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A COMMENT ON THE JÄGER-KAČUR LINEARIZATION SCHEME FOR STRONGLY NONLINEAR PARABOLIC EQUATIONS

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Abstract. The aim of this paper is to demonstrate how the variational equations from [11] can be formulated and solved in some abstract Banach spaces without any a priori construction of special linearization schemes. This should be useful e.g. in the analysis of heat conduction problems and modelling of flows in porous media.

Keywords: PDE's of evolution, method of Rothe

MSC 2000: 35K22, 35K55

1. Introduction

In [11] a linear approximation scheme for solving a large scale of nonlinear parabolic problems, such as heat conduction, flows in porous media etc., including degenerated cases (e.g. the Stephan phase change problem from [1]), has been suggested. This scheme uses a nonstandard time discretization with the relaxation function determined by the Jäger-Kačur iteration algorithm (introduced in [8]). An alternative approach can be based on the method of characteristics; for more references see [7] and [9]. All variational equations in [11] are formulated in very special Lebesgue and Sobolev spaces and the existence and convergence analysis makes use of some (not very natural) assumptions, forced by the linearization tricks. Our aim will be to rewrite the equations from [11] in more general spaces of abstract functions and to construct their solutions independently of the numerical properties of the above mentioned schemes.

This paper is a slightly extended version of the author's communication at the 9-th Seminar on Programs and Algorithms of Numerical Mathematics in Kořenov in June 1998, organized by the Mathematical Institute of the Academy of Sciences of the Czech Republic in Prague. It should briefly demonstrate the approach on a sufficiently general model problem, avoiding technical difficulties. Therefore most of natural generalizations (explicitly time-dependent mappings A and B, various degenerations, non-separable spaces V, more clever estimates of E_A and E_B, more realistic and better physically motivated examples etc.) are neglected. Nevertheless, no differentiability of A or B is assumed; this is important especially in problems involving phase changes, as in the Kiessl model of simultaneous moisture and heat transfer from [2] (no reasonable results with strong time derivatives of solutions can be expected). If H and V are Lebesgue or Sobolev spaces, the assumed properties of A and B can be verified using the theorems from [3] and [4].

All classes of special mappings applied in this paper are introduced in [6] or [3]. The standard notation of Lebesgue and Sobolev spaces (needed in examples only) is compatible with [13]; μ denotes the 3-dimensional Lebesgue measure and σ the 2-dimensional Hausdorff measure. The symbol * is reserved for adjoint spaces. The brief notation $\mathbb{R}_0 = \mathbb{R}_+ \cup \{0\}$ is used, too.

2. Solvability of a discrete scheme

Let H be a Banach space (the symbol $\emptyset$ is reserved for its zero element). Let $B\colon H \to H^\star$ be a radially continuous mapping. Let $B_1\colon H \to \mathbb{R}_+$ be a finite functional such that

$$B_1(v) - B_1(w) = (Bw, v-w) + E_B(v,w) \tag{1}$$

holds for all $v, w \in H$ where the real error $E_B(v,w)$ satisfies

$$E_B(v,w) \geqslant \beta\left(|v-w|_H\right) \tag{2}$$

for some function $\beta\colon \mathbb{R}_0 \to \mathbb{R}_0$; we use the notation $(.,.)$ for the duality between H and $H^\star$.

R e m a r k. Observe that B is monotone, but not strictly monotone in general. Especially, if B is a gradient of B_1 in the sense of [6], p. 89, it can be proved (see [6], p. 96) that B is monotone iff B_1 is convex.

Let V be a reflexive and separable Banach space such that the imbedding of V into H is strongly continuous. Let $A\colon V \to V^\star$ be a weakly continuous mapping. Let $A_1\colon H \to \mathbb{R}_+$ be a finite functional such that

$$A_1(v) - A_1(w) = \langle Aw, v-w\rangle + E_A(v,w) \tag{3}$$

holds for all $v, w \in V$ where the real error $E_A(v, w)$ satisfies

$$E_A(v, w) \geqslant \alpha\left(|v-w|_V\right) - \nu\beta\left(|v-w|_H\right) \tag{4}$$

for a function $\alpha\colon\ \mathbb{R}_0 \to \mathbb{R}_0$ with the limit behavior

$$\lim_{\zeta\to\infty} \frac{\alpha(\zeta)}{\zeta} = \infty \tag{5}$$

and for a non-negative real constant ν; we use the notation $\langle .\,, . \rangle$ for the duality between V and $V^\star$.

R e m a r k. In particular, A may be the gradient of A_1. Observe that A satisfies no reasonable monotonicity condition.

For a fixed $u_0 \in V$, let there exist non-negative real constants η and $\xi < 1$ such that

$$\langle Au_0, v\rangle \geqslant -\xi\alpha\left(|v|_V\right) - \eta \tag{6}$$

is valid for every $v \in V$.

R e m a r k. This property is a consequence of the fact that

$$\xi\alpha(\zeta) \geqslant |Au_0|_{V^\star}\,\zeta$$

for any $\zeta \in \mathbb{R}_+$. Indeed, in the case $|Au_0|_{V^\star} = 0$ it is sufficient to put $\xi = 0$. Let $|Au_0|_{V^\star} > 0$ and let v be an arbitrary element of V. If $\alpha\left(|v|_V\right) = 0$ then v is bounded in V by virtue of (5) and

$$\langle Au_0, v\rangle \geqslant -\,|Au_0|_{V^\star}\,|v|_V \geqslant -\eta$$

for a certain positive η. If $\alpha\left(|v|_V\right) > 0$ then we have

$$\langle Au_0, v\rangle \geqslant -\,|Au_0|_{V^\star}\,|v|_V \geqslant -\xi\alpha\left(|v|_V\right).$$

The last two estimates lead to (6).

Let us notice some properties of the mappings A and B. The equation (3) can be easily rewritten in its alternative form

$$\langle Av, v-w\rangle = A_1(v) - A_1(w) + E_A(w, v) \tag{7}$$

and similarly the equation (1) in its alternative form

$$(8)\qquad (Bv, v-w) = B_1(v) - B_1(w) + E_B(w,v);$$

the sum of (3) and (7) with respect to (4) gives

$$(9)\qquad \begin{aligned} \langle Av - Aw, v-w\rangle &= E_A(v,w) + E_A(w,v) \\ &\geqslant 2\alpha\left(|v-w|_V\right) - 2\nu\beta\left(|v-w|_H\right) \end{aligned}$$

and the sum of (1) and (8) with respect to (2) gives

$$(10)\qquad (Bv - Bw, v-w) = E_B(v,w) + E_B(w,v) \geqslant 2\beta\left(|v-w|_H\right).$$

Let us consider a time interval $I = \{t' \in \mathbb{R}_0\colon\ t' \leqslant T\}$ with $T \in \mathbb{R}_+$. Moreover, let us introduce the brief notation

$$A^*u(t) = \int_0^t Au(t')\,\mathrm{d}t'.$$

Remark. We shall demonstrate later that this Bochner integral is well-defined by virtue of the properties of $u(t')$ (see the proof of Theorem in the next section).

Let an "initial value" $u_0 \in V$ be given. Our aim is to derive the following result:

Theorem. *There exists an abstract function $u\colon\ I \to V$ satisfying the equation of evolution*

$$(11)\qquad (Bu(t) - Bu_0, v) + \langle A^*u(t), v\rangle = 0$$

for all $v \in V$ and for every time $t \in I$.

This will be verified using the method of discretization in time (for the complete proof see the next section). Following [11] let us consider the discrete analogue of (6)

$$(12)\qquad (Bu^n(t) - Bu_0, v) + \langle A^*u^n(t), v\rangle = 0$$

with the n-th element of the sequence of Rothe (n is allowed to be an arbitrary integer)

$$(13)\qquad u^n(t) = u_i \in V$$

for any $t \in I_i$ and $i \in \{1, \dots, n\}$ where $I_i = \{t' \in I\colon (i-1)h < t' \leqslant ih\}$ with $h = T/n$. If we put $t = jh$ for some $j \in \{1, \dots, n\}$ then (12) changes to the equation

$$(Bu_j - Bu_0, v) + h \sum_{i=1}^{j} \langle Au_i, v\rangle = 0 \tag{14}$$

which can be obtained as the sum of equations

$$(Bu_i - Bu_{i-1}, v) + h \langle Au_i, v\rangle = 0 \tag{15}$$

over $i \in \{1, \dots, j\}$. Moreover, we have

Lemma 1. *Let $u_{i-1} \in V$ ($i \in \{1, \dots, n\}$) be given. There exists $u_i \in V$ satisfying* (15).

P r o o f. Let us define a mapping $T_h\colon V \to V^\star$ using the relation

$$\langle T_h w, v\rangle = (Bw, v) + h \langle Aw, v\rangle - \langle Bu_{i-1}, v\rangle$$

for any $v, w \in V$ and a certain fixed $h \in \mathbb{R}_+$. Since A is weakly continuous and B is demicontinuous (by [6], p. 66, for monotone operators demicontinuity and radial continuity coincide), the weak continuity of T_h follows from the strong continuity of the imbedding of V into H. By [3], p. 46, every weakly continuous and coercive mapping from V to $V^\star$ is surjective (as a consequence of reflexivity and separability of V). Thus it remains to verify the coercivity of T_h. But for every $v \neq \emptyset$, by virtue of (9) and (10) we obtain

$$\begin{aligned}
\frac{\langle T_h v, v\rangle}{|v|_V} &= \frac{(Bv, v) + h \langle Av, v\rangle - (Bu_{i-1}, v)}{|v|_V} \\
&= \frac{E_B(v, \emptyset) + E_B(\emptyset, v) + hE_A(v, \emptyset) + hE_A(\emptyset, v) + \langle T_h \emptyset, v\rangle}{|v|_V} \\
&\geqslant \frac{2[1 - \nu h]\beta\left(|v|_H\right) + h\alpha\left(|v|_V\right)}{|v|_V} - |T_h \emptyset|_{V^\star}
\end{aligned}$$

and the coercivity of T_h for every $h < \nu^{-1}$ (for $\nu = 0$ any positive h is admissible) follows from (5). $\square$

R e m a r k. We will not discuss here the choice of appropriate iterative methods to obtain u_i from (15) as the strong limit of a sequence $\{u_i^k\}_{k=0}^\infty$ in V with $u_i^0 = u_{i-1}$. Several efficient algorithms for special problems (e.g., like our examples) are introduced in papers mentioned in introduction. Numerical experiments are presented e.g. in [1], p. 46.

The possibility of a recursive use of Lemma 1 is evident. Thus let u_j ($j \in \{1, \ldots, n\}$) be solutions of (14), whose existence is guaranteed by Lemma 1. The following lemmas give useful a priori estimates:

Lemma 2. *The estimate*

$$\tag{16} \begin{aligned} &\beta\left(|u_j - u_0|_H\right) + \sum_{i=1}^{j} \beta\left(|u_i - u_{i-1}|_H\right) + [2-\xi]h \sum_{i=1}^{j} \alpha\left(|u_i - u_0|_V\right) \\ &\leqslant \eta j h + 2\nu h \sum_{i=1}^{j} \beta\left(|u_i - u_0|_H\right) \end{aligned}$$

is true.

P r o o f. Let us set $v = u_i - u_0$ in (15). From the estimates based on (1), (8) and (2),

$$\begin{aligned} &\sum_{i=1}^{j} (Bu_i - Bu_{i-1}, u_i - u_0) \\ &= \sum_{i=1}^{j} (Bu_i, u_i - u_0) - \sum_{i=1}^{j} (Bu_{i-1}, u_{i-1} - u_0) - \sum_{i=1}^{j} (Bu_{i-1}, u_i - u_{i-1}) \\ &= (Bu_j, u_j - u_0) - \sum_{i=1}^{j} B_1(u_i) + \sum_{i=1}^{j} B_1(u_{i-1}) + \sum_{i=1}^{j} E_B(u_i, u_{i-1}) \\ &= (Bu_j, u_j - u_0) - B_1(u_j) + B_1(u_0) + \sum_{i=1}^{j} E_B(u_i, u_{i-1}) \\ &= E_B(u_0, u_j) + \sum_{i=1}^{j} E_B(u_i, u_{i-1}) \\ &\geqslant \beta\left(|u_j - u_0|_H\right) + \sum_{i=1}^{j} \beta\left(|u_i - u_{i-1}|_H\right), \end{aligned}$$

and from those based on (9) and (6),

$$
\begin{aligned}
& h\sum_{i=1}^{j}\langle Au_i, u_i-u_0\rangle \\
&= h\sum_{i=1}^{j}\langle Au_i-Au_0, u_i-u_0\rangle + h\sum_{i=1}^{j}\langle Au_0, u_i-u_0\rangle \\
&= h\sum_{i=1}^{j}E_A(u_i,u_0) + h\sum_{i=1}^{j}E_A(u_0,u_i) + h\sum_{i=1}^{j}\langle Au_0, u_i-u_0\rangle \\
&\geqslant [2-\xi]h\sum_{i=1}^{j}\alpha\left(|u_i-u_0|_V\right) - 2\nu h\sum_{i=1}^{j}\beta\left(|u_i-u_0|_H\right) - \eta jh,
\end{aligned}
$$

we obtain (16). □

Lemma 3. *The estimate*

$$
\begin{aligned}
& [1-\xi]\alpha\left(|u_j-u_0|_V\right) + \sum_{i=1}^{j}\alpha\left(|u_i-u_{i-1}|_V\right) \\
&+ \left[\frac{2}{h}-\nu\right]\sum_{i=1}^{j}\beta\left(|u_i-u_{i-1}|_H\right) \\
&\leqslant \eta + \nu\beta\left(|u_j-u_0|_H\right)
\end{aligned}
\tag{17}
$$

is true.

Proof. Let us set $v=(u_i-u_{i-1})/h$ in (15). From the estimate based on (10),

$$
\begin{aligned}
& \frac{1}{h}\sum_{i=1}^{j}(Bu_i-Bu_{i-1}, u_i-u_{i-1}) \\
&= \frac{1}{h}\sum_{i=1}^{j}E_B(u_i,u_{i-1}) + \frac{1}{h}\sum_{i=1}^{j}E_B(u_{i-1},u_i) \\
&\geqslant \frac{2}{h}\sum_{i=1}^{j}\beta\left(|u_i-u_{i-1}|_H\right),
\end{aligned}
$$

and from those based on (3), (6) and (4),

$$\begin{aligned}
&\sum_{i=1}^{j} \langle Au_i, u_i - u_{i-1}\rangle \\
&\quad = \sum_{i=1}^{j} A_1(u_i) - \sum_{i=1}^{j} A_1(u_{i-1}) + \sum_{i=1}^{j} E_A(u_{i-1}, u_i) \\
&\quad = A_1(u_j) - A_1(u_0) + \sum_{i=1}^{j} E_A(u_{i-1}, u_i) \\
&\quad = \langle Au_0, u_j - u_0\rangle + E_A(u_j, u_0) + \sum_{i=1}^{j} E_A(u_{i-1}, u_i) \\
&\quad \geqslant [1-\xi]\alpha\left(|u_j - u_0|_V\right) + \sum_{i=1}^{j} \alpha\left(|u_i - u_{i-1}|_V\right) \\
&\qquad - \nu\beta\left(|u_j - u_0|_H\right) - \nu\sum_{i=1}^{j} \beta\left(|u_i - u_{i-1}|_H\right) - \eta,
\end{aligned}$$

we obtain (17). $\square$

Lemma 4. *Every sequence* $\{u^n(t)\}_{n=1}^{\infty}$ *defined by* (13) *is bounded uniformly with respect to* $t \in I$.

P r o o f. The sum of the inequality (17) (Lemma 2) and the inequality (16) (Lemma 3) multiplied by 2ν gives

$$\begin{aligned}
(18)\quad & \nu\beta\left(|u_j - u_0|_H\right) + \left[\frac{2}{h} + \nu\right] \sum_{i=1}^{j} \beta\left(|u_i - u_{i-1}|_H\right) \\
& + [1-\xi]\alpha\left(|u_j - u_0|_V\right) + 2[2-\xi]\nu h \sum_{i=1}^{j} \alpha\left(|u_i - u_0|_V\right) + \sum_{i=1}^{j} \alpha\left(|u_i - u_{i-1}|_V\right) \\
& \leqslant \eta[1 + 2\nu j h] + 4\nu^2 h \sum_{i=1}^{j} \beta\left(|u_i - u_0|_H\right).
\end{aligned}$$

Neglecting all additive terms on the left-hand side of (18) except the first and using the obvious inequality $jh \leqslant T$ we obtain

$$\nu\beta\left(|u_j - u_0|_H\right) \leqslant \eta[1 + 2\nu T] + 4\nu^2 h \sum_{i=1}^{j} \beta\left(|u_i - u_0|_H\right).$$

Let us suppose $\nu \neq 0$. If $h < (4\nu)^{-1}$ then the "discrete" Gronwall lemma (see [9], p. 29) guarantees the existence of $\beta_0 \in \mathbb{R}_+$ such that

$$\beta\left(|u_j - u_0|_H\right) \leqslant \beta_0$$

holds. Neglecting all additive terms on the left-hand side of (18) except the third and using the evident consequence of the last result

$$h \sum_{i=1}^{j} \beta\left(|u_i - u_0|_H\right) \leqslant jh\beta_0 \leqslant T\beta_0$$

we obtain

$$[1-\xi]\alpha\left(|u_j - u_0|_V\right) \leqslant \eta[1+2\nu T] + 4\nu^2 T\beta_0,$$

which is nothing else than

$$\alpha\left(|u_j - u_0|_V\right) \leqslant \alpha_0$$

with $\alpha_0 = (\eta[1+2\nu T] + 4\nu^2 T\beta_0)/[1-\xi]$. Moreover, by virtue of (5) there exists also some $\alpha_* \in \mathbb{R}_+$ such that

$$|u_j - u_0|_V \leqslant \alpha_*$$

holds. This yields the required boundedness of $\{u^n(t)\}_{n=1}^{\infty}$. □

Remark. The case $\nu = 0$ has not been discussed separately. Nevertheless, it is easy to repeat the preceding arguments literally with $\alpha_0 = \eta/[1-\xi]$; the Gronwall lemma is not needed and β_0 is not defined.

Now we are ready to complete the proof of our main result presented in the previous section:

Proof of Theorem. We shall start from Lemma 4. Since V is reflexive, a subsequence with a certain weak limit $u(t) \in V$ can be extracted from $\{u^n(t)\}_{n=1}^{\infty}$ (see the well-known Eberlein-Shmul'yan theorem e.g. in [5], p. 197); for brevity we will now use the notation $\{u^n(t)\}_{n=1}^{\infty}$ only for a subsequence with this property. The weak continuity of A implies

$$\lim_{n\to\infty} \langle Au^n(t) - Au(t), v\rangle = 0 \tag{19}$$

for all $v \in V$ and $t \in I$. Because of the strong continuity of the imbedding of V into H this limit must be strong in H. Thus from the demicontinuity of B we obtain

$$\lim_{n\to\infty} (Bu^n(t) - Bu(t), v) = 0 \tag{20}$$

for all $v \in V$ and $t \in I$. As $\{Au^n(t)\}_{n=1}^{\infty}$ is bounded in $V^\star$ uniformly with respect to $t \in I$ (for details see the properties of weakly* convergent sequences in [15], p. 180), we are allowed to apply the classical Lebesgue dominated convergence theorem (see e.g. [14], p. 110) and get

$$\lim_{n\to\infty} \int_0^t \langle Au^n(t'), v\rangle \,\mathrm{d}t' = \int_0^t \lim_{n\to\infty} \langle Au^n(t'), v\rangle \,\mathrm{d}t' = \int_0^t \langle Au(t'), v\rangle \,\mathrm{d}t'$$

for all $v \in V$ and $t \in I$; this can be rewritten in the simple form

$$\lim_{n\to\infty} \langle A^* u^n(t), v\rangle = \langle A^* u(t), v\rangle\,.$$

This result together with (19) and (20) makes the limit passage from (12) to (2) possible. □

4. Illustrative examples

For an illustration how to test our assumptions in practical cases, we will present a model problem (as simple as possible) from [11]. Let us consider $H = L_2(\Omega, \mathbb{R})$, $S = L_2(\Omega, \mathbb{R}^3)$, $X = L_2(\partial\Omega, \mathbb{R})$, $M = L_\infty(\Omega, \mathbb{R}_+^{3\times 3})$, where Ω is an open set in $\mathbb{R}^3$ and $\mathbb{R}_+^{3\times 3}$ contains all positive definite matrices from $\mathbb{R}^{3\times 3}$, and a subspace V of $W_2^1(\Omega, \mathbb{R})$ of functions satisfying the essential boundary conditions. Let Ω be chosen in such a way that the imbedding of V into H is strongly continuous (cf. the well-known Rellich theorem) and the trace theorem

$$|v|_X^2 \leqslant \delta\, |\nabla v|_H^2 + \frac{\lambda}{\delta}\, |v|_H^2 \tag{21}$$

holds for any $v \in V$ and $\delta \in \mathbb{R}_+$ with a positive constant λ (independent of v and δ); moreover, let the imbedding of X into H be strongly continuous, too.

Remark. The geometrical meaning of the assumed properties of Ω is studied in [13], pp. 62, 220.

Example 1. Let us consider an integrable mapping $b\colon\ \mathbb{R} \to \mathbb{R}$ such that

$$\varepsilon \leqslant b(z) \leqslant \overline{\varepsilon}$$

holds for all $z \in \mathbb{R}$ where ε and $\overline{\varepsilon}$ are certain positive constants. For any $v, w \in H$ let us define

$$(Bv, w) = \int_\Omega b_1(v(x))w(x)\,\mathrm{d}\mu$$

(the notation x is used only for points of Ω or $\partial\Omega$ in this example) where

$$b_1(y) = \int_0^y b(z)\,\mathrm{d}z$$

for each real y. Moreover, let us consider some $\kappa \in M$ and an integrable mapping $a\colon \mathbb{R} \to \mathbb{R}$ such that

$$\gamma \leqslant a(z) \leqslant \overline{\gamma}$$

holds for all $z \in \mathbb{R}$ where γ and $\overline{\gamma}$ are real constants and $\overline{\gamma}$ is positive. Without loss of generality we can assume $-\gamma \leqslant \overline{\gamma}$. For any $v, w \in H$ let us define

$$\langle Av, w\rangle = \sum_{k,l=1}^{3} \int_\Omega \kappa_{kl}(x) \frac{\partial v}{\partial x_k}(x) \frac{\partial w}{\partial x_l}(x)\,\mathrm{d}\mu + \int_{\partial\Omega} a_1(v(x))w(x)\,\mathrm{d}\sigma$$

where

$$a_1(y) = \int_0^y a(z)\,\mathrm{d}z$$

for each real y. For an arbitrary $v \in H$ let us introduce the functional

$$B_1(v) = \int_\Omega \int_0^{v(x)} b_1(y)\,\mathrm{d}y\,\mathrm{d}\mu$$

and for an arbitrary $v \in V$ a similar one

$$A_1(v) = \frac{1}{2} \sum_{k,l=1}^{3} \int_\Omega \kappa_{kl}(x) \frac{\partial v}{\partial x_k}(x) \frac{\partial v}{\partial x_l}(x)\,\mathrm{d}\mu + \int_\Omega \int_0^{v(x)} a_1(y)\,\mathrm{d}y\,\mathrm{d}\sigma.$$

For all $v, w \in H$ we can calculate the weak differential (in the sense of [5], p. 171)

$$\begin{aligned}
\mathrm{D}B_1(v, w) &= \frac{\mathrm{d}}{\mathrm{d}s} \left\{ \int_\Omega \int_0^{v(x)+sw(x)} b_1(y)\,\mathrm{d}y\,\mathrm{d}\mu \right\}_{s=0} \\
&= \left\{ \int_\Omega b_1(v(x) + sw(x))w(x)\,\mathrm{d}\mu \right\}_{s=0} \\
&= \int_\Omega b_1(v(x))w(x)\,\mathrm{d}\mu = (Bv, w)
\end{aligned}$$

and the difference

$$\begin{aligned}
B_1(v) - B_1(w) &= \int_\Omega \int_{w(x)}^{v(x)} b_1(y)\,\mathrm{d}y\,\mathrm{d}\mu \\
&= \int_\Omega \int_{w(x)}^{v(x)} b_1(w(x))\,\mathrm{d}y\,\mathrm{d}\mu + \int_\Omega \int_{w(x)}^{v(x)} [b_1(y) - b_1(w(x))]\,\mathrm{d}y\,\mathrm{d}\mu \\
&= \int_\Omega b_1(w(x))[v(x) - w(x)]\,\mathrm{d}\mu + \int_\Omega \int_{w(x)}^{v(x)} \int_{w(x)}^{y} b(z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}\mu
\end{aligned}$$

or (more briefly)

$$B_1(v) - B_1(w) = (Bw, v - w) + E_B(v, w)$$

with

$$E_B(v, w) = \int_\Omega \int_{w(x)}^{v(x)} \int_{w(x)}^{y} b(z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}\mu.$$

Since for $y \neq w(x)$ we have $\operatorname{sgn}(y - w(x)) = \operatorname{sgn}(v(x) - w(x))$, we can estimate

$$\begin{aligned} E_B(v, w) &\geqslant \varepsilon \int_\Omega \int_{w(x)}^{v(x)} \int_{w(x)}^{y} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}\mu \\ &= \varepsilon \int_\Omega \int_{w(x)}^{v(x)} [y - w(x)]\,\mathrm{d}y\,\mathrm{d}\mu \\ &= \frac{\varepsilon}{2} \int_\Omega [v(x) - w(x)]^2\,\mathrm{d}\mu = \frac{\varepsilon}{2}\, |v - w|_H^2 \,. \end{aligned}$$

Thus we can set $\beta(\zeta) = \varepsilon\zeta^2/2$ for each $\zeta \in \mathbb{R}_0$. Moreover, the estimate

$$|(B(v + sw) - B(v + rw), w)| = \left| \int_\Omega \int_{v(x)+rw(x)}^{v(x)+sw(x)} b(z)\,\mathrm{d}z\,w(x)\,\mathrm{d}\mu \right| \leqslant \overline{\varepsilon}\,|s - r|\,|w|_H^2$$

valid for each $r, s \in \mathbb{R}$ and every $v, w \in H$ verifies the radial continuity of B. The weak continuity of A is evident: we have

$$\begin{aligned} \langle Av - Av^n, w\rangle \;=\; & \sum_{k,l=1}^{3} \int_\Omega \kappa_{kl}(x) \left[\frac{\partial v}{\partial x_k}(x) - \frac{\partial v^n}{\partial x_k}(x) \right] \frac{\partial w}{\partial x_l}(x)\,\mathrm{d}\mu \\ & + \int_{\partial\Omega} [a_1(v(x)) - a_1(v^n(x))] w(x)\,\mathrm{d}\sigma \end{aligned}$$

for every $v, w \in V$ and any sequence $\{v^n(t)\}_{n=1}^{\infty}$ in V with a weak limit v; if $n \to \infty$ then the first term vanishes because of its linearity and the second due to the estimate

$$\left| \int_{\partial\Omega} [a_1(v(x)) - a_1(v^n(x))] w(x)\,\mathrm{d}\sigma \right| = \left| \int_{\partial\Omega} \int_{v^n(x)}^{v(x)} a(z)\,\mathrm{d}z\,w(x)\,\mathrm{d}\sigma \right| \leqslant \overline{\gamma}\,|v - v^n|_X\,|w|_X$$

and to the strong continuity of the imbedding of X into H. For all $v, w \in V$ we can also calculate the weak differential

$$
\begin{aligned}
\mathrm{D}A_1(v,w) &= \frac{1}{2}\sum_{i=1}^{n}\frac{\mathrm{d}}{\mathrm{d}s}\left\{\int_\Omega \kappa_{kl}(x)\left[\frac{\partial v}{\partial x_k}(x)+s\frac{\partial w}{\partial x_k}(x)\right]\left[\frac{\partial v}{\partial x_l}(x)+s\frac{\partial w}{\partial x_l}(x)\right]\,\mathrm{d}\mu\right\}_{s=0} \\
&\quad+\frac{\mathrm{d}}{\mathrm{d}s}\left\{\int_\Omega\int_0^{v(x)+sw(x)} a_1(y)\,\mathrm{d}y\,\mathrm{d}\sigma\right\}_{s=0} \\
&= \sum_{i=1}^{n}\left\{\int_\Omega \kappa_{kl}(x)\left[\frac{\partial v}{\partial x_k}(x)+s\frac{\partial w}{\partial x_k}(x)\right]\frac{\partial w}{\partial x_l}(x)\,\mathrm{d}\mu\right\}_{s=0} \\
&\quad+\left\{\int_\Omega a_1(v(x)+sw(x))w(x)\,\mathrm{d}\sigma\right\}_{s=0} \\
&= \sum_{i=1}^{n}\int_\Omega \kappa_{kl}(x)\frac{\partial v}{\partial x_k}(x)\frac{\partial w}{\partial x_l}(x)\,\mathrm{d}\mu+\int_\Omega a_1(v(x))w(x)\,\mathrm{d}\sigma=\langle Av,w\rangle
\end{aligned}
$$

and the difference

$$
\begin{aligned}
A_1(v)-A_1(w) &= \frac{1}{2}\sum_{k,l=1}^{n}\int_\Omega \kappa_{kl}(x)\frac{\partial v}{\partial x_k}(x)\frac{\partial v}{\partial x_l}(x)\,\mathrm{d}\mu-\frac{1}{2}\sum_{k,l=1}^{n}\int_\Omega \kappa_{kl}(x)\frac{\partial w}{\partial x_k}(x)\frac{\partial w}{\partial x_l}(x)\,\mathrm{d}\mu \\
&\quad+\int_{\partial\Omega}\int_{w(x)}^{v(x)} a_1(y)\,\mathrm{d}y\,\mathrm{d}\sigma \\
&= \sum_{k,l=1}^{n}\int_\Omega \kappa_{kl}(x)\frac{\partial w}{\partial x_k}(x)\left[\frac{\partial v}{\partial x_l}(x)-\frac{\partial w}{\partial x_l}(x)\right]\,\mathrm{d}\mu \\
&\quad+\frac{1}{2}\sum_{k,l=1}^{n}\int_\Omega \kappa_{kl}(x)\left[\frac{\partial v}{\partial x_k}(x)-\frac{\partial w}{\partial x_k}(x)\right]\left[\frac{\partial v}{\partial x_l}(x)-\frac{\partial w}{\partial x_l}(x)\right]\,\mathrm{d}\mu \\
&\quad+\int_{\partial\Omega}\int_{w(x)}^{v(x)} a_1(w(x))\,\mathrm{d}y\,\mathrm{d}\sigma+\int_{\partial\Omega}\int_{w(x)}^{v(x)}[a_1(y)-a_1(w(x))]\,\mathrm{d}y\,\mathrm{d}\sigma \\
&= \sum_{k,l=1}^{n}\int_\Omega \kappa_{kl}(x)\frac{\partial w}{\partial x_k}(x)\left[\frac{\partial v}{\partial x_l}(x)-\frac{\partial w}{\partial x_l}(x)\right]\,\mathrm{d}\mu \\
&\quad+\frac{1}{2}\sum_{k,l=1}^{n}\int_\Omega \kappa_{kl}(x)\left[\frac{\partial v}{\partial x_k}(x)-\frac{\partial w}{\partial x_k}(x)\right]\left[\frac{\partial v}{\partial x_l}(x)-\frac{\partial w}{\partial x_l}(x)\right]\,\mathrm{d}\mu \\
&\quad+\int_{\partial\Omega} a_1(w(x))[v(x)-w(x)]\,\mathrm{d}\sigma+\int_{\partial\Omega}\int_{w(x)}^{v(x)}\int_{w(x)}^{y} a(z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}\sigma
\end{aligned}
$$

or (more briefly)

$$
A_1(v)-A_1(w)=\langle Aw,v-w\rangle+E_A(v,w)
$$

with

$$E_A(v,w) = \frac{1}{2}\sum_{k,l=1}^{3}\int_\Omega \kappa_{kl}(x)\left[\frac{\partial v}{\partial x_k}(x) - \frac{\partial w}{\partial x_k}(x)\right]\left[\frac{\partial v}{\partial x_l}(x) - \frac{\partial w}{\partial x_l}(x)\right]\,\mathrm{d}\mu$$
$$+ \int_{\partial\Omega}\int_{w(x)}^{v(x)}\int_{w(x)}^{y} a(z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}\sigma.$$

Since we have $\operatorname{sgn}(y - w(x)) = \operatorname{sgn}(v(x) - w(x))$ for $y \neq w(x)$, we can estimate

$$\begin{aligned}
E_A(v,w) &\geqslant \frac{\psi}{2}\sum_{k=1}^{3}\int_\Omega\left[\frac{\partial v}{\partial x_k}(x) - \frac{\partial w}{\partial x_k}(x)\right]^2\mathrm{d}\mu - \overline{\gamma}\int_{\partial\Omega}\int_{w(x)}^{v(x)}\int_{w(x)}^{y}\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}\sigma \\
&\geqslant \frac{\psi}{2}\sum_{k=1}^{3}\int_\Omega\left[\frac{\partial v}{\partial x_k}(x) - \frac{\partial w}{\partial x_k}(x)\right]^2\mathrm{d}\mu - \overline{\gamma}\int_{\partial\Omega}\int_{w(x)}^{v(x)}[y - w(x)]\,\mathrm{d}y\,\mathrm{d}\sigma \\
&= \frac{\psi}{2}\,|\nabla(v-w)|_S^2 - \frac{\overline{\gamma}}{2}\,|v-w|_X^2 \\
&\geqslant \frac{\psi}{2}\,|\nabla(v-w)|_S^2 - \frac{\psi}{4}\,|\nabla(v-w)|_S^2 - \frac{\lambda\overline{\gamma}^2}{\psi}\,|v-w|_H^2 \\
&= \frac{\psi}{4}\,|\nabla(v-w)|_S^2 - \frac{\lambda\overline{\gamma}^2}{\psi}\,|v-w|_H^2 \\
&= \frac{\psi}{4}\,|v-w|_V^2 - \left(\frac{\psi}{4} + \frac{\lambda\overline{\gamma}^2}{\psi}\right)|v-w|_H^2\,;
\end{aligned}$$

the existence of a constant $\psi \in \mathbb{R}_+$ follows directly from the inclusion $\kappa \in M$ and λ comes from (21) with $\delta = \psi/4$. Thus we can set $\alpha(\zeta) = \psi\zeta^2/4$ for any $\zeta \in \mathbb{R}_0$ and

$$\nu = \frac{2}{\varepsilon}\left(\frac{\psi}{4} + \frac{\lambda\overline{\gamma}^2}{\psi}\right).$$

Remark. Observe that zero values of β (this is the case of not strictly monotone mapping B) have not been allowed in Example 1. Example 2 demonstrates that under slightly modified assumptions such case can be handled, too.

Instead of (21) we shall now assume that the inequality of Friedrichs type

(22) $$|v|_V^2 \leqslant \omega\left(|\nabla v|_S^2 + |v|_X^2\right)$$

holds for any $v \in V$ with a positive constant ω (independent of v).

Remark. The geometrical interpretation of (22) is analyzed in [13], p. 154. Recall that the validity of the trace theorem in the form (21) is not required, but the strong continuity of the imbedding of X into H must be still satisfied (it is needed for the weak continuity of A).

E x a m p l e 2. Let us repeat Example 1 with a non-negative number ε and a positive number γ. We can omit most steps which are the same as in Example 1. A significant difference occurs only in the estimate for $E_A(v,w)$ with $v,w\in V$ which has the form

$$\begin{aligned}
E_A(v,w) &\geqslant \frac{\psi}{2}\sum_{k=1}^{3}\int_{\Omega}\left[\frac{\partial v}{\partial x_k}(x)-\frac{\partial w}{\partial x_k}(x)\right]^2 \mathrm{d}\mu+\gamma\int_{\partial\Omega}\int_{w(x)}^{v(x)}\int_{w(x)}^{y}\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}\sigma\\
&\geqslant \frac{\psi}{2}\sum_{k=1}^{3}\int_{\Omega}\left[\frac{\partial v}{\partial x_k}(x)-\frac{\partial w}{\partial x_k}(x)\right]^2 \mathrm{d}\mu+\gamma\int_{\partial\Omega}\int_{w(x)}^{v(x)}[y-w(x)]\,\mathrm{d}y\,\mathrm{d}\sigma\\
&= \frac{\psi}{2}\,|\nabla(v-w)|_S^2+\frac{\gamma}{2}\,|v-w|_X^2\\
&\geqslant \frac{1}{2}\min(\psi,\gamma)\left(|\nabla(v-w)|_S^2+|v-w|_X^2\right)\\
&\geqslant \frac{1}{2\omega}\min(\psi,\gamma)\,|v-w|_V^2\,;
\end{aligned}$$

ω comes from (22). Thus we can set $\alpha(\zeta)=\min(\psi,\gamma)\zeta^2/(2\omega)$ for any $\zeta\in\mathbb{R}_0$ and $\nu=0$.

R e m a r k. An easy generalization is available: if γ is non-negative and the definition of V guarantees zero values of $v\in V$ at least on a certain part of $\partial\Omega$ of non-zero Hausdorff measure, then a modified version of the Friedrichs inequality (22) (see e.g. [6], p. 36)

$$|v|_V^2\leqslant\omega\,|\nabla v|_S^2$$

can be applied to get analogous results. This technique is useful for mixed problems with boundary conditions of Dirichlet type.

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