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AN EXISTENCE THEOREM FOR THE BOUSSINESQ EQUATIONS WITH NON-DIRICHLET BOUNDARY CONDITIONS

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Abstract. The evolution Boussinesq equations describe the evolution of the temperature and velocity fields of viscous incompressible Newtonian fluids. Very often, they are a reasonable model to render relevant phenomena of flows in which the thermal effects play an essential role. In the paper we prescribe non-Dirichlet boundary conditions on a part of the boundary and prove the existence and uniqueness of solutions to the Boussinesq equations on a (short) time interval. The length of the time interval depends only on certain norms of the given data. In the proof we use a fixed point theorem method in Sobolev spaces with non-integer order derivatives. The proof is performed for Lipschitz domains and a wide class of data.

Keywords: Boussinesq equations, non-Dirichlet boundary conditions, Sobolev space with non-integer order derivatives, Schauder principle

MSC 2000: 35Q35, 35Q30

Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with a Lipschitz boundary $\partial\Omega$ and let Γ_1, Γ_2, Γ_3, Γ_4 and Γ_5 be open disjoint subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \overline{\Gamma}_3 \cup \overline{\Gamma}_4 \cup \overline{\Gamma}_5$, $\Gamma_1 \neq \emptyset$, $\Gamma_3 \neq \emptyset$ and the 2-dimensional measure of $\partial\Omega \backslash (\Gamma_1 \cup \Gamma_2)$ and $\partial\Omega \backslash (\Gamma_3 \cup \Gamma_4 \cup \Gamma_5)$ equals zero. Let $T > 0$ and let $\boldsymbol{n} = (n_1, n_2, n_3)$ denote the unit outward normal vector defined a.e. on $\partial\Omega$. Let $T_{ij}(\boldsymbol{u}, p) = -p\delta_{ij} + \nu \partial u_i / \partial x_j$ for $1 \leqslant i, j \leqslant 3$ and $\nu > 0$.

The evolution Boussinesq equations for the velocity $\tilde{\boldsymbol{u}}$, the pressure p and the temperature $\tilde{\vartheta}$ of the fluid are

$$\frac{\partial \tilde{\boldsymbol{u}}}{\partial t} - \nu \Delta \tilde{\boldsymbol{u}} + (\tilde{\boldsymbol{u}} \cdot \nabla)\tilde{\boldsymbol{u}} + \frac{1}{\varrho_{\mathrm{ref}}} \nabla p = \beta \boldsymbol{g}(\tilde{\vartheta} - \vartheta_{\mathrm{ref}}), \tag{1}$$

$$\operatorname{div} \tilde{\boldsymbol{u}} = 0, \tag{2}$$

$$\frac{\partial \tilde{\vartheta}}{\partial t} - \kappa \Delta \tilde{\vartheta} + (\tilde{\boldsymbol{u}} \cdot \nabla)\tilde{\vartheta} = Q. \tag{3}$$

We have introduced the thermal diffusion coefficient κ, the kinematic viscosity ν, the acceleration due to gravity $\boldsymbol{g}$, the termal expansion coefficient β, the reference density and temperature ϱ_{ref} and ϑ_{ref} and the source term Q. Without loss of generality we set β and ϱ_{ref} to be equal to unity. The system (1)–(3) is considered in the time cylinder $\Omega \times (0, T)$.

The evolution Boussinesq equations (1)–(3) describe the velocity and temperature fields of viscous incompressible Newtonian fluids where the thermal effects play an essential role. The density is constant everywhere except in the body force term of the momentum equation (1), where it is temperature dependent according to the law $\varrho = \varrho_{\mathrm{ref}}[1 - \beta(\tilde{\vartheta} - \vartheta_{\mathrm{ref}})]$. The viscous dissipation in the energy equation (3) is neglected. For more details concerning the Boussinesq equations see [1], [2], [15], [19].

The equations (1)–(3) are combined with the set of boundary conditions:

$$\tilde{\boldsymbol{u}} = \boldsymbol{\varphi} \qquad \text{on } \Gamma_1 \times (0, T), \tag{4}$$

$$T_{ij}(\tilde{\boldsymbol{u}}, p) n_j = \sigma_i \qquad \text{on } \Gamma_2 \times (0, T), \tag{5}$$

$$\tilde{\vartheta} = \varrho \qquad \text{on } \Gamma_3 \times (0, T), \tag{6}$$

$$\frac{\partial \tilde{\vartheta}}{\partial \boldsymbol{n}} = \omega \qquad \text{on } \Gamma_4 \times (0, T), \tag{7}$$

$$\frac{\partial \tilde{\vartheta}}{\partial \boldsymbol{n}} + \gamma(\tilde{\vartheta} - \vartheta_\delta) = 0 \qquad \text{on } \Gamma_5 \times (0, T), \tag{8}$$

where γ is the heat transfer coefficient and ϑ_δ is a given temperature. The initial conditions are

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}, 0) = \tilde{\boldsymbol{u}}_0(\boldsymbol{x}) \qquad \text{in } \Omega, \tag{9}$$

$$\tilde{\vartheta}(\boldsymbol{x}, 0) = \tilde{\vartheta}_0(\boldsymbol{x}) \qquad \text{in } \Omega. \tag{10}$$

The Dirichlet boundary conditions are used on the part Γ_1 of the boundary. On Γ_2 we prescribe the non-Dirichlet conditions (5). These conditions can be derived directly from the momentum equation (1) as natural boundary conditions and are sometimes referred to as "do nothing" boundary conditions (for $\boldsymbol{\sigma} = \mathbf{0}$). They are probably the best possible general-purpose boundary conditions to be used along the outflow boundaries, which seems to be supported from the mathematical point of view by their simplicity (see [7]). They have been applied in many numerical calculations especially at the outlets of tubes and channels (see e.g. [16], [17], [18]). For the temperature we use the Dirichlet boundary conditions on Γ_3, the Neumann boundary conditions on Γ_4 and the Newton boundary conditions on Γ_5 (see [4], [14]).

In the paper we prove an existence theorem for the system (1)–(10). In the proof of the existence theorem, the boundary conditions (5) are the main problem (see e.g. [8], [9], [10]) since backward flows cannot be excluded on Γ_2. The backward flows can bring more kinetic energy into the domain than is the amount of the energy carried outwards. Therefore, the amount of kinetic energy in the domain cannot be controlled and the method which is usually used for the case of the Dirichlet boundary conditions (see [20]) cannot be applied.

In the paper, Theorem 3.1 is the main result. We use a fixed point theorem method in the Sobolev spaces with non-integer order derivatives and prove the existence and uniqueness of solutions to the system (1)–(10) on a (short) time interval. The proof is performed for Lipschitz domains and a wide class of initial data. The length of the time interval on which the solution exists depends only on certain norms of the data and therefore the existence of unique solutions can be extended to a maximal time interval.

Remark 0.1. Let us note that the same or similar proof also works if some other non-Dirichlet boundary conditions are applied on Γ_2. As an example, let us mention the pressure drop problem with the boundary conditions ($\tilde{\boldsymbol{u}}_{\boldsymbol{n}}$, $\tilde{\boldsymbol{u}}_{\boldsymbol{\tau}}$ are the normal and tangential components of $\tilde{\boldsymbol{u}}$, P is a prescribed value)

$$(*) \qquad p - \nu \frac{\partial \tilde{\boldsymbol{u}}_{\boldsymbol{n}}}{\partial \boldsymbol{n}} = P, \qquad \frac{\partial \tilde{\boldsymbol{u}}_{\boldsymbol{\tau}}}{\partial \boldsymbol{n}} = 0$$

applied on Γ_2. These boundary conditions are convenient as outflow boundary conditions in some special types of geometries. For the thorough discussion of the pressure drop problem and the conditions $(*)$ see [8].

1. Preliminaries

Let $\nu > 0$, $\kappa > 0$ and $\alpha \in (0,1)$. Let $\boldsymbol{E}(\overline{\Omega}) = \{\boldsymbol{\varphi} \in [C^\infty(\overline{\Omega})]^3;\ \operatorname{div} \boldsymbol{\varphi} = 0 \text{ in } \Omega,\ \operatorname{supp} \boldsymbol{\varphi} \cap \Gamma_1 = \emptyset\}$ and let $\boldsymbol{V}$ and $\boldsymbol{H}$ be defined as the closure of $\boldsymbol{E}(\overline{\Omega})$ in the norm of the space $[W^{1,2}(\Omega)]^3$ and $[L^2(\Omega)]^3$, respectively. Let $E^\theta(\overline{\Omega}) = \{\varphi \in C^\infty(\overline{\Omega});\ \operatorname{supp} \varphi \cap \Gamma_3 = \emptyset\}$ and let V^θ be defined as the closure of $E^\theta(\overline{\Omega})$ in the norm of the space $W^{1,2}(\Omega)$. Similarly, $\tilde{\boldsymbol{E}}(\overline{\Omega}) = \{\boldsymbol{\varphi} \in [C^\infty(\overline{\Omega})]^3;\ \operatorname{div}\boldsymbol{\varphi} = 0 \text{ in } \Omega\}$ and $\tilde{\boldsymbol{V}}$ and $\tilde{\boldsymbol{H}}$ are defined as the closure of $\tilde{\boldsymbol{E}}(\overline{\Omega})$ in the norm of the space $[W^{1,2}(\Omega)]^3$ and $[L^2(\Omega)]^3$, respectively.

$\boldsymbol{V}$ is a Hilbert space endowed with the scalar product $((\boldsymbol{u},\boldsymbol{v})) = \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,\mathrm{d}x$. $\boldsymbol{H}$ and $\tilde{\boldsymbol{H}}$ are Hilbert spaces with respect to the scalar product $(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega u_i v_i\,\mathrm{d}x$, $\tilde{\boldsymbol{V}}$ is a Hilbert space with the scalar product $((\boldsymbol{u},\boldsymbol{v}))_{1,2} = (\boldsymbol{u},\boldsymbol{v})_{[W^{1,2}(\Omega)]^3}$ and V^θ is a Hilbert space with the scalar product $((\psi,\vartheta))_\theta = \int_\Omega \frac{\partial\psi}{\partial x_j}\frac{\partial\vartheta}{\partial x_j}\,\mathrm{d}x$. The scalar product both in $\boldsymbol{L}^2(\Omega)$ and $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$. In the paper, A^* denotes the dual space of a Hilbert space A. The norms of $\boldsymbol{V}$, V^θ are denoted by $\|\cdot\|$, $\|\cdot\|_\theta$.

By the Riesz representation theorem, we can identify $\boldsymbol{H}$ and $\boldsymbol{H}^*$, $L^2(\Omega)$ and $L^2(\Omega)^*$, and have the inclusions

$$\begin{gathered}
\boldsymbol{V} \subset \boldsymbol{H} \equiv \boldsymbol{H}^* \subset \boldsymbol{V}^*,\\
V^\theta \subset L^2(\Omega) \equiv L^2(\Omega)^* \subset (V^\theta)^*,\\
\tilde{\boldsymbol{V}} \subset \tilde{\boldsymbol{H}} \equiv \tilde{\boldsymbol{H}}^* \subset \tilde{\boldsymbol{V}}^*,\\
W^{1,2}(\Omega) \subset L^2(\Omega) \equiv L^2(\Omega)^* \subset W^{1,2}(\Omega)^*,
\end{gathered}$$

where each space is dense in the next one and the imbeddings are continuous.

We define Banach spaces $\boldsymbol{X}$, $\boldsymbol{X}_0$, X^θ, X^θ_0, $\tilde{\boldsymbol{X}}$, $\tilde{X}^\theta$, $\boldsymbol{Y}$, Y^θ, $\boldsymbol{X}_\alpha$ and X^θ_α. Let $D(0,T)$ be the space of $C^\infty([0,T])$ functions with a compact support contained in $(0,T)$. Then $\boldsymbol{X} = \{\boldsymbol{u} \in L^2(0,T,\boldsymbol{V});\ \boldsymbol{u}' \in L^2(0,T,\boldsymbol{V}),\ \boldsymbol{u}'' \in L^2(0,T,\boldsymbol{V}^*)\}$ and $\|\boldsymbol{u}\|_X = \|\boldsymbol{u}'\|_{L^2(0,T,\boldsymbol{V})} + \|\boldsymbol{u}''\|_{L^2(0,T,\boldsymbol{V}^*)} + \|\boldsymbol{u}(0)\|$, where $\boldsymbol{u}'$ is the derivative of $\boldsymbol{u}$ in the sense of distribution and $\boldsymbol{u}''$ is the Schwartz derivative of $\boldsymbol{u}'$ in the sense of the imbedding $\boldsymbol{V} \subset \boldsymbol{H} \equiv \boldsymbol{H}^* \subset \boldsymbol{V}^*$. This means precisely that

$$\int_0^T \big((\boldsymbol{v},\boldsymbol{u}(t))\big)\boldsymbol{\Phi}'(t)\,\mathrm{d}t = -\int_0^T \big((\boldsymbol{v},\boldsymbol{u}'(t))\big)\boldsymbol{\Phi}(t)\,\mathrm{d}t \quad \forall \boldsymbol{v} \in \boldsymbol{V},\ \ \forall \boldsymbol{\Phi} \in D(0,T)$$

and

$$\int_0^T (\boldsymbol{u}'(t),\boldsymbol{v})\boldsymbol{\Phi}'(t)\,\mathrm{d}t = -\int_0^T \langle \boldsymbol{u}''(t),\boldsymbol{v}\rangle\,\boldsymbol{\Phi}(t)\,\mathrm{d}t \quad \forall \boldsymbol{v} \in \boldsymbol{V},\ \ \forall \boldsymbol{\Phi} \in D(0,T),$$

where $\langle \boldsymbol{u}''(t),\boldsymbol{v}\rangle = \langle \boldsymbol{u}''(t),\boldsymbol{v}\rangle_{\langle \boldsymbol{V}^*,\boldsymbol{V}\rangle}$ and $\langle \boldsymbol{V}^*,\boldsymbol{V}\rangle$ is the duality between $\boldsymbol{V}^*$ and $\boldsymbol{V}$. For more details see [20]. Let $\boldsymbol{X}_0 = \{\boldsymbol{u} \in \boldsymbol{X};\ \boldsymbol{u}(0) = \boldsymbol{0}\}$, $X^\theta = \{\vartheta \in L^2(0,T,V^\theta);\ \vartheta' \in L^2(0,T,V^\theta),\ \vartheta'' \in L^2(0,T,(V^\theta)^*)\}$ and $\|\vartheta\|_{X^\theta} = \|\vartheta'\|_{L^2(0,T,V^\theta)} + \|\vartheta''\|_{L^2(0,T,(V^\theta)^*)} + \|\vartheta(0)\|_\theta$, where ϑ' is the derivative of ϑ in the sense of distribution and ϑ'' is the Schwartz derivative of ϑ' in the sense of the imbedding $V^\theta \subset L^2(\Omega) \equiv L^2(\Omega)^* \subset (V^\theta)^*$. Let $X^\theta_0 = \{\vartheta \in X;\ \vartheta(0) = 0\}$.

Similarly we define $\tilde{\boldsymbol{X}} = \{\boldsymbol{u} \in L^2(0,T,\tilde{\boldsymbol{V}});\ \boldsymbol{u}' \in L^2(0,T,\tilde{\boldsymbol{V}}),\ \boldsymbol{u}'' \in L^2(0,T,\tilde{\boldsymbol{V}}^*)\}$ and $\|\boldsymbol{u}\|_{\tilde{X}} = \|\boldsymbol{u}'\|_{L^2(0,T,\tilde{\boldsymbol{V}})} + \|\boldsymbol{u}''\|_{L^2(0,T,\tilde{\boldsymbol{V}}^*)} + \|\boldsymbol{u}(0)\|_{\tilde{\boldsymbol{V}}}$. Again, $\boldsymbol{u}'$ means the derivative of $\boldsymbol{u}$ in the sense of distribution and $\boldsymbol{u}''$ is the Schwartz derivative of $\boldsymbol{u}'$ in the sense of the imbedding $\tilde{\boldsymbol{V}} \subset \tilde{\boldsymbol{H}} \equiv \tilde{\boldsymbol{H}}^* \subset \tilde{\boldsymbol{V}}^*$. $\tilde{X}^\theta = \{\vartheta \in L^2(0,T,W^{1,2}(\Omega));\ \vartheta' \in L^2(0,T,W^{1,2}(\Omega));\ \vartheta'' \in L^2(0,T,W^{1,2}(\Omega)^*)\}$, $\|\vartheta\|_{\tilde{X}^\theta} = \|\vartheta'\|_{L^2(0,T,W^{1,2}(\Omega))} + \|\vartheta''\|_{L^2(0,T,W^{1,2}(\Omega)^*)} + \|\vartheta(0)\|_{W^{1,2}(\Omega)}$ and ϑ' means the derivative of ϑ in the sense of distribution and ϑ'' is the Schwartz derivative of ϑ' in the sense of the imbedding $W^{1,2}(\Omega) \subset L^2(\Omega) \equiv L^2(\Omega)^* \subset W^{1,2}(\Omega)^*$.

We define $\boldsymbol{Y} = \{[[\boldsymbol{f}, \boldsymbol{w}]];\ \boldsymbol{f} \in C([0,T], \boldsymbol{V}^*),\ \boldsymbol{f}' \in L^2(0,T,\boldsymbol{V}^*),\ \boldsymbol{w} \in \boldsymbol{V},\ \boldsymbol{f}(0) - \nu((\boldsymbol{w},\cdot)) \in \boldsymbol{H}\}$, $\|[[\boldsymbol{f},\boldsymbol{w}]]\|_{\boldsymbol{Y}} = \|\boldsymbol{f}'\|_{L^2(0,T,\boldsymbol{V}^*)} + \|\boldsymbol{w}\| + \|\boldsymbol{f}(0) - \nu((\boldsymbol{w},\cdot))\|_{\boldsymbol{H}}$, where $\boldsymbol{f}'$ is the derivative of $\boldsymbol{f}$ in the sense of distribution. Similarly, $Y^\theta = \{[[f,w]];\ f \in C([0,T],(V^\theta)^*), f' \in L^2(0,T,(V^\theta)^*), w \in \boldsymbol{V}^\theta, f(0) - \kappa((w,\cdot))_\theta \in L^2(\Omega)\}$, $\|[[f,w]]\|_{Y^\theta} = \|f'\|_{L^2(0,T,(V^\theta)^*)} + \|w\|_\theta + \|f(0) - \kappa((w,\cdot))_\theta\|_{L^2(\Omega)}$, where f' is the derivative of f in the sense of distribution.

Let $X_\alpha = \{\boldsymbol{u} \in L^2(0,T,\boldsymbol{W}^{\alpha,2}(\Omega));\ \boldsymbol{u}' \in L^2(0,T,\boldsymbol{W}^{\alpha,2}(\Omega)) \cap L^4(0,T,\boldsymbol{L}^2(\Omega)),\ \boldsymbol{u}(0) = \boldsymbol{0}\}$ and $\|\boldsymbol{u}\|_{X_\alpha} = \|\boldsymbol{u}'\|_{L^2(0,T,\boldsymbol{W}^{\alpha,2}(\Omega))} + \|\boldsymbol{u}'\|_{L^4(0,T,\boldsymbol{L}^2(\Omega))}$, where $\boldsymbol{u}'$ is the derivative of $\boldsymbol{u}$ in the sense of distribution, $\boldsymbol{W}^{\alpha,2}(\Omega) = [W^{\alpha,2}(\Omega)]^3$ and $W^{\alpha,2}(\Omega)$ are Sobolev spaces with non-integer order derivatives (for their definition and properties see [12] or [13]). $X_\alpha^\theta = \{\vartheta \in L^2(0,T,W^{\alpha,2}(\Omega));\ \vartheta' \in L^2(0,T,W^{\alpha,2}(\Omega)) \cap L^4(0,T,L^2(\Omega)),\ \vartheta(0) = 0\}$ and $\|\vartheta\|_{X_\alpha^\theta} = \|\vartheta'\|_{L^2(0,T,W^{\alpha,2}(\Omega))} + \|\vartheta'\|_{L^4(0,T,L^2(\Omega))}$, where ϑ' is the derivative of ϑ in the sense of distribution.

In the paper we will use the following imbedding theorems:

$$\boldsymbol{W}^{1,2}(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega),\ \|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} \leqslant c_1\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,2}(\Omega)},\ \forall \boldsymbol{u} \in \boldsymbol{W}^{1,2}(\Omega), \tag{11}$$

$$\boldsymbol{W}^{1,2}(\Omega) \hookrightarrow \boldsymbol{L}^4(\partial\Omega),\ \|\boldsymbol{u}\|_{\boldsymbol{L}^4(\partial\Omega)} \leqslant c_2\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,2}(\Omega)},\ \forall \boldsymbol{u} \in \boldsymbol{W}^{1,2}(\Omega), \tag{12}$$

$$\|\boldsymbol{u}\|_{\tilde{\boldsymbol{H}}} \leqslant c_3\|\boldsymbol{u}\|_{\tilde{\boldsymbol{V}}},\ \forall \boldsymbol{u} \in \tilde{\boldsymbol{V}}, \tag{13}$$

$$\boldsymbol{W}^{7/8,2}(\Omega) \hookrightarrow \boldsymbol{L}^4(\Omega),\ \|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)} \leqslant c_4\|\boldsymbol{u}\|_{\boldsymbol{W}^{7/8,2}(\Omega)},\ \forall \boldsymbol{u} \in \boldsymbol{W}^{7/8,2}(\Omega), \tag{14}$$

$$\boldsymbol{W}^{7/8,2}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Gamma_2),\ \|\boldsymbol{u}\|_{\boldsymbol{L}^3(\Gamma_2)} \leqslant c_5\|\boldsymbol{u}\|_{\boldsymbol{W}^{7/8,2}(\Omega)},\ \forall \boldsymbol{u} \in \boldsymbol{W}^{7/8,2}(\Omega), \tag{15}$$

$$\boldsymbol{W}^{5/6,2}(\Omega) \hookrightarrow \boldsymbol{L}^4(\Omega),\ \|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)} \leqslant c_6\|\boldsymbol{u}\|_{\boldsymbol{W}^{5/6,2}(\Omega)},\ \forall \boldsymbol{u} \in \boldsymbol{W}^{5/6,2}(\Omega), \tag{16}$$

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{5/6,2}(\Omega)} \leqslant c_7\|\boldsymbol{u}\|^{20/21}_{\boldsymbol{W}^{7/8,2}(\Omega)}\|\boldsymbol{u}\|^{1/21}_{L^2(\Omega)},\ \forall \boldsymbol{u} \in \boldsymbol{W}^{7/8,2}(\Omega), \tag{17}$$

$$\boldsymbol{W}^{5/6,2}(\Omega) \hookrightarrow \boldsymbol{L}^{12/5}(\Gamma_2),\ \|\boldsymbol{u}\|_{\boldsymbol{L}^{12/5}(\Gamma_2)} \leqslant c_8\|\boldsymbol{u}\|_{\boldsymbol{W}^{5/6,2}(\Omega)},\ \forall \boldsymbol{u} \in \boldsymbol{W}^{5/6,2}(\Omega). \tag{18}$$

R e m a r k 1.1. Let us comment upon the items (11)–(18). It follows immediately from Theorem 8.3.3.(i) in [12] that for $k > 3/4$, $W^{k,2}(\Omega) \hookrightarrow L^4(\Omega)$. Consequently, (14) and (16) hold. (11)–(13) are classical results, see [12]. The interpolation inequality (17) follows from the definition of the spaces $\boldsymbol{W}^{\alpha,2}(\Omega)$ in [13]. The items

(15) and (18) follow from Theorem 8.3.3.(i) in [12] and Theorem 1.5.1.2 in [6]:

$$\boldsymbol{W}^{7/8.2}(\Omega) \hookrightarrow \boldsymbol{W}^{3/8.3}(\Omega) \hookrightarrow \boldsymbol{W}^{1/24.3}(\Gamma_2) \hookrightarrow \boldsymbol{L}^3(\Gamma_2)$$

and

$$\boldsymbol{W}^{5/6.2}(\Omega) \hookrightarrow \boldsymbol{W}^{7/12,12/5}(\Omega) \hookrightarrow \boldsymbol{W}^{1/6,12/5}(\Gamma_2) \hookrightarrow \boldsymbol{L}^{12/5}(\Gamma_2).$$

To simplify the paper we use the constant c instead of the constants c_1–c_8 from (11)–(18). In the paper, the constant c grows with the time interval T. We now define

$$\begin{aligned} b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) &= \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i \,\mathrm{d}x, \\ C(\boldsymbol{u},\vartheta,\varphi) &= \int_\Omega u_j \frac{\partial \vartheta}{\partial x_j} \varphi \,\mathrm{d}x \end{aligned}$$

for every $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{W}^{1,2}(\Omega)$ and every $\vartheta,\varphi \in W^{1,2}(\Omega)$ and

$$\begin{aligned} b_r(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) &= \int_{\Gamma_2} u_j n_j v_i w_i \,\mathrm{d}S - \int_\Omega u_j v_i \frac{\partial w_i}{\partial x_j} \,\mathrm{d}x, \\ C_r(\boldsymbol{u},\vartheta,\varphi) &= \int_{\Gamma_4\cup\Gamma_5} u_j n_j \vartheta\varphi \,\mathrm{d}S - \int_\Omega u_j \vartheta \frac{\partial \varphi}{\partial x_j} \,\mathrm{d}x \end{aligned}$$

for every $\boldsymbol{u},\boldsymbol{v} \in \boldsymbol{W}^{7/8,2}(\Omega)$, $\boldsymbol{w} \in \boldsymbol{W}^{1,2}(\Omega)$ and every $\vartheta \in W^{7/8,2}(\Omega)$, $\varphi \in W^{1,2}(\Omega)$. Clearly,

$$b_r(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}), \quad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{W}^{1,2}(\Omega), \boldsymbol{w} \in \boldsymbol{V}, \tag{19}$$

$$C_r(\boldsymbol{u},\vartheta,\varphi) = C(\boldsymbol{u},\vartheta,\varphi), \quad \forall \boldsymbol{u} \in \boldsymbol{W}^{1,2}(\Omega), \vartheta \in W^{1,2}(\Omega), \varphi \in V^\theta. \tag{20}$$

2. Weak formulation

Let us suppose that $\tilde{\boldsymbol{u}}$, p and $\tilde{\vartheta}$ are smooth solutions to the system (1)–(10). It is a standard procedure to show that

$$\begin{aligned} &(\tilde{\boldsymbol{u}}', \boldsymbol{v}) + \nu((\tilde{\boldsymbol{u}}, \boldsymbol{v})) + b(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}}, \boldsymbol{v}) \\ &\qquad = \int_{\Gamma_2} \sigma_i v_i \,\mathrm{d}S + ((\tilde{\vartheta} - \vartheta_{\mathrm{ref}})\boldsymbol{g}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \\ &(\tilde{\vartheta}', \varphi) + \kappa((\tilde{\vartheta}, \varphi))_\theta + C(\tilde{\boldsymbol{u}}, \tilde{\vartheta}, \varphi) \\ &\qquad = (Q, \varphi) + \int_{\Gamma_4} \kappa\omega\varphi \,\mathrm{d}S - \int_{\Gamma_5} \kappa\gamma(\tilde{\vartheta} - \vartheta_\delta)\varphi \,\mathrm{d}S \qquad \forall \varphi \in V^\theta. \end{aligned}$$

Let us denote

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle = \int_{\Gamma_2} \sigma_i v_i \,\mathrm{d}S \qquad \forall \boldsymbol{v} \in \boldsymbol{V},$$

$$\langle f^\theta, \varphi\rangle = (Q, \varphi) + \int_{\Gamma_4} \kappa\omega\varphi \,\mathrm{d}S \qquad \forall \varphi \in V^\theta.$$

It is clear that if the data are sufficiently smooth then $\boldsymbol{f} \in C([0,T], \boldsymbol{V}^*)$, $\boldsymbol{f}' \in L^2(0,T,\boldsymbol{V}^*)$, $f^\theta \in C([0,T],(V^\theta)^*)$ and $(f^\theta)' \in L^2(0,T,(V^\theta)^*)$. This suggests the following weak formulation of the system (1)–(10):

Problem 2.1 (weak formulation). Let ν, κ and γ be positive constants and let

(21) $\quad \boldsymbol{f} \in C([0,T], \boldsymbol{V}^*),\ \boldsymbol{f}' \in L^2(0,T,\boldsymbol{V}^*),$

(22) $\quad f^\theta \in C([0,T],(V^\theta)^*),\ (f^\theta)' \in L^2(0,T,(V^\theta)^*),$

(23) $\quad \tilde{\boldsymbol{u}}_0 \in \tilde{\boldsymbol{V}},\ \tilde{\vartheta}_0 \in W^{1,2}(\Omega),$

(24) $\quad \boldsymbol{\varphi} \in \tilde{\boldsymbol{X}},\ \varrho \in \tilde{X}^\theta,$

(25) $\quad \boldsymbol{\varphi}(0) = \tilde{\boldsymbol{u}}_0 \quad \text{on } \Gamma_1,$

(26) $\quad \varrho(0) = \tilde{\vartheta}_0 \quad \text{on } \Gamma_3,$

(27) $\quad \boldsymbol{f}(0) - \nu((\tilde{\boldsymbol{u}}_0,\cdot)) - b(\tilde{\boldsymbol{u}}_0,\tilde{\boldsymbol{u}}_0,\cdot) \in \boldsymbol{H},$

(28) $\quad \boldsymbol{g} \in \boldsymbol{L}^\infty(\Omega),$

(29) $\quad \vartheta_{\mathrm{ref}}, \vartheta'_{\mathrm{ref}} \in L^2(0,T,L^2(\Omega)),$

(30) $\quad \vartheta_\delta, \vartheta'_\delta \in L^2(0,T,L^2(\Gamma_5)),$

(31) $$f^\theta(0) - \kappa((\tilde{\vartheta}_0,\cdot))_\theta - C(\tilde{\boldsymbol{u}}_0,\tilde{\vartheta}_0,\cdot) - \int_{\Gamma_5} \kappa\gamma(\tilde{\vartheta}_0 - \vartheta_\delta(0))\cdot \,\mathrm{d}S \in L^2(\Omega).$$

Let us denote $\boldsymbol{u}_0 = \tilde{\boldsymbol{u}}_0 - \boldsymbol{\varphi}(0)$ and $\vartheta_0 = \tilde{\vartheta}_0 - \varrho(0)$. Clearly, $\boldsymbol{u}_0 \in \boldsymbol{V}$ and $\vartheta_0 \in V^\theta$. Find $\boldsymbol{u} \in \boldsymbol{X}_0$ and $\vartheta \in X_0^\theta$ satisfying, for every $\boldsymbol{v} \in \boldsymbol{V}$ and $\varphi \in V^\theta$,

(32) $$(\boldsymbol{u}' + \boldsymbol{\varphi}', \boldsymbol{v}) + \nu\big((\boldsymbol{u} + \boldsymbol{\varphi} + \boldsymbol{u}_0, \boldsymbol{v})\big) + b(\boldsymbol{u} + \boldsymbol{\varphi} + \boldsymbol{u}_0, \boldsymbol{u} + \boldsymbol{\varphi} + \boldsymbol{u}_0, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle + \big((\vartheta + \varrho + \vartheta_0 - \vartheta_{\mathrm{ref}})\boldsymbol{g}, \boldsymbol{v}\big),$$

(33) $$(\vartheta' + \varrho', \varphi) + \kappa\big((\vartheta + \varrho + \vartheta_0, \varphi)\big)_\theta + C(\boldsymbol{u} + \boldsymbol{\varphi} + \boldsymbol{u}_0, \vartheta + \varrho + \vartheta_0, \varphi) = \langle f^\theta, \varphi\rangle - \int_{\Gamma_5} \kappa\gamma(\vartheta + \varrho + \vartheta_0 - \vartheta_\delta)\varphi \,\mathrm{d}S.$$

The pair $\tilde{\boldsymbol{u}} = \boldsymbol{u} + \boldsymbol{\varphi} + \boldsymbol{u}_0$, $\tilde{\vartheta} = \vartheta + \varrho + \vartheta_0$ is called the weak solution of the system (1)–(10).

R e m a r k 2.2. The conditions (25) and (26) are compatibility conditions. The conditions (27) and (31) follow from the fact that $\tilde{\boldsymbol{u}} \in \tilde{\boldsymbol{X}}$ and $\tilde{\vartheta} \in \tilde{X}^\theta$. The functions $\boldsymbol{g}$, $\vartheta_{\rm ref}$ and ϑ_δ defined in (28), (29) and (30) are supposed to depend on the time and space variables.

R e m a r k 2.3. Since $\boldsymbol{u} + \boldsymbol{u}_0 + \boldsymbol{\varphi} \in \tilde{\boldsymbol{X}}$ and $\vartheta + \varrho + \vartheta_0 \in \tilde{X}^\theta$, it follows from (19), (20) and the definitions of $\tilde{\boldsymbol{X}}$ and $\tilde{X}^\theta$ that Problem 2.1 does not change if $b(\boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0, \boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0, \boldsymbol{v})$ in (32) or $C(\boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0, \vartheta+\varrho+\vartheta_0, \varphi)$ in (33) is replaced by $b_r(\boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0, \boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0, \boldsymbol{v})$ or $C_r(\boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0, \vartheta+\varrho+\vartheta_0, \varphi)$, respectively. It will be more convenient, since we are working with Sobolev spaces with non-integer order derivatives, to use the following form of (32) and (33):

$$
\begin{aligned}
(\boldsymbol{u}', \boldsymbol{v}) + \nu\big((\boldsymbol{u}, \boldsymbol{v})\big) &= \langle \boldsymbol{f}, \boldsymbol{v}\rangle + \big((\vartheta+\varrho+\vartheta_0-\vartheta_{\rm ref})\boldsymbol{g}, \boldsymbol{v}\big) - (\boldsymbol{\varphi}', \boldsymbol{v}) - \nu\big((\boldsymbol{\varphi}+\boldsymbol{u}_0, \boldsymbol{v})\big)\\
&\quad - b_r(\boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0, \boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0, \boldsymbol{v}),\\
(\vartheta', \varphi) + \kappa\big((\vartheta, \varphi)\big)_\theta &= \big\langle f^\theta, \varphi\big\rangle - \int_{\Gamma_5} \kappa\gamma\, (\vartheta+\varrho+\vartheta_0-\vartheta_\delta)\varphi \,\mathrm{d}S - (\varrho', \varphi)\\
&\quad - \kappa((\varrho+\vartheta_0, \varphi))_\theta - C_r(\boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0, \vartheta+\varrho+\vartheta_0, \varphi).
\end{aligned}
$$

R e m a r k 2.4. The function $\boldsymbol{f}$ in (21) is sufficiently general. It is possible to show that the choice $\boldsymbol{f} = \boldsymbol{0}$ leads to the "do nothing" boundary conditions on Γ_2, that is $T_{ij}(\tilde{\boldsymbol{u}}, p)n_j = 0$. If we set $\langle \boldsymbol{f}, \boldsymbol{v}\rangle = \int_{\Gamma_2} \sigma_i v_i \,\mathrm{d}S$ for every $\boldsymbol{v} \in \boldsymbol{V}$, then we get the boundary conditions (5) on Γ_2. Similarly, the choice $\langle \boldsymbol{f}, \boldsymbol{v}\rangle = -P\int_{\Gamma_2} v_i n_i \,\mathrm{d}S$ for every $\boldsymbol{v} \in \boldsymbol{V}$ leads to the boundary conditions $(*)$ on Γ_2 (see Remark 0.1 and [8]).

3. Existence theorem

Let ν, κ and γ be positive constants and let $\boldsymbol{f}, f^\theta, \tilde{\boldsymbol{u}}_0, \tilde{\vartheta}_0, \boldsymbol{\varphi}, \varrho, \boldsymbol{g}, \vartheta_{\rm ref}, \vartheta_\delta$ satisfy (21)–(31).

Theorem 3.1. *There exists a positive number* $T_* = \min(T, T_1)$, $T_1 = T_1(\boldsymbol{f}, f^\theta, \tilde{\boldsymbol{u}}_0, \tilde{\vartheta}_0, \boldsymbol{\varphi}, \varrho, \boldsymbol{g}, \vartheta_{\rm ref}, \vartheta_\delta)$ *such that there exists a solution* $\boldsymbol{u} \in \boldsymbol{X}_0$, $\vartheta \in X_0^\theta$ *of Problem* 2.1 *on* $(0, T_*)$.

Theorem 3.1 is the main result of the paper.

Definition 3.2. The map S from $\boldsymbol{X}$ to $\boldsymbol{Y}$ is defined by $S(\boldsymbol{u}) = [[\boldsymbol{f}, \boldsymbol{w}]]$, $\forall \boldsymbol{u} \in X$, where $\langle \boldsymbol{f}(t), \cdot\rangle = (\boldsymbol{u}'(t), \cdot) + \nu((\boldsymbol{u}(t), \cdot))$ for every $t \in [0, T]$ and $\boldsymbol{w} = \boldsymbol{u}(0)$.

Definition 3.3. The map S^θ from X^θ to Y^θ is defined by $S^\theta(\vartheta) = [[f^\theta, \psi]]$, $\forall \vartheta \in X^\theta$, $\big\langle f^\theta(t), \cdot\big\rangle = (\vartheta'(t), \cdot) + \kappa((\vartheta(t), \cdot))$ for every $t \in [0, T]$ and $\psi = \vartheta(0)$.

Definition 3.4. The mapping $P\colon \boldsymbol{X}_{7/8}\times X^{\theta}_{7/8}\to \boldsymbol{Y}$ is defined by $P(\boldsymbol{u},\vartheta)=[[\boldsymbol{F},\boldsymbol{0}]]$ for every $\boldsymbol{u}\in \boldsymbol{X}_{7/8}$ and $\vartheta\in X^{\theta}_{7/8}$, where

$$\begin{aligned}(34)\qquad \langle \boldsymbol{F},\boldsymbol{v}\rangle = {} & \langle \boldsymbol{f},\boldsymbol{v}\rangle + \big((\vartheta+\varrho+\vartheta_0-\vartheta_{\mathrm{ref}})\boldsymbol{g},\boldsymbol{v}\big)-(\boldsymbol{\varphi}',\boldsymbol{v})\\ & -\nu\big((\boldsymbol{\varphi}+\boldsymbol{u}_0,\boldsymbol{v})\big)\\ & - b_r(\boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0,\boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0,\boldsymbol{v})\quad \forall \boldsymbol{v}\in\boldsymbol{V}.\end{aligned}$$

Definition 3.5. The mapping $P^{\theta}\colon \boldsymbol{X}_{7/8}\times X^{\theta}_{7/8}\to Y^{\theta}$ is defined by $P^{\theta}(\boldsymbol{u},\vartheta)=[[F^{\theta},0]]$ for every $\boldsymbol{u}\in\boldsymbol{X}_{7/8}$ and $\vartheta\in X^{\theta}_{7/8}$, where

$$\begin{aligned}\langle F^{\theta},\varphi\rangle = \langle f^{\theta},\varphi\rangle & - \int_{\Gamma_5}\kappa\gamma(\vartheta+\varrho+\vartheta_0-\vartheta_\delta)\varphi\,\mathrm{d}S\\ & -(\varrho',\varphi)-\kappa\big((\varrho+\vartheta_0,\varphi)\big)_\theta\\ & - C_r(\boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0,\vartheta+\varrho+\vartheta_0,\varphi)\qquad \forall\varphi\in V^{\theta}.\end{aligned}$$

Definition 3.6. Let $\alpha\in(0,1)$. Then Q_α denotes the imbedding of $\boldsymbol{X}_0$ into $\boldsymbol{X}_\alpha$ and Q^{θ}_α the imbedding of X^{θ}_0 into X^{θ}_α.

To prove Theorem 3.1, we will show that there exists a positive number $T_*=\min(T,T_1)$, $T_1=T_1(\boldsymbol{f},f^{\theta},\tilde{\boldsymbol{u}}_0,\tilde{\vartheta}_0,\boldsymbol{\varphi},\varrho,\boldsymbol{g},\vartheta_{\mathrm{ref}},\vartheta_\delta)$, such that there exists a fixed point of the mapping

$$Z\colon (\boldsymbol{u},\vartheta)\in\boldsymbol{X}_{7/8}\times X^{\theta}_{7/8}\longmapsto (Q_{7/8}S^{-1}P(\boldsymbol{u},\vartheta),Q^{\theta}_{7/8}(S^{\theta})^{-1}P^{\theta}(\boldsymbol{u},\vartheta))\in\boldsymbol{X}_{7/8}\times X^{\theta}_{7/8},$$

where Z is defined on $(0,T_*)$. In the following lemmas we first show some useful properties of P, S, Q_α, P^{θ}, S^{θ} and Q^{θ}_α.

Lemma 3.7. *Let $T>0$, $\alpha\in(0,1)$, $M=\{\boldsymbol{u}\in L^2(0,T,\boldsymbol{V});\ \boldsymbol{u}'\in L^2(0,T,\boldsymbol{V}^*)\}$, where $\boldsymbol{u}'$ is the Schwartz derivative of $\boldsymbol{u}$ in the sense of the imbedding $\boldsymbol{V}\subset\boldsymbol{H}\equiv\boldsymbol{H}^*\subset\boldsymbol{V}^*$ and $\|\boldsymbol{u}\|_M=\|\boldsymbol{u}\|_{L^2(0,T,\boldsymbol{V})}+\|\boldsymbol{u}'\|_{L^2(0,T,\boldsymbol{V}^*)}$ for every $\boldsymbol{u}\in M$. Then $M\hookrightarrow\hookrightarrow L^2(0,T,\boldsymbol{W}^{\alpha,2}(\Omega))$.*

Proof. Let $\{\boldsymbol{u}_m\}_{m=1}^{\infty}\in M$ and $\|\boldsymbol{u}_m\|_M\leqslant c<\infty$, $\forall m\in N$. Then there exists a subsequence of $\{\boldsymbol{u}_m\}_{m=1}^{\infty}$ (we denote it again by $\{\boldsymbol{u}_m\}_{m=1}^{\infty}$) such that $\boldsymbol{u}_m\to\boldsymbol{u}$ in $L^2(0,T,\boldsymbol{H})$ (see Theorem 2.1, Chapter 3 in [20]) and $\boldsymbol{u}_m\to\boldsymbol{u}$ in the weak topology of $L^2(0,T,\boldsymbol{V})$. Since $\boldsymbol{V}\hookrightarrow\hookrightarrow\boldsymbol{W}^{\alpha,2}(\Omega)\hookrightarrow\boldsymbol{L}^2(\Omega)$, it follows from [20], Chapter 3, Lemma 2.1 that for every $\eta>0$ there exists a $c_\eta>0$ such that $\|\boldsymbol{u}\|_{\boldsymbol{W}^{\alpha,2}(\Omega)}\leqslant\eta\|\boldsymbol{u}\|+c_\eta\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}$ for every $\boldsymbol{u}\in\boldsymbol{V}$. Therefore, we have for every $m\in N$ and a.e. $t\in(0,T)$ that $\boldsymbol{u}(t),\boldsymbol{u}_m(t)\in\boldsymbol{V}$, $\|\boldsymbol{u}(t)-\boldsymbol{u}_m(t)\|_{\boldsymbol{W}^{\alpha,2}(\Omega)}\leqslant\eta\|\boldsymbol{u}(t)-\boldsymbol{u}_m(t)\|+$

$c_\eta \|\boldsymbol{u}(t) - \boldsymbol{u}_m(t)\|_{\boldsymbol{L}^2(\Omega)}$ and

$$\begin{aligned} &\left(\int_0^T \|\boldsymbol{u}(t) - \boldsymbol{u}_m(t)\|^2_{\boldsymbol{W}^{\alpha,2}(\Omega)}\, \mathrm{d}t\right)^{1/2} \\ &\leqslant \left(\int_0^T (\eta\|\boldsymbol{u}(t) - \boldsymbol{u}_m(t)\| + c_\eta\|\boldsymbol{u}(t) - \boldsymbol{u}_m(t)\|_{\boldsymbol{H}})^2\, \mathrm{d}t\right)^{1/2} \\ &\leqslant \left(\int_0^T \eta^2\|\boldsymbol{u}(t) - \boldsymbol{u}_m(t)\|^2\, \mathrm{d}t\right)^{1/2} + \left(\int_0^T c_\eta^2\|\boldsymbol{u}(t) - \boldsymbol{u}_m(t)\|^2_{\boldsymbol{H}}\, \mathrm{d}t\right)^{1/2} \\ &= \eta\|\boldsymbol{u} - \boldsymbol{u}_m\|_{L^2(0,T,\boldsymbol{V})} + c_\eta\|\boldsymbol{u} - \boldsymbol{u}_m\|_{L^2(0,T,\boldsymbol{H})}. \end{aligned}$$

Since $\|\boldsymbol{u} - \boldsymbol{u}_m\|_{L^2(0,T,\boldsymbol{V})}$ is a bounded sequence and $\|\boldsymbol{u} - \boldsymbol{u}_m\|_{L^2(0,T,\boldsymbol{H})} \to 0$ for $m \to \infty$, it follows that $\limsup_{m\to\infty} \|\boldsymbol{u} - \boldsymbol{u}_m\|_{L^2(0,T,\boldsymbol{W}^{\alpha,2}(\Omega))} \leqslant c\eta$. Now, η is an arbitrary positive number and therefore $\boldsymbol{u}_m \to \boldsymbol{u}$ in $L^2(0,T,\boldsymbol{W}^{\alpha,2}(\Omega))$. The proof is complete.

□

Lemma 3.8. *Let $T > 0$, $\alpha \in (0,1)$ and let M be defined as in Lemma 3.7. Then $M \hookrightarrow\hookrightarrow L^4(0,T,\boldsymbol{H})$.*

Proof. As in Lemma 3.7 we consider the sequence $\{\boldsymbol{u}_m\}_{m=1}^\infty \in M$, where $\|\boldsymbol{u}_m\|_M = \|\boldsymbol{u}_m\|_{L^2(0,T,\boldsymbol{V})} + \|\boldsymbol{u}'_m\|_{L^2(0,T,\boldsymbol{V}^*)} \leqslant c < \infty$, $\forall m \in N$ and $\boldsymbol{u}_m \to \boldsymbol{u}$ in $L^2(0,T,\boldsymbol{H})$. To prove Lemma 3.8, we will show that $\|\boldsymbol{u}_m - \boldsymbol{u}\|_{L^4(0,T,\boldsymbol{H})} \to 0$ for $m \to \infty$. First, it is easy to realize that for every $m \in N$ there exists a $t_m \in [0,T]$ such that $\|\boldsymbol{u}_m(t_m)\|_{\boldsymbol{H}} \leqslant c_1$, where c_1 does not depend on m. Next, we can also suppose that $\|\boldsymbol{u}_m\|_{L^\infty(0,T,\boldsymbol{H})} \leqslant c_1$ for all $m \in N$. This follows from the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_m(t)\|^2_{\boldsymbol{H}} = 2\left\langle \boldsymbol{u}'_m(t), \boldsymbol{u}_m(t)\right\rangle \quad \forall m \in N, \text{ a.e. } t \in (0,T)$$

and from the inequalities

$$\begin{aligned} &\|\boldsymbol{u}_m(t)\|^2_{\boldsymbol{H}} \leqslant \|\boldsymbol{u}_m(t_m)\|^2_{\boldsymbol{H}} + \left|2\int_{t_m}^t \left\langle \boldsymbol{u}'_m(s), \boldsymbol{u}_m(s)\right\rangle\, \mathrm{d}s\right| \\ &\leqslant c_1^2 + 2\|\boldsymbol{u}'_m\|_{L^2(0,T,\boldsymbol{V}^*)}\|\boldsymbol{u}_m\|_{L^2(0,T,\boldsymbol{V})} \leqslant c_1^2 + 2c^2, \quad \forall m \in N,\ t \in [0,T]. \end{aligned}$$

Therefore, we can write

$$\begin{aligned} \|\boldsymbol{u}_m - \boldsymbol{u}\|_{L^4(0,T,\boldsymbol{H})} &= \left(\int_0^T \|\boldsymbol{u}_m(t) - \boldsymbol{u}(t)\|^4_{\boldsymbol{H}}\, \mathrm{d}t\right)^{1/4} \\ &\leqslant \left(\|\boldsymbol{u}_m - \boldsymbol{u}\|^2_{L^\infty(0,T,\boldsymbol{H})} \int_0^T \|\boldsymbol{u}_m(t) - \boldsymbol{u}(t)\|^2_{\boldsymbol{H}}\, \mathrm{d}t\right)^{1/4} \\ &= \|\boldsymbol{u}_m - \boldsymbol{u}\|^{1/2}_{L^\infty(0,T,\boldsymbol{H})}\|\boldsymbol{u}_m - \boldsymbol{u}\|^{1/2}_{L^2(0,T,\boldsymbol{H})} \end{aligned}$$

and the proof follows easily. □

Lemma 3.9. *Let $T>0$ and $\alpha\in(0,1)$. Then $\boldsymbol{X}_0 \hookrightarrow\hookrightarrow \boldsymbol{X}_\alpha$.*

Proof. The proof is a consequence of Lemmas 3.7 and 3.8: Obviously, $\boldsymbol{X}_0\subset\boldsymbol{X}_\alpha$. Let now $\{\boldsymbol{u}_m\}_{m=1}^\infty$ be a bounded sequence in $\boldsymbol{X}_0$, that is $\|\boldsymbol{u}_m'\|_{L^2(0,T,\boldsymbol{V})}+\|\boldsymbol{u}_m''\|_{L^2(0,T,\boldsymbol{V}^*)}\leqslant c<\infty$, $\forall m\in N$. As a consequence of Lemma 3.7 and Lemma 3.8 we can suppose that there exists $\boldsymbol{U}\in M$ (for M see Lemma 3.7) such that $\boldsymbol{U}\in L^2(0,T,\boldsymbol{W}^{\alpha,2}(\Omega))\cap L^4(0,T,\boldsymbol{H})$ and $\boldsymbol{u}_m'\to\boldsymbol{U}$ both in $L^2(0,T,\boldsymbol{W}^{\alpha,2}(\Omega))$ and $L^4(0,T,\boldsymbol{H})$. Let $\boldsymbol{u}(t)=\int_0^t\boldsymbol{U}(s)\,\mathrm{d}s$, $\forall t\in[0,T]$. One can see then that $\boldsymbol{u}\in\boldsymbol{X}_0\subset\boldsymbol{X}_\alpha$ and $\|\boldsymbol{u}-\boldsymbol{u}_m\|_{\boldsymbol{X}_\alpha}=\|\boldsymbol{u}'-\boldsymbol{u}_m'\|_{L^2(0,T,\boldsymbol{W}^{\alpha,2}(\Omega))}+\|\boldsymbol{u}'-\boldsymbol{u}_m'\|_{L^4(0,T,\boldsymbol{L}^2(\Omega))}\to 0$ for $m\to\infty$. The proof is complete. □

Lemma 3.10. *Let $T>0$ and $\alpha\in(0,1)$. Then $X_0^\theta \hookrightarrow\hookrightarrow X_\alpha^\theta$.*

Proof. The same as for Lemmas 3.7, 3.8 and 3.9. □

Lemma 3.11. *The operator S from Definition 3.2 is a linear continuous one-to-one operator from $\boldsymbol{X}$ onto $\boldsymbol{Y}$. Moreover, $\|\boldsymbol{u}\|_{\boldsymbol{X}}\leqslant c\|S\boldsymbol{u}\|_{\boldsymbol{Y}}$, where $c=2+\nu^{1/2}+\nu^{-1/2}+\nu^{-1}$.*

Proof. For the proof see e.g. [11]. □

Lemma 3.12. *The operator S^θ from Definition 3.3 is a linear continuous one-to-one operator from X^θ onto Y^θ. Moreover, $\|\vartheta\|_{X^\theta}\leqslant c\|S^\theta\vartheta\|_{Y^\theta}$, where $c=2+\kappa^{1/2}+\kappa^{-1/2}+\kappa^{-1}$.*

Proof. The same as for Lemma 3.11. □

Lemma 3.13. *The mapping P from Definition 3.4 maps $\boldsymbol{X}_{7/8}\times X_{7/8}^\theta$ into $\boldsymbol{Y}$, $\|P(\boldsymbol{u},\vartheta)\|_{\boldsymbol{Y}}\leqslant c(1+T^{1/84}\|\vartheta\|_{X_{7/8}^\theta}+T^{1/42}\|\boldsymbol{u}\|_{\boldsymbol{X}_{7/8}}^2)$, where c depends only on $\|\boldsymbol{g}\|_{\boldsymbol{L}^\infty(\Omega)}$, $\|\boldsymbol{f}'\|_{L^2(0,T,\boldsymbol{V}^*)}$, $\|\varrho\|_{\tilde{X}^\theta}$, $\|\vartheta_{\mathrm{ref}}'\|_{L^2(0,T,L^2(\Omega))}$, $\|\boldsymbol{\varphi}\|_{\tilde{\boldsymbol{X}}}$, $\|\boldsymbol{f}(0)-\nu((\tilde{\boldsymbol{u}}_0,\cdot))-b(\tilde{\boldsymbol{u}}_0,\tilde{\boldsymbol{u}}_0,\cdot)\|_{\boldsymbol{H}}$, $\|\tilde{\vartheta}_0-\vartheta_{\mathrm{ref}}(0)\|_{L^2(\Omega)}$, $\|\boldsymbol{u}_0\|$ and the constants c_1–c_8 from (11)–(18).*

Proof. Let $\boldsymbol{u}\in\boldsymbol{X}_{7/8}$ and $\vartheta\in X_{7/8}^\theta$. It follows from (19), (23) and (27) that $\boldsymbol{f}(0)-\nu((\tilde{\boldsymbol{u}}_0,\cdot))-b_r(\tilde{\boldsymbol{u}}_0,\tilde{\boldsymbol{u}}_0,\cdot)\in\boldsymbol{H}$. We have from (23), (24), (28) and (29) that $(\boldsymbol{\varphi}'(0),\cdot)\in\boldsymbol{H}$ and $((\tilde{\vartheta}_0-\vartheta_{\mathrm{ref}}(0))\boldsymbol{g},\cdot)\in\boldsymbol{H}$. Therefore

$$
\begin{aligned}
(35)\qquad \|\boldsymbol{F}(0)\|_{\boldsymbol{H}} &= \|\boldsymbol{f}(0)-(\boldsymbol{\varphi}'(0),\cdot)-\nu((\tilde{\boldsymbol{u}}_0,\cdot))\\
&\qquad -b_r(\tilde{\boldsymbol{u}}_0,\tilde{\boldsymbol{u}}_0,\cdot)+\big((\tilde{\vartheta}_0-\vartheta_{\mathrm{ref}}(0))\boldsymbol{g},\cdot\big)\|_{\boldsymbol{H}}\\
&\leqslant c(\|\boldsymbol{f}(0)-\nu((\tilde{\boldsymbol{u}}_0,\cdot))-b(\tilde{\boldsymbol{u}}_0,\tilde{\boldsymbol{u}}_0,\cdot)\|_{\boldsymbol{H}}\\
&\qquad +\|\boldsymbol{\varphi}\|_{\tilde{\boldsymbol{X}}}+\|\tilde{\vartheta}_0-\vartheta_{\mathrm{ref}}(0)\|_{L^2(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^\infty(\Omega)}),
\end{aligned}
$$

where $\boldsymbol{F}$ was defined in (34). We will now show that $\boldsymbol{F} \in L^2(0,T,\boldsymbol{V}^*)$ by estimating $\|\boldsymbol{F}\|_{L^2(0,T,\boldsymbol{V}^*)}$. Clearly,

$$\|(\boldsymbol{\varphi}',\cdot)\|_{L^2(0,T,\boldsymbol{V}^*)} \leqslant c\|\boldsymbol{\varphi}'\|_{L^2(0,T,\tilde{\boldsymbol{V}})}, \tag{36}$$

$$\|\nu((\boldsymbol{u}_0,\cdot))\|_{L^2(0,T,\boldsymbol{V}^*)} \leqslant \left(\int_0^T \nu^2\|\boldsymbol{u}_0\|^2\,\mathrm{d}t\right)^{1/2} = \nu T^{1/2}\|\boldsymbol{u}_0\|, \tag{37}$$

$$\|\nu((\boldsymbol{\varphi},\cdot))\|_{L^2(0,T,\boldsymbol{V}^*)} \leqslant \nu\|\boldsymbol{\varphi}\|_{L^2(0,T,\tilde{\boldsymbol{V}})}, \tag{38}$$

$$\begin{aligned}\big\|\big((\vartheta+\varrho+\vartheta_0-\vartheta_{\mathrm{ref}})\boldsymbol{g},\cdot\big)\big\|_{L^2(0,T,\boldsymbol{V}^*)} &\leqslant cT^{1/2}\|\boldsymbol{g}\|_{\boldsymbol{L}^\infty(\Omega)}\\ &\times\left(\|\vartheta\|_{X^\theta_{7/8}}+\|\varrho\|_{\tilde{X}^\theta}+\|\vartheta_0\|_\theta+\|\vartheta_{\mathrm{ref}}\|_{L^2(0,T,L^2(\Omega))}\right).\end{aligned} \tag{39}$$

Let us denote for simplicity $\boldsymbol{w}=\boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0$. Let $\boldsymbol{v}\in\boldsymbol{V}$. Then $\forall t\in[0,T]$

$$\begin{aligned}|b_r(\boldsymbol{w}(t),\boldsymbol{w}(t),\boldsymbol{v})| &\leqslant \int_\Omega |w_j(t)w_i(t)\frac{\partial v_i}{\partial x_j}|\,\mathrm{d}x+\int_{\Gamma_2}|w_j(t)n_jw_i(t)v_i|\,\mathrm{d}S\\ &\leqslant c\left(\|\boldsymbol{w}(t)\|^2_{\boldsymbol{L}^4(\Omega)}+\|\boldsymbol{w}(t)\|^2_{\boldsymbol{L}^3(\Gamma_2)}\right)\|\boldsymbol{v}\|\\ &\leqslant c\left(\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{7/8,2}(\Omega)}+\|\boldsymbol{\varphi}(t)\|_{\tilde{\boldsymbol{V}}}+\|\boldsymbol{u}_0\|\right)^2\|\boldsymbol{v}\|\\ &\leqslant c\left(T^{1/2}\|\boldsymbol{u}'\|_{L^2(0,T,\boldsymbol{W}^{7/8,2}(\Omega))}+\|\boldsymbol{\varphi}(t)\|_{\tilde{\boldsymbol{V}}}+\|\boldsymbol{u}_0\|\right)^2\|\boldsymbol{v}\|.\end{aligned}$$

Consequently,

$$\begin{aligned}&\|b_r(\boldsymbol{w},\boldsymbol{w},\cdot)\|_{L^2(0,T,\boldsymbol{V}^*)}\\ &\quad\leqslant\left(\int_0^T c^2(T^{1/2}\|\boldsymbol{u}'\|_{L^2(0,T,\boldsymbol{W}^{7/8,2}(\Omega))}+\|\boldsymbol{\varphi}(t)\|_{\tilde{\boldsymbol{V}}}+\|\boldsymbol{u}_0\|)^4\,\mathrm{d}t\right)^{1/2}\\ &\quad\leqslant c\left(T^{3/2}\|\boldsymbol{u}'\|^2_{L^2(0,T,\boldsymbol{W}^{7/8,2}(\Omega))}+\|\boldsymbol{\varphi}\|^2_{L^4(0,T,\tilde{\boldsymbol{V}})}+T^{1/2}\|\boldsymbol{u}_0\|^2\right)\\ &\quad\leqslant c\left(T^{3/2}\|\boldsymbol{u}\|^2_{\boldsymbol{X}_{7/8}}+\|\boldsymbol{\varphi}\|^2_{\tilde{\boldsymbol{X}}}+T^{1/2}\|\boldsymbol{u}_0\|^2\right).\end{aligned} \tag{40}$$

The estimate of the norm of $\boldsymbol{F}$ in $L^2(0,T,\boldsymbol{V}^*)$ follows from (36)–(40).

Let us now estimate $\boldsymbol{F}'$ in the norm of $L^2(0,T,\boldsymbol{V}^*)$. To do this, we note first that the derivative of $b_r(\boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0,\boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0,\cdot)$ in $L^2(0,T,\boldsymbol{V}^*)$ is $b_r(\boldsymbol{u}'+\boldsymbol{\varphi}',\boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0,\cdot)+b_r(\boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0,\boldsymbol{u}'+\boldsymbol{\varphi}',\cdot)$. This follows for example from Theorem (1.7), p. 153, in [5]. Therefore, we will estimate $\|b_r(\boldsymbol{u}'+\boldsymbol{\varphi}',\boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0,\cdot)\|_{L^2(0,T,\boldsymbol{V}^*)}$. The estimate of $\|b_r(\boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0,\boldsymbol{u}'+\boldsymbol{\varphi}',\cdot)\|_{L^2(0,T,\boldsymbol{V}^*)}$ proceeds in the same way. For a.e. $t\in(0,T)$

and for every $\boldsymbol{v}\in\boldsymbol{V}$ we have

$$\begin{aligned}
&|b_r\big(\boldsymbol{u}'(t)+\boldsymbol{\varphi}'(t),\boldsymbol{u}(t)+\boldsymbol{\varphi}(t)+\boldsymbol{u}_0,\boldsymbol{v}\big)|\\
&\qquad=\bigg|\int_\Omega \big(u_i'(t)+\varphi_i'(t)\big)\big(u_j(t)+\varphi_j(t)+u_{0j}\big)\frac{\partial v_i}{\partial x_j}\,\mathrm{d}x\\
&\qquad\quad+\int_{\Gamma_2}\big(u_j'(t)+\varphi_j'(t)\big)n_j\big(u_i(t)+\varphi_i(t)+u_{0i}\big)v_i\,\mathrm{d}S\bigg|\\
&\qquad\leqslant\|\boldsymbol{v}\|\|\boldsymbol{u}'(t)+\boldsymbol{\varphi}'(t)\|_{\boldsymbol{L}^4(\Omega)}\|\boldsymbol{u}(t)+\boldsymbol{\varphi}(t)+\boldsymbol{u}_0\|_{\boldsymbol{L}^4(\Omega)}\\
&\qquad\quad+\|\boldsymbol{v}\|_{\boldsymbol{L}^4(\partial\Omega)}\|\boldsymbol{u}'(t)+\boldsymbol{\varphi}'(t)\|_{\boldsymbol{L}^{12/5}(\Gamma_2)}\|\boldsymbol{u}(t)+\boldsymbol{\varphi}(t)+\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\Gamma_2)}\\
&\qquad\leqslant c\|\boldsymbol{v}\|\,\big(\|\boldsymbol{u}'(t)\|_{\boldsymbol{W}^{5/6,2}(\Omega)}+\|\boldsymbol{\varphi}'(t)\|_{\tilde{\boldsymbol{V}}}\big)\\
&\qquad\quad\times\big(\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{7/8,2}(\Omega)}+\|\boldsymbol{\varphi}(t)\|_{\tilde{\boldsymbol{V}}}+\|\boldsymbol{u}_0\|\big)\\
&\qquad\leqslant c\|\boldsymbol{v}\|\big(\|\boldsymbol{u}'(t)\|^{20/21}_{\boldsymbol{W}^{7/8,2}(\Omega)}\|\boldsymbol{u}'(t)\|^{1/21}_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{\varphi}'(t)\|_{\tilde{\boldsymbol{V}}}\big)\\
&\qquad\quad\times\Big(T^{1/2}\|\boldsymbol{u}\|_{\boldsymbol{X}_{7/8}}+(1+T^{1/2})\|\boldsymbol{\varphi}\|_{\tilde{\boldsymbol{X}}}+\|\boldsymbol{u}_0\|\Big)
\end{aligned}$$

and this implies that

$$\begin{aligned}
(41)\quad&\|b_r(\boldsymbol{u}'+\boldsymbol{\varphi}',\boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0,\cdot)\|_{L^2(0,T,\boldsymbol{V}^*)}\\
&\leqslant c\Big(T^{1/2}\|\boldsymbol{u}\|_{\boldsymbol{X}_{7/8}}+(1+T^{1/2})\|\boldsymbol{\varphi}\|_{\tilde{\boldsymbol{X}}}+\|\boldsymbol{u}_0\|\Big)\\
&\quad\times\bigg(\int_0^T\Big(\|\boldsymbol{u}'(t)\|^{20/21}_{\boldsymbol{W}^{7/8,2}(\Omega)}\|\boldsymbol{u}'(t)\|^{1/21}_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{\varphi}'(t)\|_{\tilde{\boldsymbol{V}}}\Big)^2\,\mathrm{d}t\bigg)^{1/2}\\
&\leqslant c\Big(T^{1/2}\|\boldsymbol{u}\|_{\boldsymbol{X}_{7/8}}+(1+T^{1/2})\|\boldsymbol{\varphi}\|_{\tilde{\boldsymbol{X}}}+\|\boldsymbol{u}_0\|\Big)\\
&\quad\times\bigg(\Big[\int_0^T\|\boldsymbol{u}'(t)\|^2_{\boldsymbol{W}^{7/8,2}(\Omega)}\,\mathrm{d}t\Big]^{10/21}\Big[\int_0^T\|\boldsymbol{u}'(t)\|^4_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t\Big]^{1/84}T^{1/84}\\
&\quad+\Big[\int_0^T\|\boldsymbol{\varphi}'(t)\|^2_{\tilde{\boldsymbol{V}}}\,\mathrm{d}t\Big]^{1/2}\bigg)\\
&\leqslant c\Big(T^{1/2}\|\boldsymbol{u}\|_{\boldsymbol{X}_{7/8}}+(1+T^{1/2})\|\boldsymbol{\varphi}\|_{\tilde{\boldsymbol{X}}}+\|\boldsymbol{u}_0\|\Big)\Big(\|\boldsymbol{u}\|_{\boldsymbol{X}_{7/8}}T^{1/84}+\|\boldsymbol{\varphi}\|_{\tilde{\boldsymbol{X}}}\Big)\\
&\leqslant c\Big(T^{1/42}\|\boldsymbol{u}\|^2_{\boldsymbol{X}_{7/8}}+\|\boldsymbol{\varphi}\|^2_{\tilde{\boldsymbol{X}}}+\|\boldsymbol{u}_0\|^2\Big).
\end{aligned}$$

In the second inequality of (41) we have used the fact that $\boldsymbol{u}'\in L^4(0,T,\boldsymbol{L}^2(\Omega))$. To complete the estimate of $\boldsymbol{F}'$ in the norm of $L^2(0,T,\boldsymbol{V}^*)$, it is sufficient to realize that $\boldsymbol{\varphi}''$ or $\nu((\boldsymbol{\varphi}',\cdot))$ is respectively the derivative of $(\boldsymbol{\varphi}',\cdot)$ or $\nu\big((\boldsymbol{\varphi}+\boldsymbol{u}_0,\cdot)\big)$ in $L^2(0,T,\boldsymbol{V}^*)$ and

$$\|\boldsymbol{\varphi}''\|_{L^2(0,T,\boldsymbol{V}^*)}\leqslant\|\varphi\|_{\tilde{\boldsymbol{X}}},\tag{42}$$

$$\|\nu((\boldsymbol{\varphi}',\cdot))\|_{L^2(0,T,\boldsymbol{V}^*)}\leqslant\nu\|\varphi\|_{\tilde{\boldsymbol{X}}}.\tag{43}$$

Since $\big((\vartheta'+\varrho'-\vartheta'_{\rm ref})\boldsymbol{g},\cdot\big)$ is the derivative of $\big((\vartheta+\varrho+\vartheta_0-\vartheta_{\rm ref})\boldsymbol{g},\cdot\big)$ in $L^2(0,T,\boldsymbol{V}^*)$ and

$$\begin{aligned}\Big(\int_0^T \|\vartheta'(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\Big)^{1/2} & \\ &\leqslant c\Big(\int_0^T \|\vartheta'(t)\|^2_{W^{5/6,2}(\Omega)}\,\mathrm{d}t\Big)^{1/2}\\ &\leqslant c\Big(\int_0^T \|\vartheta'(t)\|^{40/21}_{\boldsymbol{W}^{7/8,2}(\Omega)}\|\vartheta'(t)\|^{2/21}_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t\Big)^{1/2}\\ &\leqslant c\Big(\int_0^T \|\vartheta'(t)\|^2_{\boldsymbol{W}^{7/8,2}(\Omega)}\,\mathrm{d}t\Big)^{10/21}\Big(\int_0^T \|\vartheta'(t)\|^4_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t\Big)^{1/84}T^{1/84}\\ &= cT^{1/84}\|\vartheta'\|^{20/21}_{L^2(0,T,W^{7/8,2}(\Omega))}\|\vartheta'\|^{1/21}_{L^4(0,T,L^2(\Omega))}\leqslant cT^{1/84}\|\vartheta\|_{X^\theta_{7/8}},\end{aligned}$$

we have

$$\begin{aligned}(44)\qquad \|\big((\vartheta'+\varrho'-\vartheta'_{\rm ref})\boldsymbol{g},\cdot\big)\|_{L^2(0,T,\boldsymbol{V}^*)} \leqslant c\|\boldsymbol{g}\|_{\boldsymbol{L}^\infty(\Omega)}\big(&T^{1/84}\|\vartheta\|_{X^\theta_{7/8}}\\ &+\|\varrho\|_{\tilde X^\theta}+\|\vartheta'_{\rm ref}\|_{L^2(0,T,L^2(\Omega))}\big).\end{aligned}$$

It now follows from (35), (41), (42), (43) and (44) that

$$\|P(\boldsymbol{u},\vartheta)\|_{\boldsymbol{Y}}\leqslant c\Big(1+T^{1/84}\|\vartheta\|_{X^\theta_{7/8}}+T^{1/42}\|\boldsymbol{u}\|^2_{\boldsymbol{X}_{7/8}}\Big),$$

where c depends only on $\|\boldsymbol{f}(0)-\nu((\tilde{\boldsymbol{u}}_0,\cdot))-b(\tilde{\boldsymbol{u}}_0,\tilde{\boldsymbol{u}}_0,\cdot)\|_{\boldsymbol{H}}$, $\|\boldsymbol{f}'\|_{L^2(0,T,\boldsymbol{V}^*)}$, $\|\boldsymbol{g}\|_{\boldsymbol{L}^\infty(\Omega)}$, $\|\tilde\vartheta_0-\vartheta_{\rm ref}(0)\|_{L^2(\Omega)}$, $\|\varrho\|_{\tilde X^\theta}$, $\|\vartheta'_{\rm ref}\|_{L^2(0,T,L^2(\Omega))}$, $\|\boldsymbol{\varphi}\|_{\tilde{\boldsymbol{X}}}$, $\|\boldsymbol{u}_0\|$ and the constants c_1–c_8 from (11)–(18). The proof is complete. □

Lemma 3.14. *The mapping P^θ from Definition 3.5 maps $\boldsymbol{X}_{7/8}\times X^\theta_{7/8}$ into Y^θ and $\|P^\theta(\boldsymbol{u},\vartheta)\|_{Y^\theta}\leqslant c\big(1+T^{1/42}\|\vartheta\|^2_{X^\theta_{7/8}}+T^{1/42}\|\boldsymbol{u}\|^2_{\boldsymbol{X}_{7/8}}\big)$, where c depends on $\|\varrho\|_{\tilde X^\theta}$, $\|\boldsymbol{\varphi}\|_{\tilde{\boldsymbol{X}}}$, $\|\vartheta_0\|_\theta$, $\|f^\theta(0)-\int_{\Gamma_5}\kappa\gamma(\tilde\vartheta_0-\vartheta_\delta(0))\cdot\mathrm{d}S-\kappa((\tilde\vartheta_0,\cdot))_\theta-C(\tilde{\boldsymbol{u}}_0,\tilde\vartheta_0,\cdot)\|_{L^2(\Omega)}$, $\|\varrho\|_{\tilde X^\theta}$, $\|\boldsymbol{u}_0\|$, $\|\vartheta'_\delta\|_{L^2(0,T,L^2(\Gamma_5))}$ and the constants c_1–c_8 from (11)–(18).*

P r o o f. The same as for Lemma 3.13. □

Lemma 3.15. *Let $\boldsymbol{u}\in\boldsymbol{X}_{7/8}$, $\vartheta\in X^\theta_{7/8}$ and let $\boldsymbol{w}\in\boldsymbol{X}_0$ solve the equation*

$$\begin{aligned}(45)\quad (\boldsymbol{w}',\boldsymbol{v})+\nu((\boldsymbol{w},\boldsymbol{v}))=\langle\boldsymbol{f},\boldsymbol{v}\rangle\ &+\big((\vartheta+\varrho+\vartheta_0-\vartheta_{\rm ref})\boldsymbol{g},\boldsymbol{v}\big)\\ &-(\boldsymbol{\varphi}',\boldsymbol{v})-\nu\big((\boldsymbol{\varphi}+\boldsymbol{u}_0,\boldsymbol{v})\big)\\ &-b_r(\boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0,\boldsymbol{u}+\boldsymbol{\varphi}+\boldsymbol{u}_0,\boldsymbol{v}),\quad\forall\boldsymbol{v}\in\boldsymbol{V}.\end{aligned}$$

Then $\|\boldsymbol{w}\|_{\boldsymbol{X}_{7/8}} \leqslant c(1 + \|\boldsymbol{w}\|_{\boldsymbol{X}_0})$.

P r o o f. Clearly,

$$\|\boldsymbol{w}'\|_{L^\infty(0,T,\boldsymbol{H})} \leqslant \|\boldsymbol{w}'(0)\|_{\boldsymbol{H}} + \|\boldsymbol{w}\|_{\boldsymbol{X}_0}.$$

Therefore

$$\begin{aligned} \|\boldsymbol{w}\|_{\boldsymbol{X}_{7/8}} &= \|\boldsymbol{w}'\|_{L^2(0,T,\boldsymbol{W}^{7/8,2}(\Omega))} + \|\boldsymbol{w}'\|_{L^4(0,T,\boldsymbol{L}^2(\Omega))} \\ &\leqslant c\|\boldsymbol{w}'\|_{L^2(0,T,\boldsymbol{V})} + T^{1/4}\|\boldsymbol{w}'\|_{L^\infty(0,T,\boldsymbol{H})} \\ &\leqslant c\|\boldsymbol{w}\|_{\boldsymbol{X}_0} + T^{1/4}\left(\|\boldsymbol{w}'(0)\|_{\boldsymbol{H}} + \|\boldsymbol{w}\|_{\boldsymbol{X}_0}\right). \end{aligned} \tag{46}$$

It follows from (45) that $(\boldsymbol{w}'(0),\boldsymbol{v}) = \langle \boldsymbol{f}(0),\boldsymbol{v}\rangle + \big((\tilde{\vartheta}_0 - \vartheta_{\text{ref}}(0))\boldsymbol{g},\boldsymbol{v}\big) - (\boldsymbol{\varphi}'(0),\boldsymbol{v}) - \nu((\tilde{\boldsymbol{u}}_0,\boldsymbol{v})) - b(\tilde{\boldsymbol{u}}_0,\tilde{\boldsymbol{u}}_0,\boldsymbol{v})$ for every $\boldsymbol{v} \in \boldsymbol{V}$. We get from here and from (35) that $\|\boldsymbol{w}'(0)\|_{\boldsymbol{H}} \leqslant \|\boldsymbol{f}(0) - \nu((\tilde{\boldsymbol{u}}_0,\cdot)) - b(\tilde{\boldsymbol{u}}_0,\tilde{\boldsymbol{u}}_0,\cdot)\|_{\boldsymbol{H}} + \|\boldsymbol{\varphi}\|_{\tilde{\boldsymbol{X}}} + \|\tilde{\vartheta}_0 - \vartheta_{\text{ref}}(0)\|_{L^2(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^\infty(\Omega)}$. The proof now follows from (46). □

Lemma 3.16. *Let* $\boldsymbol{u} \in \boldsymbol{X}_{7/8}$, $\vartheta \in X^\theta_{7/8}$ *and let* $w \in X^\theta_0$ *solve the equation*

$$\begin{aligned} (w',\varphi) + \kappa((w,\varphi))_\theta = \langle f^\theta,\varphi\rangle &- \int_{\Gamma_5} \kappa\gamma(\vartheta + \varrho + \vartheta_0 - \vartheta_\delta)\varphi\,\mathrm{d}S \\ &- (\varrho',\varphi) - \kappa((\varrho + \vartheta_0,\varphi))_\theta \\ &- C_r(\boldsymbol{u} + \boldsymbol{\varphi} + \boldsymbol{u}_0, \vartheta + \varrho + \vartheta_0, \varphi) \qquad \forall \varphi \in V^\theta. \end{aligned}$$

Then $\|w\|_{X_{7/8}} \leqslant c(1 + \|w\|_{X^\theta_0})$.

P r o o f. The same as for Lemma 3.15. □

P r o o f of Theorem 3.1. It follows from Lemmas 3.11, 3.12, 3.13, 3.14, 3.15 and 3.16 that for every $\boldsymbol{u} \in \boldsymbol{X}_{7/8}$ and $\vartheta \in X^\theta_{7/8}$ we have

$$\begin{aligned} &\|Q_{7/8}S^{-1}P(\boldsymbol{u},\vartheta)\|_{\boldsymbol{X}_{7/8}} + \|Q^\theta_{7/8}(S^\theta)^{-1}P^\theta(\boldsymbol{u},\vartheta))\|_{X^\theta_{7/8}} \\ &\qquad \leqslant c\left(1 + T^{1/42}\|\boldsymbol{u}\|^2_{\boldsymbol{X}_{7/8}} + T^{1/42}\|\vartheta\|^2_{X^\theta_{7/8}}\right), \end{aligned} \tag{47}$$

where c depends only on $\|\vartheta_0\|_\theta$, $\|\vartheta'_\delta\|_{L^2(0,T,L^2(\Gamma_5))}$, $\|\boldsymbol{g}\|_{\boldsymbol{L}^\infty(\Omega)}$, $\|\tilde{\vartheta}_0 - \vartheta_{\text{ref}}(0)\|_{L^2(\Omega)}$, $\|\varrho\|_{\tilde{X}^\theta}$, $\|\vartheta'_{\text{ref}}\|_{L^2(0,T,L^2(\Omega))}$, $\|\boldsymbol{u}_0\|$, $\|\boldsymbol{f}'\|_{L^2(0,T,\boldsymbol{V}^*)}$, $\|\boldsymbol{\varphi}\|_{\tilde{\boldsymbol{X}}}$, $\|\boldsymbol{u}_0\|$, $\|\boldsymbol{f}(0) - \nu((\tilde{\boldsymbol{u}}_0,\cdot)) - b(\tilde{\boldsymbol{u}}_0,\tilde{\boldsymbol{u}}_0,\cdot)\|_{\boldsymbol{H}}$, $\|f^\theta(0) - \int_{\Gamma_5}\kappa\gamma(\tilde{\vartheta}_0 - \vartheta_\delta)\cdot\mathrm{d}S - \kappa((\tilde{\vartheta}_0,\cdot))_\theta - C(\tilde{\boldsymbol{u}}_0,\tilde{\vartheta}_0,\cdot)\|_{L^2(\Omega)}$ and the constants c_1–c_8 from (11) – (18). It follows from Lemmas 3.9 and 3.10 and from (47) that the mapping Z (for its definition see the text following Definition 3.6) is compact and maps the convex set $H = \{(\boldsymbol{u},\vartheta) \in \boldsymbol{X}_{7/8} \times X^\theta_{7/8};\ \|\boldsymbol{u}\|_{\boldsymbol{X}_{7/8}} + \|\vartheta\|_{X^\theta_{7/8}} \leqslant 2c\}$

into itself if it is defined on the time interval $(0,T_*)$, where $T_* = \min(T,T_1)$ and $T_1 = (1/2c)^{84}$. The mapping Z is continuous on H — this can be proved using the same method as in the proof of Lemma 3.13. It follows from the Schauder principle (see e.g. [3], Theorem 7.5.5) that there exists a fixed point of Z in H which is a solution of Problem 2.1. The proof of Theorem 3.1 is complete. $\square$

4. Uniqueness theorem

Theorem 4.1. *The solution $\boldsymbol{u} \in \boldsymbol{X}_0$, $\vartheta \in X_0^\theta$ from Theorem 3.1 is unique.*

Proof (briefly). Let us assume that $\boldsymbol{u}_1 \in \boldsymbol{X}_0$, $\vartheta_1 \in X_0^\theta$ and $\boldsymbol{u}_2 \in \boldsymbol{X}_0$, $\vartheta_2 \in X_0^\theta$ are two solutions from Theorem 3.1. The difference $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $\vartheta = \vartheta_1 - \vartheta_2$ satisfies, for every $\boldsymbol{v} \in \boldsymbol{V}$ and $\psi \in V^\theta$, the identities

$$\begin{aligned}
&(\boldsymbol{u}', \boldsymbol{v}) + \nu((\boldsymbol{u}, \boldsymbol{v})) = (\vartheta \boldsymbol{g}, \boldsymbol{v}) - b(\boldsymbol{u}, \boldsymbol{u}_2 + \boldsymbol{\varphi} + \boldsymbol{u}_0, \boldsymbol{v}) - b(\boldsymbol{u}_1 + \boldsymbol{\varphi} + \boldsymbol{u}_0, \boldsymbol{u}, \boldsymbol{v}),\\
&(\vartheta', \psi) + \kappa((\vartheta, \psi))_\theta = -\int_{\Gamma_5} \kappa\gamma\vartheta\psi \,\mathrm{d}S - C(\boldsymbol{u}, \vartheta_2 + \varrho + \vartheta_0, \psi) - C(\boldsymbol{u}_1 + \boldsymbol{\varphi} + \boldsymbol{u}_0, \vartheta, \psi).
\end{aligned}$$

Therefore we get

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_{\boldsymbol{H}}^2 + \nu\|\boldsymbol{u}\|^2 \leqslant c\|\vartheta\|_{L^2(\Omega)}^2 &+ c\|\boldsymbol{u}\|_{\boldsymbol{H}}^2 + c(\|\boldsymbol{u}_1\|_{L^\infty(0,T,\boldsymbol{V})}\\
&+ \|\boldsymbol{\varphi}\|_{L^\infty(0,T,\tilde{\boldsymbol{V}})} + \|\boldsymbol{u}_0\|\\
&+ \|\boldsymbol{u}_2\|_{L^\infty(0,T,\boldsymbol{V})})\|\boldsymbol{u}\|^{7/4}\|\boldsymbol{u}\|_{\boldsymbol{H}}^{1/4},
\end{aligned} \tag{48}$$

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vartheta\|_{L^2(\Omega)}^2 + \kappa\|\vartheta\|^2 &\leqslant c\|\vartheta\|_{L^2(\Omega)}^2 + \frac{\kappa}{3}\|\vartheta\|^2 + c(\|\boldsymbol{u}_1\|_{L^\infty(0,T,\boldsymbol{V})}\\
&\quad + \|\boldsymbol{\varphi}\|_{L^\infty(0,T,\tilde{\boldsymbol{V}})} + \|\boldsymbol{u}_0\|)\|\vartheta\|^{7/4}\|\vartheta\|_{L^2(\Omega)}^{1/4}\\
&\quad + c(\|\vartheta_2\|_{L^\infty(0,T,V^\theta)} + \|\varrho\|_{L^\infty(0,T,W^{1,2}(\Omega))}\\
&\quad + \|\vartheta_0\|_{V^\theta})\|\boldsymbol{u}\|^{3/4}\|\boldsymbol{u}\|_{\boldsymbol{H}}^{1/4}\|\vartheta\|^{3/4}\|\vartheta\|_{L^2(\Omega)}^{1/4}\\
&\leqslant \big(c\|\vartheta\|_{L^2(\Omega)}^2 + \frac{\kappa}{3}\|\vartheta\|^2\big) + \big(c\|\vartheta\|_{L^2(\Omega)}^2 + \frac{\kappa}{3}\|\vartheta\|^2\big)\\
&\quad + \big(\frac{\nu}{6}\|\boldsymbol{u}\|^2 + c\|\boldsymbol{u}\|_{\boldsymbol{H}}^2 + \frac{\kappa}{3}\|\vartheta\|^2 + c\|\vartheta\|_{L^2(\Omega)}^2\big),
\end{aligned} \tag{49}$$

where the constant c depends only on the data and the solutions $\boldsymbol{u}_1$, ϑ_1 and $\boldsymbol{u}_2$, ϑ_2. It follows from (48) and (49) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_{\boldsymbol{H}}^2 + \frac{\nu}{3}\|\boldsymbol{u}\|^2 \leqslant c\|\boldsymbol{u}\|_{\boldsymbol{H}}^2 + c\|\vartheta\|_{L^2(\Omega)}^2, \tag{50}$$

(51) $$\frac{\mathrm{d}}{\mathrm{d}t}\|\vartheta\|^2_{L^2(\Omega)} \leqslant c\|\vartheta\|^2_{L^2(\Omega)} + \frac{\nu}{3}\|\boldsymbol{u}\|^2 + c\|\boldsymbol{u}\|^2_{\boldsymbol{H}}$$

and therefore by summing (50) and (51) we conclude that

(52) $$\frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{u}\|^2_{\boldsymbol{H}} + \|\vartheta\|^2_{L^2(\Omega)}) \leqslant c(\|\boldsymbol{u}\|^2_{\boldsymbol{H}} + \|\vartheta\|^2_{L^2(\Omega)}).$$

Since $\boldsymbol{u}(0) = \boldsymbol{0}$ and $\vartheta(0) = 0$ the proof follows immediately from (52) and the Gronwall lemma. □

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