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A SURVEY OF RESULTS ON NONLINEAR VENTTSEL PROBLEMS

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Abstract. We review the recent results for boundary value problems with boundary conditions given by second-order integral-differential operators. Particular attention has been paid to nonlinear problems (without integral terms in the boundary conditions) for elliptic and parabolic equations. For these problems we formulate some statements concerning a priori estimates and the existence theorems in Sobolev and Hölder spaces.

Keywords: Venttsel boundary conditions, elliptic equations, parabolic equations, a priori estimates, existence theorems, boundary value problems

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Introduction

The theory of solvability of the boundary value problems for nondivergent elliptic and parabolic equations describing stationary and nonstationary diffusion processes has been actively developing over the last two decades.

First and foremost, the classical boundary value problems have been studied for these equations. Nowadays, sufficiently complete results on solvability of the Dirichlet and the oblique derivative problems have been obtained in Hölder and Sobolev spaces. Surveys of these results and an extensive bibliography can be found in [8], [15], [17–18]. In addition, papers cited in the above-mentioned reviews we should mention also the publications [23–24] and [26].

In 1959, A. D. Venttsel introduced a *new class* of boundary conditions for elliptic equations given by second-order integral-differential operators (see [27], and also [12], [28]). From the probabilistic point of view, Venttsel boundary conditions are the *most general admissible boundary conditions*, since they include Dirichlet, Neumann, oblique derivative and mixed boundary conditions as special cases. Note also that the problems with Venttsel type conditions have applications to various fields of science and technology. Among them are the water wave theory and a model for heat transfer, as well as engineering problems of "hydraulic fracturing" in oil wells and aspects of financial mathematics (see [10], [13–14], [16], [25]).

In recent years, the studies of Venttsel problems have continued along diversity of avenues. In particular, a survey of results on linear problems, having an integral term in the boundary condition, can be found in [11]. We mention also the paper [19] which deals with applications of the potential theory to Venttsel problems.

Let us consider in more detail the nonlinear Venttsel problem for parabolic and elliptic equations. Such a problem (without integral terms) describes the diffusion process in a bounded domain combining boundary reflection and diffusion along a surface. This situation arises when the boundary is covered with a thin layer of a material having higher permeability. The boundary condition in that case is given by a nonlinear nondivergent equation of the second order with the principal term being a parabolic (elliptic) operator in tangential variables.

The results already obtained for Venttsel problems mainly concern the *nondegenerate* case, when the thickness of the boundary layer is a positive constant.

The examination of Venttsel problems for general second order elliptic equations was initiated by N. S. Trudinger in the late 1980's. The classical solvability of the problem for a *quasilinear* uniformly elliptic equation with a quasilinear uniformly elliptic Venttsel condition was established by Y. Luo and N. S. Trudinger in the papers [20–22]. This result was generalized in [7]. In this paper we prove the existence theorem in Sobolev spaces as well as the classical solvability result under the assumption that the boundary condition has *quadratic growth* with respect to the tangential gradient components. The restrictions on all the data of the problem are optimal in [7]. In particular, in the case of Sobolev's solutions, summable singularities with respect to independent variables are also admissible.

We also studied the nonstationary nondegenerate quasilinear Venttsel problem in the series of papers ([1], [6], [8–9]). Solvability in Sobolev and Hölder spaces was established under the weakest (natural) structure conditions similar to those in the above-described stationary case.

Thus, solvability results on nondegenerate Venttsel problems have been obtained under the equally general assumptions as for the classical boundary value problems. The case of *degenerate* Venttsel problems is more difficult and far from having been solved completely at the moment.

The classical existence and uniqueness of solutions of stationary *linear* degenerate problems was established in [22].

In the papers [4–5] we obtained some a priori estimates for solutions of nonlinear degenerate Venttsel problems. The notion of a *uniformly degenerating* boundary condition, describing the situation when the thickness of the boundary layer can vanishes, was introduced in the process. Namely, we suggest that the elliptic (parabolic) terms in the boundary operators contain a scalar multiplier that may vanish on some subsets of the boundary, while the first-order terms involved in the boundary conditions produce a nondegenerate nontangential operator. Experts believe that this type of degeneracy is natural for diffusion processes.

In [5], global Hölder estimates for Sobolev's solutions of stationary and nonstationary quasilinear problems were established by using a suitable combination of the techniques for the oblique derivative and the nondegenerate Venttsel problems. Again, as in the nondegenerate case, all results were obtained in [5] under the weakest (natural) assumptions.

Finally, gradient estimates for classical solutions to stationary degenerate Venttsel problems for fully nonlinear and quasilinear equations (under different assumptions on the smoothness of a solution) were obtained in [4]. Unfortunately, the arguments from [7] concerning gradient bounds are not applicable to a degenerate boundary condition. Instead of this, we use in [4] the ideas from [18]. It requires the smoothness of the right-hand side of the equation and the boundary condition for both fully nonlinear and quasilinear cases. The question of gradient estimates for solutions from Sobolev spaces is still an open one.

This paper is organized as follows. Section 1 is devoted to nondegenerate problems. For reasons of space, we formulate here the theorems concerning a priori estimates and the existence results in Sobolev and Hölder spaces only for solutions of nonstationary problems and provide all necessary references to analogous results for stationary problems. The case of the degenerate boundary condition is considered in Section 2, where we state some a priori estimates for solutions of nonstationary and stationary problems for quasilinear equations. References to the corresponding results for stationary problems for fully nonlinear equations can also be found there.

Notation. $x = (x_1, \dots, x_{n-1}, x_n)$ is a vector in $\mathbb{R}^n$ with the Euclidean norm $|x|$; $(x;t)$ is a point in $\mathbb{R}^{n+1}$; Ω is a bounded domain in $\mathbb{R}^n$ and $\partial\Omega$ is its boundary; $\mathbf{n}(x) = (\mathbf{n}_i(x))$ is the unit vector of the outward normal to $\partial\Omega$ at the point x. For a cylinder $Q = \Omega \times]0,T[$ we denote by $\partial''Q = \partial\Omega \times]0,T[$ its lateral surface and by $\partial' Q = \partial''Q \cup \{\overline{\Omega} \times \{0\}\}$ its parabolic boundary. We denote by $\Omega^+(Q^+)$ the part of Ω (of Q) lying in the halfspace $x_n > 0$ and by $\Gamma(\Omega^+)(\Gamma(Q^+))$ the part of $\partial\Omega^+$ (of $\partial''Q^+$) lying on the hyperplane $x_n = 0$. We define $B_R = \{x \in \mathbb{R}^n : |x| < R\}$, $\Gamma_R = \Gamma(B_R^+)$.

The indices i, j vary from 1 to n, while the indices l, m vary from 1 to $n-1$. Repeated indices indicate summation. The exponent q satisfies $n < q < \infty$.

The symbol D_i denotes the operator of differentiation with respect to x_i; in particular, $Du = (D_1 u, \dots, D_n u)$ is the gradient of u. Let d_i be the tangential differential operator on $\partial\Omega$, i.e., $d_i = D_i - \mathbf{n}_i\mathbf{n}_j D_j$. Then $du = (d_i u)$ is the tangential gradient of u on $\partial\Omega$; in particular, $du = (D_1 u, \dots, D_{n-1}u, 0)$ on $\Gamma(\Omega^+)$; $u_t = \frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial \mathbf{n}}$ is the normal derivative of u. Setting $p' = p - (p\mathbf{n})\mathbf{n}$ for a vector $p \in \mathbb{R}^n$ we define operators $\delta = \frac{\partial}{\partial u} + \frac{p}{|p|^2}\frac{\partial}{\partial x}$, $\delta' = \frac{\partial}{\partial u} + \frac{p'}{|p'|^2}\frac{\partial}{\partial x'}$.

We denote by $\|\cdot\|_{p,Q}$ the norm in $L_p(Q)$. We also introduce the following spaces:
$W_p^{2,1}(Q)$ with the norm $\|u\|_{W_p^{2,1}(Q)} = \|u_t\|_{p,Q} + \|D(Du)\|_{p,Q} + \|u\|_{p,Q}$;
$W_p^{2,1}(\partial''Q)$ with the norm $\|u\|_{W_p^{2,1}(\partial''Q)} = \|u_t\|_{p,\partial''Q} + \|d(du)\|_{p,\partial''Q} + \|u\|_{p,\partial''Q}$;
$V_p(Q)$ with the norm $\|u\|_{V_p(Q)} = \|u\|_{W_p^{2,1}(Q)} + \|u\|_{W_{p-1}^{2,1}(\partial''Q)}$;
the space $C(\overline{\Omega})$ $\big(C(\overline{Q})\big)$ of continuous functions on $\overline{\Omega}$ $(\overline{Q})$ with the norm $\|\cdot\|_\Omega$ $\big(\|\cdot\|_Q$, respectively$\big)$; the space $C^k(\overline{\Omega})$ of functions which have continuous derivatives up to the k-th order;
the Hölder spaces $C^\gamma(\overline{\Omega})$ and $C^{2+\gamma}(\overline{Q})$ $(0<\gamma<1)$ with the norms

$$\|u\|_{C^\gamma(\overline{\Omega})} = \|u\|_\Omega + [u]_{\gamma,\Omega},$$
$$\|u\|_{C^{2+\gamma}(\overline{Q})} = \|u\|_Q + \|D(Du)\|_Q + \|u_t\|_Q + [D(Du)]_{\gamma,Q} + [u_t]_{\gamma,Q},$$

where $[\cdot]_{\gamma,\Omega}$ is the corresponding Hölder constant with exponent γ, while $[\cdot]_{\gamma,Q}$ stands for the Hölder constant with exponent γ with respect to the parabolic distance $d_{\mathrm{par}}((x^1;t^1),(x^2;t^2)) = |x^1 - x^2| + |t^1 - t^2|^{1/2}$.

We set $f_+ = \max\{f, 0\}$, $f_- = \max\{-f, 0\}$, $\operatorname*{osc}_\Omega f = \sup_\Omega f - \inf_\Omega f$. We denote by $\operatorname{tr}(a)$ the trace of a matrix (a).

1. Nondegenerate case

We consider the initial-boundary value problem

$$u_t - a^{ij}(x,t,u,Du)D_iD_ju = a(x,t,u,Du) \qquad \text{in } Q = \Omega\times\,]0,T[\,, \tag{1.1}$$

$$u_t - \alpha^{ij}(x,t,u,du)d_id_ju = \alpha(x,t,u,Du) \quad \text{on } \partial''Q, \tag{1.2}$$

$$u|_{t=0} = 0 \qquad \text{in} \qquad \Omega. \tag{1.3}$$

Assume that (a^{ij}) is a symmetric matrix and the following natural structure conditions hold for all $(x,t)\in Q$, $z\in\mathbb{R}^1$, $p\in\mathbb{R}^n$:

$$\nu|\xi|^2 \leqslant a^{ij}(x,t,z,p)\xi_i\xi_j \leqslant \nu^{-1}|\xi|^2, \qquad \forall \xi\in\mathbb{R}^n; \tag{A0}$$

$$|a(x,t,z,p)| \leqslant \mu|p|^2 + b(x,t)|p| + \Phi_1(x,t); \tag{A1}$$

(A2) $a^{ij}(x,t,z,p)$ have the first-order derivatives with respect to x, z, p;

$$(1+|p|)\cdot\left|\frac{\partial a^{ij}(x,t,z,p)}{\partial p}\right| \leqslant \mu, \quad (1+|p|)|\delta a^{ij}(x,t,z,p)| \leqslant \mu|p| + \Phi_2(x,t); \tag{A3}$$

$$b,\ \Phi_1,\ \Phi_2 \in L_{q+2}(Q). \tag{A4}$$

Here ν, μ are positive constants.

The boundary condition (1.2) is assumed to be a *uniformly parabolic* Venttsel condition, i.e., (α^{ij}) is a symmetric matrix, the function $\alpha(x,t,z,p)$ is differentiable with respect to p and the following conditions hold for $(x,t)\in\partial''Q$, $z\in\mathbb{R}^1$, $p\in\mathbb{R}^n$:

(B0) $$\nu_1|\xi|^2 \leqslant \alpha^{ij}(x,t,z,p')\xi_i\xi_j \leqslant \nu_1^{-1}|\xi|^2, \qquad \forall\xi\in\mathbb{R}^n,\ \xi\perp\mathbf{n}(x);$$

(B) $$0\leqslant -\frac{\partial\alpha(x,t,z,p)}{\partial p}\mathbf{n}(x)\leqslant\beta(x,t).$$

In addition we require for any $(x,t)\in\partial''Q$, $z\in\mathbb{R}^1$, $p\in\mathbb{R}^n$ that the following natural structure conditions be fulfilled:

(B1) $$|\alpha(x,t,z,p')|\leqslant\mu_1|p'|^2+\beta(x,t)|p'|+\Theta_1(x,t);$$

(B2) $\alpha^{ij}(x,t,z,p')$ have the first-order derivatives with respect to x, z, p';

(B3) $$(1+|p'|)\cdot\left|\frac{\partial\alpha^{ij}(x,t,z,p')}{\partial p'}\right|\leqslant\mu_1,$$
$$(1+|p'|)\left|\delta'\alpha^{ij}(x,t,z,p')\right|\leqslant\mu_1|p'|+\Theta_2(x,t);$$

(B4) $$\beta,\ \Theta_1,\ \Theta_2\in L_{q+1}(\partial''Q).$$

Here ν_1, μ_1 are positive constants.

Theorem 1.1 (maximum principle). *Let $\partial\Omega\in W^2_{n+1}$. Assume that a function $u\in V_{n+1}(Q)\cap C(\overline{Q})$ satisfies*

(1.4) $$u_t-a^{ij}(x,t)D_iD_ju+b^i(x,t)D_iu+c(x,t)u=f(x,t)\quad \text{in } Q,$$

(1.5) $$u_t-\alpha^{sm}(x,t)D_sD_mu+\beta^i(x,t)D_iu+\gamma(x,t)u=\theta(x,t)\qquad\text{on}\quad\partial''Q.$$

Also we assume that the coefficients in (1.4), (1.5) *satisfy the following conditions:*

$$a^{ij}=a^{ji},\qquad \nu|\xi|^2\leqslant a^{ij}\xi_i\xi_j\leqslant\nu^{-1}|\xi|^2\quad\text{for } \xi\in\mathbb{R}^n;$$
$$\alpha^{sm}=\alpha^{ms},\quad \nu_1|\xi|^2\leqslant\alpha^{sm}\xi_s\xi_m\leqslant\nu_1^{-1}|\xi|^2\quad\text{for } \xi\in\mathbb{R}^n,\xi\perp\mathbf{n}(x);$$
$$0\leqslant\beta^i(x,t)\mathbf{n}_i(x)\quad\text{for } (x,t)\in\partial''Q;$$
$$f_+,\ |(b^i(x,t))|,\ c_-\in L_{n+1}(Q);\quad \theta_+,\ |(\beta^i(x,t))|,\ \gamma_-\in L_n(\partial''Q),$$

where ν, ν_1 are positive constants. Then u satisfies the estimate

$$\sup_Q u\leqslant C_1\Big\{\|f_+\|_{n+1,Q}+\|\theta_+\|_{n,\partial''Q}+\sup_\Omega u(\cdot,0)\Big\},$$

where C_1 depends only on n, ν, ν_1, T, diam(Ω), *the characteristics of $\partial\Omega$, the numbers $\|(b^i)\|_{n+1,Q}$, $\|(\beta^i)\|_{n,\partial''Q}$, and on the moduli of absolute continuity of the* *functions* $b^i(x,t)$, $c_-(x,t)$ *in the space* $L_{n+1}(Q)$ *and of the functions* $\beta^i(x,t)$, $\gamma_-(x,t)$ *in the space* $L_n(\partial''Q)$.

This theorem was proved in [8]. The proof is based on the Aleksandrov type local maximum principle (see [1], Theorem 1). The stationary analogue of the local theorem can be found in [6] (see Theorem 3). Note also that the stationary local maximum principle was proved in [20] only for $\beta^m = 0$.

Theorem 1.2 (Hölder estimates of solutions). *Suppose that* $\partial\Omega \in W^2_{n+2}$ *and a function* $u \in V_{n+1}(Q) \cap C(\overline{Q})$ *satisfies* (1.1)–(1.2) *and the initial condition* $u|_{t=0} = \varphi$ *in* Ω.

Let the following conditions hold:

(1) $|u| \leqslant M_0$ *in* $\overline{Q}$;
(2) $[\varphi]_{\gamma_1,\Omega} \leqslant M_{\gamma_1}$ *with* $\gamma_1 = const > 0$;
(3) *the conditions* (A0), (A1), (B), (B0), (B1) *are fulfilled for* $|z| \leqslant M_0$;
(4) $b \in L_{n+2}(Q)$, $\Phi_1 \in L_{n+1}(Q)$, $\beta \in L_{n+1}(\partial''Q)$, $\Theta_1 \in L_n(\partial''Q)$.

Then there exists $\gamma_2 > 0$, *completely determined by* n, ν, ν_1, γ_1 *and* $\partial\Omega$, *such that*

$$[u]_{\gamma_2,Q} \leqslant M_{\gamma_2},$$

and M_{γ_2} *depends on the same arguments as* γ_2 *and, in addition, on* μ, μ_1, M_0, M_{γ_1}, $\|\Phi_1\|_{n+1,Q}$, $\|\Theta_1\|_{n,\partial''Q}$, *and on the moduli of absolute continuity of* $b(x,t)$ *in* $L_{n+2}(Q)$ *and of* $\beta(x,t)$ *in* $L_{n+1}(\partial''Q)$.

To prove this estimate we refer the reader to [6] (see Theorem 1). The elliptic analogue of Theorem 1.2 can also be found in [6] (see Theorem 1$'$). It generalizes the results of [20–21] to the case of unbounded b, β, Φ_1 and Θ_1.

Theorem 1.3 (Gradient estimates for solutions). *Let* $\partial\Omega \in W^2_{q+2}$, *and let a function* $u \in V_{q+2}(Q)$ *be a solution to the problem* (1.1)–(1.3) *such that* $\|u\|_Q \leqslant M_0$. *We also assume that the conditions* (A0)–(A4), (B), (B0)–(B4) *hold for* $|z| \leqslant M_0$.

Then

$$\|Du\|_Q \leqslant C_2, \qquad [Du]_{\gamma,Q} \leqslant C_3, \qquad \|u\|_{V_{q+2}(Q)} \leqslant C_4,$$

where the constants $\gamma \in \,]0,1[$, $C_2 - C_4$ *depend on* n, ν, ν_1, q, μ, μ_1, $\|b\|_{q+2,Q}$, $\|\beta\|_{q+1,\partial''Q}$, $\|\Phi_h\|_{q+2,Q}$, $\|\Theta_h\|_{q+1,\partial''Q}$ $(h = 1,2)$, M_0, *and on the properties of* $\partial''Q$.

The above estimates were established in [9] (see Theorem 3.1 and Corollary 3.1). The corresponding results for the stationary problem were obtained in [7]. Note that the gradient estimates for solutions of stationary problems were established in [21] under the hypotheses more limiting than ours: the right-hand sides of the equation and the boundary condition were assumed to be differentiable with respect to all variables, the function $\alpha(x, z, p)$ had to be nondegenerate with respect to the normal component of the gradient and could have at most the linear growth according to p. Note also that the arguments from [7, 9] can be extended to the case of fully nonlinear Venttsel boundary value problems.

If the gradient estimates have been established, the investigation of our problem reduces to the application of the solvability results for the classical boundary value problems.

Theorem 1.4 (Global solvability in Sobolev spaces). *Let the following conditions hold:*

(1) $\partial\Omega \in W^2_{q+2}$;
(2) *the conditions* (A0)–(A4), (B), (B0)–(B4) *are fulfilled;*
(3) *the functions* $a^{ij}(x, t, z, p)$ *and* $\alpha^{ij}(x, t, z, p')$ *are continuous with respect to all their arguments and the functions* $a(\cdot, z, p)$ *and* $\alpha(\cdot, z, p)$ *regarded as elements of the spaces* $L_{q+2}(Q)$ *and* $L_{q+1}(\partial''Q)$ *are continuous with respect to* (z, p).

Then the problem (1.1)–(1.3) *has a solution* $u \in V_{q+2}(Q)$.

Theorem 1.5 (Global solvability in Hölder spaces). *Let the following conditions hold:*

(1) $\partial\Omega \in C^{2+\widetilde{\gamma}}$, $\widetilde{\gamma} \in]0, 1[$;
(2) *the conditions* (A0), (A2)–(A3), (B0), (B2)–(B3), *as well as the following structure conditions are fulfilled:*

(A1′) $$|a(x, t, z, p)| \leqslant \mu(|p|^2 + 1) \quad \text{for} \quad (x, t) \in Q, z \in \mathbb{R}^1, p \in \mathbb{R}^n;$$

(A4′) $$\Phi_2 \in L_{q+2}(Q);$$

(B′) $$0 \leqslant -\frac{\partial\alpha(x, t, z, p)}{\partial p}\mathbf{n}(x) \leqslant \chi^{-1} \quad \text{for} \quad (x, t) \in \partial''Q, z \in \mathbb{R}^1, p \in \mathbb{R}^n;$$

(B1′) $$|\alpha(x, t, z, p')| \leqslant \mu_1(|p'|^2 + 1) \quad \text{for} \quad (x, t) \in \partial''Q, z \in \mathbb{R}^1, p \in \mathbb{R}^n;$$

(B4′) $$\Theta_2 \in L_{q+1}(\partial''Q),$$

where μ, μ_1, $\chi = const > 0$.

(3) *the functions* a^{ij}, a, α^{ij}, α *satisfy the Hölder condition with the exponent* $\widetilde{\gamma}$ *in the variables* x, z, p *and with the exponent* $\widetilde{\gamma}/2$ *in the variable* t.

Then the problem (1.1)–(1.3) *has a solution* $u \in C^{2+\widetilde{\gamma}}(\overline{Q})$.

These assertions were obtained in [9] (see Theorems 1 and 4.1, respectively). The elliptic analogues of the existence theorems which sufficiently improve the results of [21] can be found in [7].

In the cylinder $Q = \Omega \times]0, T[$ we consider the equation

$$u_t - a^{ij}(x,t,u,Du)D_iD_ju + a(x,t,u,Du) = 0 \tag{2.1}$$

and suppose that

$$u|_{t=0} = \varphi(x) \qquad \text{in } \Omega, \tag{2.2}$$

$$\begin{aligned} &\tau(x,t)\big[u_t - \alpha^{ij}(x,t,u,du)d_id_ju + \alpha_1(x,t,u,du)\big] \\ &+ \alpha_2(x,t,u,Du) = 0 \quad \text{on } \partial''Q. \end{aligned} \tag{2.3}$$

We assume that the boundary condition (2.3) is a *uniformly degenerating* parabolic Venttsel condition, i.e., (α^{ij}) is a symmetric matrix, the function $\alpha_2(x,t,z,p)$ is differentiable with respect to p and for $(x,t) \in \partial''Q$, $z \in \mathbb{R}^1$, $p \in \mathbb{R}^n$ the condition (B0) as well as the following conditions are fulfilled:

(D) τ is a nonnegative function on $\partial''Q$ satisfying the Lipschitz condition with the constant λ;

$$|\alpha_1(x,t,z,p')| \leqslant \mu_1(|p'|^2 + 1); \tag{B1-1}$$

$$0 < \chi \leqslant \frac{\partial \alpha_2(x,t,z,p)}{\partial p}\mathbf{n}(x) \leqslant \chi^{-1}; \tag{$\widetilde{\text{B}}$}$$

$$|\alpha_2(x,t,z,p)| \leqslant \chi^{-1}(|p| + 1). \tag{B1-2}$$

Here μ_1, χ are positive constants.

Theorem 2.1 (An Aleksandrov type local maximum principle). *Suppose that a function $u \in W^{2,1}_{n+1}(Q^+) \cap W^{2,1}_{\infty}(\Gamma(Q^+)) \cap C(\overline{Q^+})$ satisfies*

$$u_t - a^{ij}(x,t)D_iD_ju + b^i(x,t)D_iu + c(x,t)u = f(x,t) \quad \text{in } Q^+, \tag{2.4}$$

$$\tau(x,t)u_t - \alpha^{sm}(x,t)D_sD_mu + \beta^i(x,t)D_iu + \gamma(x,t)u = \theta(x,t) \text{ on } \Gamma(Q^+). \tag{2.5}$$

Assume also that the coefficients in (2.4), (2.5) satisfy the following conditions:

$$\begin{gathered} a^{ij} = a^{ji},\ a^{ij}\xi_i\xi_j \geqslant 0 \quad \text{for } \xi \in \mathbb{R}^n,\ \operatorname{tr}(a^{ij}) > 0 \quad \text{a.e. in } Q^+,\ c \geqslant 0; \\ \alpha^{sm} = \alpha^{ms}, \qquad \alpha^{sm}\xi_s\xi_m \geqslant 0 \qquad \text{for } \xi \in \mathbb{R}^{n-1}, \quad \tau \geqslant 0, \\ \beta^n \leqslant -\chi < 0; \quad \frac{|(\beta^m)|}{|\beta^n|} \leqslant \chi_1; \qquad \chi_1 = \mathit{const} > 0; \quad \gamma \geqslant 0. \end{gathered}$$

If, in addition, $u \leqslant 0$ on $\partial'Q \backslash \Gamma(Q^+)$, then

$$u \leqslant C_5\Big[\operatorname{diam}(\Omega^+)\Big\|\frac{\theta_+}{\chi}\Big\|_{\infty,\Gamma(Q^+)} + (\operatorname{diam}(\Omega^+))^{n/(n+1)}\Big\|\frac{f_+}{\Delta}\Big\|_{n+1,Q^+}\Big],$$

where

$$C_5 = C_5(n, \chi_1, (\mathrm{diam}(\Omega))^{-1/(n+1)} \|h\|_{n+1,Q^+}),$$
$$h = \frac{|(b^i)|}{\Delta}, \qquad \Delta = (\det(a^{ij}))^{1/(n+1)}$$

and we set $\frac{0}{0} = 0$ *if such an indeterminacy arises.*

This estimate was established in [1] (see Theorem 2). The stationary analogue of this theorem can be found in [5, Theorem 4.3]. In addition, we note that the stationary theorem was proved in [20] only for uniformly elliptic operators in the domain; it was assumed also that $b = 0$.

Theorem 2.2 (Hölder estimates of solutions). *Let* $\partial\Omega \in W^2_\infty$, *and let* $u \in W^{2,1}_{n+1}(Q) \cap W^{2,1}_\infty(\partial'' Q) \cap C(\overline{Q})$ *be a solution of* (2.1)–(2.3). *Assume also that the following conditions hold:*

(1) $|u| \leqslant M_0$ *in* $\overline{Q}$*;*
(2) $[\varphi]_{\gamma_3,\Omega} \leqslant M_{\gamma_3}$ *with* $\gamma_3 = \mathit{const} > 0$*;*
(3) *the conditions* (A0), (A1), (D), $(\widetilde{\mathrm{B}})$, (B0), (B1-1)–(B1-2) *are fulfilled for* $|z| \leqslant M_0$*;*
(4) $b \in L_{n+2}(Q)$, $\Phi_1 \in L_{n+1}(Q)$.

Then there exists $\gamma_4 > 0$, *completely determined by* n, ν, ν_1, γ_3, χ, λ *and by the properties of* $\partial\Omega$, *such that*

$$[u]_{\gamma_4,Q} \leqslant M_{\gamma_4},$$

and M_{γ_4} *depends on the same arguments as* γ_4 *and, in addition, on* μ, μ_1, M_0, M_{γ_3}, $\|\Phi_1\|_{n+1,Q}$, *and on the modulus of absolute continuity of* $b(x,t)$ *in* $L_{n+2}(Q)$.

This statement was proved in [5] (see Theorem 1.1). For the stationary analogue of Theorem 2.2 we refer the reader also to [5, Theorem 4.1].

As discussed in introduction, the gradient estimates have been obtained, for the moment, only for classical solutions of stationary degenerate problems. Now we state the relevant theorems.

Assuming here and in the sequel that $\partial\Omega \in C^2$, we consider the problem

$$-a^{ij}(x,u,Du) D_i D_j u + a(x,u,Du) = 0 \qquad \text{in } \Omega, \tag{2.6}$$
$$-\tau^2(x)\big[\alpha^{ij}(x,u,du) d_i d_j u + \alpha_1(x,u,du)\big] + \alpha_2(x,u,Du) = 0 \quad \text{on } \partial\Omega. \tag{2.7}$$

Suppose that (a^{ij}), (α^{ij}) are symmetric matrices and the following structure conditions are satisfied:

$$\text{for all } x \in \Omega, \quad z \in \mathbb{R}^1, \quad p \in \mathbb{R}^n,$$

(A2′) a^{ij}, a are differentiable with respect to all their arguments;

$$\text{(A3-0)} \qquad (1+|p|) \cdot \Big|\frac{\partial a^{ij}(x,z,p)}{\partial p}\Big|, |\delta a^{ij}(x,z,p)|, \ |\delta' a^{ij}(x,z,p)| \leqslant \mu;$$

$$\text{(A3-1)} \qquad (1+|p|) \cdot \Big|\frac{\partial a(x,z,p)}{\partial p}\Big|, |\delta a(x,z,p)|, |\delta' a(x,z,p)| \leqslant \mu(|p|^2+1);$$

$$\text{for all } x \in \partial\Omega, \ z \in \mathbb{R}^1, \ p \in \mathbb{R}^n,$$

(B2′) α^{ij}, α_1, α_2 are differentiable with respect to all their arguments;

$$\text{(B3-0)} \qquad (1+|p'|) \cdot \Big|\frac{\partial \alpha^{ij}(x,z,p')}{\partial p'}\Big|, \ |\delta' \alpha^{ij}(x,z,p')| \leqslant \mu_1;$$

$$\text{(B3-1)} \qquad (1+|p'|) \cdot \Big|\frac{\partial \alpha_1(x,z,p')}{\partial p'}\Big|, |\delta' \alpha_1(x,z,p')| \leqslant \mu_1(|p'|^2+1);$$

$$\text{(B3-2)} \qquad (1+|p|) \cdot \Big|\frac{\partial \alpha_2(x,z,p)}{\partial p}\Big|, |\delta' \alpha_2(x,z,p)| \leqslant \chi^{-1}(|p|+1).$$

Here μ, μ_1, χ are positive constants.

As the interior gradient estimates for solutions of (2.6) are known, we have only to obtain the boundary ones. So, flattening the boundary we may consider only a solution u of a local problem:

$$\text{(2.8)} \qquad -a^{ij}(x,u,Du)D_iD_ju + a(x,u,Du) = 0 \quad \text{in } B_1^+,$$

$$\text{(2.9)} \quad -\tau^2(x)\big[\alpha^{sm}(x,u,D'u)D_sD_mu + \alpha_1(x,u,D'u)\big] + \alpha_2(x,u,Du) = 0 \quad \text{on } \Gamma_1.$$

Theorem 2.3 (A local estimate of $|D'u|$). *Let a function $u \in C^2(B_1^+ \cup \Gamma_1)$ be a solution to the problem* (2.8)–(2.9) *such that $\|u\|_{B_1^+} \leqslant M_0$. Suppose that functions a^{ij}, α_2 are continuously differentiable with respect to p while functions α^{sm} are continuously differentiable with respect to p'. Assume also that the conditions* (A0), (A1′), (A2′), (A3-0)–(A3-1), (D), ($\widetilde{\text{B}}$), (B1-1)–(B1-2), (B2′), (B3-0)–(B3-2) *hold for $|z| \leqslant M_0$.*

Then there exist positive constants $\gamma_5 \in \,]0,1[$ and C_6 such that for any $R \leqslant 1/2$

$$\operatorname*{osc}_{B_R^+} D'u \leqslant C_6 R^{\gamma_5}.$$

Here $\gamma_5 = \gamma_5(n,\nu,\nu_1,\chi,\lambda)$ while C_6 depends on the same arguments as γ_5 and, in addition, on μ, μ_1, M_0.

Corollary. *Under the hypotheses of Theorem* 2.3 *there exists a positive constant* $\hat{C}$ *such that*

$$\|Du\|_{C^{\gamma_5}(\overline{B^+_{1/4}})} \leqslant \hat{C}.$$

Here $\hat{C}$ *is determined by the same parameters as* C_6 *from Theorem* 2.3.

Gradient estimates for C^2 solutions of stationary quasilinear problems were established in [4] (see Theorems 2.1, 2.2 and Corollary 2.1). For the detailed calculations we refer the reader to [2–3]. It should be mentioned that similar results were obtained in [2–4] for solutions of fully nonlinear problems under the assumption: a solution u satisfies $u \in C^3(B^+_1 \cup \Gamma_1)$.

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