H}v(A)$$

holds.

R e m a r k. On the right-hand side, the modification $\mathscr{H}$ is applied to a vector defined on R_h.

P r o o f. We have the equalities

$$\sum_{B\in T_{1h}} u(B)\mathscr{H}\mathscr{F}v(B) = \sum_{B\in T_{1h}} u(B)\sum_{i=1}^{6} c_i\mathscr{H}v(K_iB) = \sum_{i=1}^{6} c_i \sum_{B\in T_{1h}} u(B)\mathscr{H}v(K_iB)$$
$$= \sum_{i=1}^{6} c_i \sum_{B\in T_{ih}} u(K_i^{-1}B_i)\mathscr{H}v(B_i) = \sum_{i=1}^{6}\sum_{B\in T_{ih}} \mathscr{P}u(B_i)\mathscr{H}v(B_i)$$
$$= \sum_{A\in R_h} \mathscr{P}u(A)\mathscr{H}v(A).$$

The last equality is a consequence of the fact that on the interfaces of the triangles T_i the values are added. □

We will use the discretization of the Laplace operator on the mesh R_h^r with the usual five-point scheme with appropriate modifications on the boundary for the Dirichlet or Neumann boundary conditions.

Similarly, we discretize the Laplace operator on R_h with the seven-point scheme on a triangular mesh. We will use the operator $-\Delta$ because of its positiveness. We have

$$-\Delta_h u(V_{i,j}) = \frac{2}{3h^2}\big[6\,u(V_{i,j}) - u(V_{i-2,j}) - u(V_{i+2,j}) - u(V_{i-1,j+1})$$
$$- u(V_{i+1,j+1}) - u(V_{i-1,j-1}) - u(V_{i+1,j-1})\big].$$

At the points adjacent to the boundary or on the boundary, the scheme is modified by the skew-symmetric prolongation for the Dirichlet boundary condition and the symmetric prolongation for the Neumann boundary condition. For these cases, we thus have the following stencils (apart from the factor $\frac{2}{3h^2}$) for the straight parts of the boundary (Fig. 2) and for the parts of the boundary near vertices (Fig. 3).

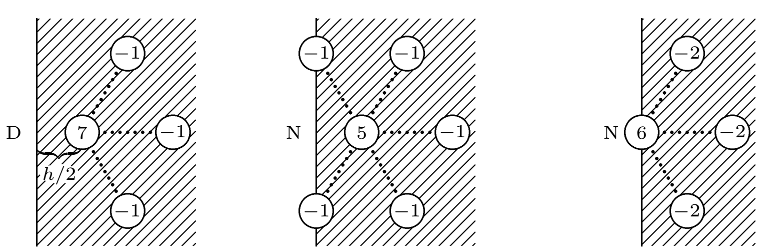

Figure 2. Discretization stencils near the straight boundary for Dirichlet (D) and Neumann (N) boundary condition.

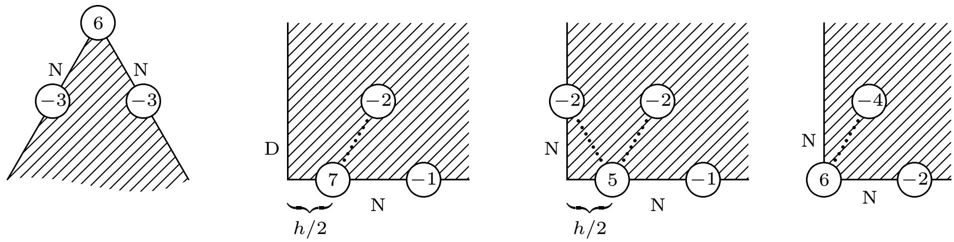

Figure 3. Discretization stencils near the vertices for Dirichlet (D) and Neumann (N) boundary condition and their combination.

The construction of the stencils for other possible cases is left to the reader.

1. Dirichlet boundary conditions

We will consider separately the eigenfunctions on T_h with skew symmetry or symmetry with respect to the y-axis.

First, for the skew-symmetric case, we recall that the values of the functions $\sin \frac{k\pi x}{\sqrt{3}} \sin l\pi y$, $k = 1, \dots, 3N-1$, $l = 1, \dots, N-1$ on R_h^r are eigenvectors of the five-point Laplace operator discretization on R_h^r with Dirichlet boundary conditions.

The values of the functions $u_{k,l}(x,y) = \sin \frac{k\pi x}{\sqrt{3}} \sin l\pi y$ for $k = 1, \dots, 3l-1$, $l = 1, \dots, N-1$, and $k = 3l$, $l = 1, \dots, [N/2]$ ([ ] denotes the integer part) on R_h are the eigenvectors of the seven-point Laplace operator discretization on the triangular mesh R_h with Dirichlet boundary conditions. This fact is easily established by direct calculation. Their number is equal to the number of points of R_h°.

We will show that these eigenvectors form an orthogonal system on R_h and, because their number is equal to the number of points of R_h°, that they thus form a complete system of eigenvectors. First let us note that the values of the functions $u_{k',l'}(x,y) = \sin \frac{k'\pi x}{\sqrt{3}} \sin l'\pi y$, where $k' = 3N-k$, $l' = N-l$, are equal to $(-1)^N u_{k,l}(x,y)$ on R_h. For the values on $R_h^r \backslash R_h$ one has $u_{k,l}(x,y) = -(-1)^N u_{k',l'}(x,y)$. Therefore we have

$$\sum_{(x,y)\in R_h} u_{k,l}(x,y) u_{m,n}(x,y)$$
$$= \frac{1}{4} \sum_{(x,y)\in R_h^r} [u_{k,l}(x,y) + (-1)^N u_{k',l'}(x,y)][u_{m,n}(x,y) + (-1)^N u_{m',n'}(x,y)]$$

and this is equal to zero for $(k,l) \neq (m,n)$ as a consequence of the orthogonality of the eigenfunctions on R_h^r. On the other hand, for $(k,l) = (m,n)$ the value is obviously positive.

Thus, we have a complete system of eigenvectors on R_h for the case of the Dirichlet conditions on all sides of R_h.

For this skew-symmetric case we set in (2) $c_i = 1$ for $i = 1, 3, 4, 6$ and $c_i = -1$ for $i = 2, 5$. For $(x, y) \in T_{1h}$ and $k = 1, \ldots, l-1$; $l = 1, \ldots, N-1$, k and l of the same parity, we define

$$
\begin{aligned}
(3) \qquad U_{k,l}(x,y) = \mathscr{F} u_{k,l}(x,y) &= \sin \frac{k\pi x}{\sqrt{3}} \sin l\pi y \\
&- \sin \frac{k\pi}{2\sqrt{3}}(-x - \sqrt{3}y + \sqrt{3}) \sin \frac{l\pi}{2}(-\sqrt{3}x + y + 1) \\
&+ \sin \frac{k\pi}{2\sqrt{3}}(x - \sqrt{3}y + \sqrt{3}) \sin \frac{l\pi}{2}(\sqrt{3}x + y + 1) \\
&+ \sin \frac{k\pi}{2\sqrt{3}}(-x + \sqrt{3}y + \sqrt{3}) \sin \frac{l\pi}{2}(-\sqrt{3}x - y + 1) \\
&- \sin \frac{k\pi}{2\sqrt{3}}(x + \sqrt{3}y + \sqrt{3}) \sin \frac{l\pi}{2}(\sqrt{3}x - y + 1) \\
&+ \sin \frac{k\pi(\sqrt{3} - x)}{\sqrt{3}} \sin l\pi(1 - y).
\end{aligned}
$$

With the help of manipulations similar to that in [1] we simplify this expression obtaining

$$
\begin{aligned}
(4) \qquad U_{k,l}(x,y) = {} & 2 \sin \frac{k\pi x}{\sqrt{3}} \sin l\pi y \\
& - 2(-1)^{(k+l)/2} \sin \frac{\pi x}{2\sqrt{3}}\,(k + 3l) \sin \frac{\pi y}{2}\,(k - l) \\
& + 2(-1)^{(k-l)/2} \sin \frac{\pi x}{2\sqrt{3}}\,(k - 3l) \sin \frac{\pi y}{2}\,(k + l).
\end{aligned}
$$

The number of admissible pairs (k, l) is equal to the number of points of T°_{1h}.

Now, for the symmetric case, the eigenvectors of the discrete Laplace operator on R_h are the values of the functions $v_{k,l}(x,y) = \cos \frac{k\pi x}{\sqrt{3}} \sin l\pi y$, $k = 0, \ldots, 3l-1$, $l = 1, \ldots, N-1$ and $k = 3l$, $l = 1, \ldots, [(N-1)/2]$. For $k' = 3N - k$, $l' = N - l$, we now have $v_{k',l'}(x,y) = -(-1)^N v_{k,l}(x,y)$ on R_h. Similarly to the previous case, we prove that these functions are for $(k,l) \neq (m,n)$ orthogonal on R_h and that they are nonzero. Instead of the usual scalar product, one must consider the sum

$$
\sum_{(x,y) \in R_h} v_{k,l}(x,y) \mathscr{H} v_{m,n}(x,y)
$$

because of the presence of cosines. We thus have a complete system of eigenvectors on R_h for the Dirichlet boundary conditions on the horizontal sides and Neumann boundary conditions on the vertical sides.

We prolong now the vector v defined on T_{1h} and with zero components on $T^r_{1h} \cup T^c_{1h}$. The prolongation $\mathscr{P}v$ is given by (2) with $c_i = 1$ for $i = 1,4,5$ and $c_i = -1$ for $i = 2,3,6$ and the corresponding folding $\mathscr{F}$ uses the same coefficients.

For $(x,y) \in T_{1h}$ and $k = 0,\dots,l-1$, $l = 1,\dots,N-1$, k and l of the same parity, we define

$$
\begin{aligned}
(5)\quad V_{k,l}(x,y) &= \mathscr{F}v_{k,l}(x,y) \\
&= \cos\frac{k\pi x}{\sqrt{3}}\sin l\pi y - \cos\frac{k\pi}{2\sqrt{3}}(-x-\sqrt{3}y+\sqrt{3})\sin\frac{l\pi}{2}(-\sqrt{3}x+y+1) \\
&\quad - \cos\frac{k\pi}{2\sqrt{3}}(x-\sqrt{3}y+\sqrt{3})\sin\frac{l\pi}{2}(\sqrt{3}x+y+1) \\
&\quad + \cos\frac{k\pi}{2\sqrt{3}}(-x+\sqrt{3}y+\sqrt{3})\sin\frac{l\pi}{2}(-\sqrt{3}x-y+1) \\
&\quad + \cos\frac{k\pi}{2\sqrt{3}}(x+\sqrt{3}y+\sqrt{3})\sin\frac{l\pi}{2}(\sqrt{3}x-y+1) \\
&\quad - \cos\frac{k\pi(\sqrt{3}-x)}{\sqrt{3}}\sin l\pi(1-y).
\end{aligned}
$$

Proceeding in the same way as before, we find the expression

$$
\begin{aligned}
(6)\qquad V_{k,l}(x,y) &= 2\cos\frac{k\pi x}{\sqrt{3}}\sin l\pi y \\
&\quad + 2(-1)^{(k+l)/2}\cos\frac{\pi x}{2\sqrt{3}}(k+3l)\sin\frac{\pi y}{2}(k-l) \\
&\quad - 2(-1)^{(k-l)/2}\cos\frac{\pi x}{2\sqrt{3}}(3l-k)\sin\frac{\pi y}{2}(k+l).
\end{aligned}
$$

For this case, the number of admissible pairs (k,l) is equal to the number of the points of $T^\circ_{1h} \cup T^v_{1h}$.

It is easy to prove by direct calculation that the functions $U_{k,l}$ and $V_{k,l}$ are eigenfunctions of the operator $-\Delta_h$, the corresponding eigenvalues being

$$
(7)\qquad -\frac{4}{3h^2}\left(\cos\frac{k\pi h}{\sqrt{3}} + 2\cos\frac{k\pi h}{2\sqrt{3}}\cos\frac{l\pi\sqrt{3}h}{2} - 3\right).
$$

These values converge for $h \to 0$ to the eigenvalue $\pi^2(\frac{k^2}{3} + l^2)$ obtained in the continuous case.

It is clear that these functions vanish for $y = 0$. Since $U_{k,l}$ and $V_{k,l}$ are results of the folding $\mathscr{F}$ of the functions $\sin\frac{k\pi x}{\sqrt{3}}\sin l\pi y$ and $\cos\frac{k\pi x}{\sqrt{3}}\sin l\pi y$ they vanish on the side $y = -\sqrt{3}x + 1$ of the triangle T and thus on the side $y = \sqrt{3}x + 1$, too.

Theorem. *The values of the functions*

$$U_{k,l}(x,y),\quad k=1,\dots,l-1;\ l=1,\dots,N-1,\ k\equiv l \bmod 2,$$

and

$$V_{k,l}(x,y),\quad k=0,\dots,l-1;\ l=1,\dots,N-1,\ k\equiv l \bmod 2$$

for $(x,y)\in T_h^\circ$ form a complete orthogonal system of eigenvectors.

P r o o f. In order to prove the orthogonality (in the modified sense), we realize first that the functions $U_{k,l}$ and $V_{k,l}$ are their own prolongations, i.e. $u=\mathscr{P}u\big|_{T_{1h}}$ (cf. [1]). Therefore we have

$$\sum_{(x,y)\in T_h} V_{k,l}(x,y)V_{m,n}(x,y) = 2\sum_{(x,y)\in T_{1h}} V_{k,l}(x,y)V_{m,n}(x,y)$$
$$= 2\sum_{(x,y)\in R_h} V_{k,l}(x,y)\mathscr{H}\cos\frac{m\pi x}{\sqrt{3}}\sin n\pi y.$$

This sum is, however, equal to zero for $(k,l)\neq(m,n)$. The orthogonality of the functions $U_{k,l}$ is proved in the same way and the mutual orthogonality of $U_{k,l}$ and $V_{k,l}$ is obvious. □

We thus have nonzero eigenvectors the number of which is equal to the number of the mesh points in T_h°.

2. Neumann boundary condition

The above approach can be used also for the Laplace operator with the Neumann boundary condition on all the three sides of the triangle T. Boundary conditions of different types on different sides of the triangle are not considered here.

We now show formulae for eigenfunctions for the case of the Neumann conditions. The prolongation of the skew-symmetric part of the function is now defined by (2) with $c_i=1$ for $i=1,2,4$ and $c_i=-1$ for $i=3,5,6$, and the prolongation of the symmetric part by (2) with all $c_i=1$.

We proceed as above concluding that

$$(8)\qquad U_{k,l}(x,y) = 2\sin\frac{k\pi x}{\sqrt{3}}\cos l\pi y - 2(-1)^{(k+l)/2}\sin\frac{\pi x}{2\sqrt{3}}(k+3l)\cos\frac{\pi y}{2}(k-l) - 2(-1)^{(k-l)/2}\sin\frac{\pi x}{2\sqrt{3}}(k-3l)\cos\frac{\pi y}{2}(k+l),$$
$$k=1,2,\dots,l,\ \ l=1,2,\dots,N,\ \ k\equiv l \bmod 2$$

and

$$
\begin{aligned}
(9) \qquad V_{k,l}(x,y) = 2\cos\frac{k\pi x}{\sqrt{3}}\cos l\pi y \\
&+ 2(-1)^{(k+l)/2}\cos\frac{\pi x}{2\sqrt{3}}\,(k+3l)\cos\frac{\pi y}{2}\,(k-l) \\
&+ 2(-1)^{(k-l)/2}\cos\frac{\pi x}{2\sqrt{3}}\,(k-3l)\cos\frac{\pi y}{2}\,(k+l), \\
&k = 0,1,\ldots,l,\;\; l = 0,1,2,\ldots,N,\;\; k \equiv l \bmod 2.
\end{aligned}
$$

The system of functions (8) and (9) is a complete orthogonal system of eigenvectors of the discrete Laplace operator with the Neumann boundary conditions. The orthogonality is again understood in the modified scalar product. The eigenvalues are, as before, given by (7). Since we have now a singular problem, we obtain for $k = l = 0$ the zero eigenvalue. The proof is essentialy the same as for the case of the Dirichlet boundary conditions.

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