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ALGEBRAIC DOMAIN DECOMPOSITION SOLVER FOR LINEAR ELASTICITY

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Abstract. We generalize the overlapping Schwarz domain decomposition method to problems of linear elasticity. The convergence rate independent of the mesh size, coarse-space size, Korn's constant and essential boundary conditions is proved here. Abstract convergence bounds developed here can be used for an analysis of the method applied to singular perturbations of other elliptic problems.

Keywords: algebraic multigrid, zero energy modes, convergence theory, finite elements, computational mechanics, iterative solvers

MSC 2000: 65F10, 65N55

1. Introduction

This paper is concerned with the development and analysis of a black-box algebraic solver suitable for problems of structural mechanics. The presented method is an overlapping Schwarz domain decomposition ([7]) with a coarse space given by smoothed aggregations ([11, 15, 12]). For scalar elliptic problems, such a method is proposed and analyzed in [5]. Here, we adapt the algorithm by employing a coarse space created using zero-energy modes ([15]), and prove the optimal convergence rate independent of the H^1-coercivity (Korn's constant) and the essential boundary conditions. In a certain sense, the convergence bounds given here are also independent of the computational domain. The indirect dependence arises only through subdomains that can be generated based on the nature of the solved problem.

The direct generalization of the estimates presented in [5] to problems of linear elasticity is possible only to a certain extent. More precisely, by following the spirit of the analysis presented there, it is impossible to prove practically very important independences of the convergence rate on the above mentioned problem data.

This difficulty is avoided by establishing abstract convergence bounds that use separate assumptions on the nonsmoothed coarse space and the prolongator smoother.

When applied to a particular problem, one has to prove approximation properties of the nonsmoothed coarse space that is geometrically very simple. The independence of such approximation property of a Korn's constant and essential boundary conditions comes as a direct consequence of the fact that aggregates are disjoint sets of nodes.

The assumptions of the abstract theory are verified for problems discretized on quasiuniform P1 finite element meshes. Using algebraic tools developed here, the generalization of the theory to problems discretized by using more complex shape-regular elements and problems with jumps in coefficients is quite straigtforward.

The paper is organized as follows: in Section 2 we give an outline of an algebraic Schwarz domain decomposition method with an abstract convergence theorem. Section 3 presents the convergence theorem for our method together with the assumptions on the components of the algorithm, which are verified in Section 4. The main convergence theorem is then re-stated in Section 5. Section 6 deals with computational complexity of the proposed algorithm and Section 7 contains the results of the numerical experiments on industrial models.

2. Abstract Schwarz domain decomposition

We are interested in solving a system of linear algebraic equations

$$A\mathbf{u} = \mathbf{b},$$

where A is a symmetric positive definite matrix corresponding to a finite element discretization of a continuous problem by P1 elements.

Our method is a standard multiplicative Schwarz overlapping domain decomposition with a coarse space given by smoothed aggregations as in [15].

In an abstract setting, the overlapping Schwarz method is determined by a *coarse space* $\mathrm{Rng}(P)$ and by local subdomain spaces $\mathrm{Rng}(N_i)$, $i = 1, \ldots, J$, where P: $\mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$, $n_2 \ll n_1$, is the *prolongator* and N_i are 0/1 matrices specifying subdomains understood as sets of degrees of freedom.

Based on the stiffness matrix A, let us define subdomain selections A_i and local correction operators R_i, $i = 1, \ldots, J$, by

$$A_i = N_i^T A N_i, \quad R_i = N_i (A_i)^{-1} N_i^T.$$

Analogously for the coarse space,

$$A_0 = P^T A P, \quad R_0 = P (A_0)^{-1} P^T.$$

Further, let us set

$$T_0 = R_0 A \quad \text{and} \quad T_i = R_i A. \tag{1}$$

Note that T_0 and T_i's, $i = 1, \ldots, J$ are orthogonal projections in the A-inner product to $\mathrm{Rng}(P)$ and $\mathrm{Rng}(N_i)$, respectively.

For the sake of parallelism we introduce a coloring $\{\mathcal{C}_i\}_{i=1}^{n_c}$ of the set $\{1, \ldots, J\}$ satisfying

$$\cos(\mathrm{Rng}(N_j), \mathrm{Rng}(N_k)) = 0 \qquad \text{for every } j, k \in \mathcal{C}_i\,, \quad i = 1, \ldots, n_c\,,$$

where the cosine is measured in the A-inner product. Let us further define the constant

$$K = \max_i |\mathcal{B}_i|\,, \qquad \text{where} \quad \mathcal{B}_i = \{j\colon \cos(\mathrm{Rng}(N_i), \mathrm{Rng}(N_j)) \neq 0\}. \tag{2}$$

Obviously, $\cos(\mathrm{Rng}(N_j), \mathrm{Rng}(N_k)) = 0$ if the subdomains corresponding to the matrices N_j, N_k are not adjacent or overlapping.

With such a coloring of subdomains we are able to solve subdomain problems partially in parallel by the following algorithm.

A l g o r i t h m 2.1. Given a vector $\mathbf{x}^i$, the method returns $\mathbf{x}^{i+1}$ computed as follows:

1. Set $\mathbf{z}^0 := \mathbf{x}^i$.
2. For $i = 1, \ldots, n_c$ perform local corrections:

$$\mathbf{z}^i := \mathbf{z}^{i-1} + \sum_{j \in \mathcal{C}_i} R_j \mathbf{d}^i, \quad \text{where } \mathbf{d}^i := \mathbf{b} - A\mathbf{z}^{i-1}.$$

3. Perform one coarse-level correction:

$$\mathbf{z}^0 := \mathbf{z}^{n_c} + R_0(\mathbf{b} - A\mathbf{z}^{n_c}).$$

4. (optional) Set $\mathbf{z}^{n_c} := \mathbf{z}^0$. For $i = n_c, \ldots, 1$ do

$$\mathbf{z}^{i-1} := \mathbf{z}^i + \sum_{j \in \mathcal{C}_i} R_j \mathbf{d}^i\,, \quad \text{where } \mathbf{d}^i := \mathbf{b} - A\mathbf{z}^i.$$

5. Set $\mathbf{x}^{i+1} := \mathbf{z}^0$.

Note that this algorithm can be viewed as either a multiplicative or a hybrid one. One can easily see that for the error-propagation operator $E\colon\ \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ we have

$$E = (I - T_0) \prod_{i=1}^{n_c} \prod_{j \in \mathcal{C}_i} (I - T_j) = (I - T_0) \prod_{i=1}^{n_c} \Big(I - \sum_{j \in \mathcal{C}_i} T_j \Big). \tag{3}$$

Here, the first term is the error-propagation operator of the colored multiplicative algorithm while the second term is E for a hybrid algorithm.

It is a well known fact (see e.g. Lions lemma in Mandel, Bjørstad [2], Theorem 3.2 combined with Bramble, Pasciak, Wang, Xu [4], Theorem 2.2, improved by Xu [16], [17]) that the convergence estimates of Algorithm 2.1 are governed by the following theorem:

Theorem 2.2. *Let* $\tilde{\mathcal{Q}}_0\colon\ \mathbb{R}^{n_1} \to \mathrm{Rng}(P)$, $\tilde{\mathcal{Q}}_i\colon\ \mathbb{R}^{n_1} \to \mathrm{Rng}(N_i)$, $i = 1, \ldots, J$ *be the mappings decomposing unity in* $\mathbb{R}^{n_1}$, *and let* $C_L > 1$ *be a constant such that*

$$\sum_{i=0}^{J} \|\tilde{\mathcal{Q}}_i \mathbf{u}\|_A^2 \leqslant C_L \|\mathbf{u}\|_A^2. \tag{4}$$

Then for the error propagation operator E *of Algorithm* 2.1 *we have*

$$\|E\|_A^2 \leqslant 1 - \frac{1}{C_L [K+2]^2}, \tag{5}$$

where K *is the overlap bound in* (2).

3. Smoothed aggregation coarse space

The smoothed aggregation technique consists in constructing the *prolongator* P in the form

$$P = S\tilde{P}. \tag{6}$$

Here, $\tilde{P}$ is the *tentative prolongator* responsible for the approximation properties of the coarse space. The purpose of the *prolongator smoother* S is to enforce the smoothness of the coarse space functions. The construction of such smoother aims at reducing the energy of the tentative coarse space basis. The precise definitions of $\tilde{P}$ and S will be given in Sections 4.1 and 4.2, respectively.

To get a picture of the method, it is worth mentioning in advance that the Schwarz subdomains $\mathrm{Rng}(N_i)$ can be derived from the nonzero structure of the prolongator P.

The purpose of this section is to prove abstract convergence bounds under assumptions on the prolongator smoother S, the tentative prolongator $\tilde{P}$ and the Schwarz domain decomposition subspaces $\mathrm{Rng}(N_i)$.

Assumption 3.1. Let us denote an available upper bound of $\varrho(S^TAS)$ by $\bar{\varrho}_{S^TAS}$. Assume that the *prolongator smoother* $S\colon\ \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ satisfies

$$\|S\| \leqslant 1, \qquad \|S\|_A \leqslant 1, \tag{7}$$

$$\|(I-S)\mathbf{u}\| \leqslant \frac{C_1}{\sqrt{\bar{\varrho}_{S^TAS}}}\|\mathbf{u}\|_A. \tag{8}$$

Further, let $\tilde{Q}_0\colon\ \mathbb{R}^{n_1} \to \mathrm{Rng}(\tilde{P})$ be an operator for which the weak approximation property of the *tentative coarse space* is satisfied in the following form:

$$\|\mathbf{u} - \tilde{Q}_0\mathbf{u}\| \leqslant \frac{C_2}{\sqrt{\bar{\varrho}_{S^TAS}}}\|\mathbf{u}\|_A. \tag{9}$$

Also, we assume that there exist decomposition operators $\mathcal{Q}_i\colon\ \mathbb{R}^{n_1} \to \mathrm{Rng}(N_i)$ that decompose the unity on $\mathbb{R}^{n_1}$ to *Schwarz domain decomposition* subspaces $\mathrm{Rng}(N_i)$, $i = 1, \ldots, J$, so that for all $\mathbf{w} \in \mathbb{R}^{n_1}$ we have

$$\sum_{i=1}^{J} \|\mathcal{Q}_i\mathbf{w}\|_A^2 \leqslant C_3\left(\bar{\varrho}_{S^TAS}\|\mathbf{w}\|^2 + \|\mathbf{w}\|_A^2\right). \tag{10}$$

R e m a r k 3.2. Notice that in the weak approximation property (9) of the tentative coarse space we employ an estimate of $\varrho(S^TAS)$, rather than using $\varrho(A)$. As $\varrho(S^TAS) \ll \varrho(A)$, we are here making a weaker assumption than in the standard algebraic multigrid theory (e.g. by Bramble [3], Assumption A.6, p. 40).

The assumption (10) on Schwarz domain decomposition subspaces (cf. e.g. [7]) is here in the role of a smoothing property. If we had just a pure multiplicative Schwarz algorithm without any coarse space, the condition (10) would reduce to the assumption from Lions' lemma

$$\sum_{i=1}^{J} \|\mathcal{Q}_i\mathbf{w}\|_A^2 \leqslant \tilde{C}_3\|\mathbf{w}\|_A^2,$$

ensuring (together with the overlap bound K) convergence of the algorithm. The convergence rate of such algorithm would, however, depend on the geometry of the problem through the dependence of $\tilde{C}_3$ on the boundary conditions, on the computational domain and on the geometry of the Schwarz subdomains! By adding the coarse grid correction to the multiplicative Schwarz domain decomposition we make a weaker assumption on the Schwarz subdomains by inserting the l-2 penalizing term into the right-hand side.

In fact, the inequality (10) indicates how wide a range of low-energy errors is reduced by the coarse space correction and what errors have to be dealt with on the

fine level by the Schwarz domain decomposition; this all with respect to the smoothing properties of the prolongator smoother S. It is obvious that the more "powerful" S we have, the smaller band of low-energy errors we can reduce by the coarse space correction. However, as we will see later, we do not choose too "powerful" smoothers, we are content with such a prolongator smoother S which just ensures the energy stability of the smoothed coarse space.

Theorem 3.3. *Under Assumption* 3.1, *Algorithm* 2.1 *with the prolongator* P *of the form* (6) *converges in the energy norm at the rate*

$$\|E\|_A^2 \leqslant 1 - \frac{1}{C_L[K+2]^2},$$

where E *is the error-propagation operator in* (3), K *is the overlap bound* (2), *and* $C_L > 1$ *is a constant of the form* $C_L = (1+2C_3)(C_2+1)^2 + C_3(C_1+C_2)^2 + 2C_3$.

Before giving the proof of Theorem 3.3, we have to see what the assumptions (7), (8), and (9) imply for the properties of the smoothed coarse space. For this, we employ an operator splitting introduced by Vaněk, Brezina and Mandel in [12].

Definition 3.4. Having the operator $\tilde{Q}_0$ from (9), S from (7–8) and decomposition operators $\{\mathcal{Q}_i\}$ as in (10), we define *mappings* $Q_i\colon\ \mathbb{R}^{n_1} \to \mathrm{Rng}(N_i)$, $i = 1, \ldots, J$ and $Q_0\colon\ \mathbb{R}^{n_1} \to \mathrm{Rng}(P)$ by

$$Q_0 = S\tilde{Q}_0 \qquad \text{and} \qquad Q_i = \mathcal{Q}_i(I - Q_0).$$

Lemma 3.5. *Let the operator* $\tilde{Q}_0\colon\ \mathbb{R}^{n_1} \to \mathrm{Rng}(\tilde{P})$ *satisfy* (9) *and let the smoother* S *satisfy* (7) *and* (8). *Then for the smoothed operator* $Q_0 = S\tilde{Q}_0\colon\ \mathbb{R}^{n_1} \to \mathrm{Rng}(S\tilde{P})$ *we have*

$$\|\mathbf{u} - Q_0\mathbf{u}\| \leqslant \frac{C_1 + C_2}{\sqrt{\overline{\varrho}_{S^TAS}}}\|\mathbf{u}\|_A \tag{11}$$

and

$$\|Q_0\mathbf{u}\|_A \leqslant (C_2+1)\|\mathbf{u}\|_A. \tag{12}$$

P r o o f. We start with the weak approximation property (11) of the smoothed coarse space: applying the triangle inequality, assumption (7), assumption (9) and assumption (8) in this order we get

$$\begin{aligned}\|\mathbf{u} - Q_0\mathbf{u}\| &= \|\mathbf{u} - S\tilde{Q}_0\mathbf{u}\| = \|S(\mathbf{u} - \tilde{Q}_0\mathbf{u}) + (I-S)\mathbf{u}\| \\ &\leqslant \|S(\mathbf{u} - \tilde{Q}_0\mathbf{u})\| + \|(I-S)\mathbf{u}\| \\ &\leqslant \|\mathbf{u} - \tilde{Q}_0\mathbf{u}\| + \|(I-S)\mathbf{u}\| \leqslant \frac{C_1 + C_2}{\sqrt{\overline{\varrho}_{S^TAS}}}\|\mathbf{u}\|_A.\end{aligned}$$

Verifying the energy stability (12), we introduce the splitting of $S\tilde{Q}_0$, then apply the triangle inequality, assumption (7), and the fact that $\|S\mathbf{w}\|_A \leqslant \varrho(S^TAS)\|\mathbf{w}\|$ for all $\mathbf{w} \in \mathbb{R}^{n_1}$. By assumption (9) we then get

$$\begin{aligned}\|Q_0\mathbf{u}\|_A = \|S\tilde{Q}_0\mathbf{u}\|_A = \|S\mathbf{u} + S\tilde{Q}_0\mathbf{u} - S\mathbf{u}\|_A \leqslant \|S(\mathbf{u} - \tilde{Q}_0\mathbf{u})\|_A + \|\mathbf{u}\|_A \\ \leqslant \sqrt{\varrho(S^TAS)}\|\mathbf{u} - \tilde{Q}_0\mathbf{u}\| + \|\mathbf{u}\|_A \leqslant (C_2+1)\|\mathbf{u}\|_A.\end{aligned}$$

This concludes the proof. □

Lemma 3.6. *Under Assumption* 3.1 *the operators* Q_i, $i = 0, \ldots, J$ *from Definition* 3.4 *satisfy*

$$I = \sum_{i=0}^{J} Q_i \tag{13}$$

and for all $\mathbf{u} \in \mathbb{R}^{n_1}$ *the inequality*

$$\sum_{i=0}^{J} \|Q_i\mathbf{u}\|_A^2 \leqslant C_L\|\mathbf{u}\|_A^2 \tag{14}$$

holds with the constant $C_L = (1+2C_3)(C_2+1)^2 + C_3(C_1+C_2)^2 + 2C_3$.

P r o o f. The decomposition property of $\{Q_i\}$ in (13) follows directly from the decomposition property of $\mathcal{Q}_i$. The energy stability (14) of the decomposition mappings $\{Q_i\}$ can be shown by using Definition 3.4, estimate (12), assumptions (10), (7), and estimates (11) and (12):

$$\begin{aligned}\sum_{i=0}^{J} \|Q_i\mathbf{u}\|_A^2 &= \|S\tilde{Q}_0\mathbf{u}\|_A^2 + \sum_{i=1}^{J} \|\mathcal{Q}_i(I - Q_0)\mathbf{u}\|_A^2 \\ &\leqslant (C_2+1)^2\|\mathbf{u}\|_A^2 + C_3\left[\bar{\varrho}_{S^TAS}\|(I - S\tilde{Q}_0)\mathbf{u}\|^2 + \|(I - S\tilde{Q}_0)\mathbf{u}\|_A^2\right] \\ &\leqslant \left[(C_2+1)^2 + C_3(C_1+C_2)^2 + 2C_3\right]\|\mathbf{u}\|_A^2 + 2C_3\|S\tilde{Q}_0\mathbf{u}\|_A^2 \\ &\leqslant \left[(1+2C_3)(C_2+1)^2 + C_3(C_1+C_2)^2 + 2C_3\right]\|\mathbf{u}\|_A^2,\end{aligned}$$

which concludes the proof. □

Now we are ready to prove the Convergence Theorem 3.3.

P r o o f. The proof follows from Assumption 3.1, Lemma 3.6 and the Abstract Convergence Theorem 2.2 by setting $\tilde{\mathcal{Q}}_i \equiv Q_i$. □

4. Components of the algorithm

In the following sections we will confine ourselves to linear elasticity problems and we will discuss the three components of the algorithm—the properties of the tentative coarse space, the prolongator smoother and the Schwarz domain decomposition subspaces.

Let $\Omega \subset \mathbb{R}^d$ be the bounded computational domain, $\mathcal{T}$ the quasiuniform triangulation of Ω. Further, let $\mathrm{V}_i = P1(\mathcal{T})$, $i = 1, \ldots, d$, be the corresponding P1 finite element spaces for the i-th displacement field and

$$\mathrm{V} = \{u \in \mathrm{V}_1 \times, \ldots \times \mathrm{V}_d \colon\ B_k u = 0 \quad \text{on selected boundary nodes } k\},$$

where B_k is an arbitrary $d \times d$ matrix.

We assume that the matrix A is a stiffness matrix obtained by discretizing the continuous problem:

$$\text{for given } f \in \mathrm{V} \text{ find } u \in \mathrm{V}: \quad (u, v)_\varepsilon = (f, v), \quad \forall v \in \mathrm{V},$$

where

$$(u, v)_\varepsilon = \sum_{i,j=1}^{3} \varepsilon_{ij}(u)\varepsilon_{ij}(v), \quad \varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The induced energy semi-norm is then $|\cdot|_\varepsilon = (\cdot, \cdot)_\varepsilon^{1/2}$.

4.1. Tentative prolongator.

The aim of this section is to give a precise definition of the *tentative prolongator* $\tilde{P}$ and to verify that such a tentative prolongator satisfies assumption (9).

The tentative prolongator $\tilde{P}$ will be created by purely algebraic means based on the aggregation technique in [15]. It rests on the aggregation of all active finite element nodes to a system of disjoint aggregates $\{\mathcal{A}_i\}$, $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$, for all $i \neq j$.

Definition 4.1. Let us denote by RBM the *kernel* of $|\cdot|_\varepsilon$ in $[H^1(\Omega)]^d$,

$$\mathrm{RBM} = \{u \in [H^1(\Omega)]^d \colon\ |u|_\varepsilon(\Omega) = 0\}.$$

Further, let **RBM** be its *discrete form*, i.e. if we use the finite element interpolator $\Pi_h \colon\ \mathbb{R}^{n_1} \to \mathrm{V}$, we can find for each $\mathbf{u} \in \mathbf{RBM}$ some $u \in \mathrm{RBM}$ such that $\Pi_h \mathbf{u} = u$ on Ω apart from one layer of finite elements around the essential boundary conditions.

The reason for excluding the elements around this part of the boundary is that the finite element space V is not able to support RBM there.

A l g o r i t h m 4.2. Let $\langle \tilde{\mathbf{r}}_k \rangle_{k=1}^{n_K}$, $n_K = \dim(\mathbf{RBM})$, be a basis of RBM. We construct the tentative prolongator $\tilde{P}$ by the following algorithm:

For $i = 1, \dots, J$ do

Set $\tilde{\mathbf{r}}_k^i = \begin{cases} \tilde{\mathbf{r}}_k & \text{on degrees of freedom in } \mathcal{A}_i, \\ 0 & \text{otherwise.} \end{cases}$

Orthonormalize $\langle \tilde{\mathbf{r}}_k^i \rangle_{k=1}^{n_K}$ to get $\langle \mathbf{r}_k^i \rangle$.

Put $\mathbf{r}_k^i$ to be the $[n_K \cdot i + k]$-th column of $\tilde{P}$, $k = 1, \dots, n_K$.

end.

R e m a r k 4.3. The orthonormalization step in Algorithm 4.2 is not required by the theory, it just improves the conditioning of the matrix of the coarse problem.

In the next part, we recall some known facts about Korn's inequality which can be found for example in [9], [8], and [6]. The need for using Korn's inequality

$$\min_{k \in \mathrm{RBM}} \|u - k\|_{[H^1(B_i)]^d} \leqslant C_K(B_i) \|u\|_{\varepsilon(B_i)} \tag{15}$$

stems from the necessity to verify assumption (9) on our tentative coarse space. We want to avoid using Korn's inequality on the whole computational domain Ω, because Korn's constant C_K depends on the domain. Instead, we propose to use (15) on the tentative coarse space restricted to continuous envelopes B_i of the nodal aggregates $\mathcal{A}_i$, whose shape we can control in the code.

This of course means that the tentative coarse space restricted to $\mathcal{A}_i$ must be able to support exactly the functions from RBM, so that we can project them out in the left-hand side of (15). Note that the tentative coarse space $\mathrm{Rng}(\tilde{P})$ with $\tilde{P}$ constructed by Algorithm 4.2 has this property.

Definition 4.4. Suppose a domain D is star-shaped with respect to a ball B, i.e. for all $x \in D$, the closed convex hull of $\{x\} \cup B$ is a subset of D. Let

$$\varrho_{\max} = \sup\{\varrho\colon\ D \text{ is star-shaped with respect to a ball of diameter } \varrho\}.$$

Then the *chunkiness parameter* of D is defined by

$$\gamma(D) = \frac{\operatorname{diam}(D)}{\varrho_{\max}}.$$

To provide estimates independent of the geometry of Ω, we have to make some assumptions on the shape of the aggregates or their envelopes:

Assumption 4.5. For each aggregate $\mathcal{A}_i$, let there be a continuous subdomain $B_i \subset \Omega$ which contains all nodes listed in $\mathcal{A}_i$ and

1. there is an integer constant $N_{\mathcal{A}} > 0$ such that every point $x \in \Omega$ belongs to at most N subdomains B_i (bounded overlaps),
2. $\operatorname{diam}(B_i) \leqslant CH \quad i = 1, \ldots, J$,
3. $\operatorname{meas}(B_i) \geqslant CH^d \quad i = 1, \ldots, J$,
4. each B_i is the union of at most N_ω star-shaped domains ω_i^j with

(a) uniformly bounded chunkiness parameter $\gamma(\omega_i^j) \leqslant \gamma_\omega$,

(b) $\operatorname{meas}(\omega_i^{k+1} \cap (\bigcup_{j=1}^{k} \omega_i^j)) \geqslant c_\omega \min\{\operatorname{meas}(\omega_i^{k+1}), \operatorname{meas}(\bigcup_{j=1}^{k} \omega_i^j)\}$, $k < N_\omega$.

A simple greedy algorithm (cf. [5]) can be proposed to create the aggregates $\{\mathcal{A}_i\}$ which satisfy the above assumption. However, note that here a "hidden" geometrical dependence infiltrates our estimates, due to geometrical dependence of the envelopes B_i along the boundary of Ω: if the boundary of Ω has "chunks" of a characteristic size less than the size of an aggregate, then the aggregate (and also its envelope B_i) adjacent to this part of the boundary will have a large chunkiness parameter.

Lemma 4.6. *(Poincaré-Korn) Under Assumption* 4.5 *on B_i, we have*

$$\inf_{k_i \in \mathrm{RBM}} \|u - k_i\|^2_{[L^2(B_i)]^d} \leqslant C(N_\omega, \gamma_\omega, c_\omega) H^2 |u|^2_{\varepsilon(B_i)}. \tag{16}$$

P r o o f. It is well known [8] for a star-shaped domain Ω that Korn's constant $C_K(\Omega)$ on the factor space modulo rigid body modes (RBM's) can be controlled by the chunkiness parameter $\gamma(\Omega)$. Furthermore, for two Lipschitz domains Ω_1 and Ω_2, Korn's constant C_K on $\Omega_1 \cup \Omega_2$ can be estimated [6] as

$$\begin{aligned} C_K(\Omega_1 \cup \Omega_2) \geqslant\ & C_K(\Omega_1) + C_K(\Omega_2) \\ & + \frac{\min(\operatorname{meas}(\Omega_1), \operatorname{meas}(\Omega_2))}{\operatorname{meas}(\Omega_1 \cap \Omega_2)} \big(\sqrt{C_K(\Omega_1)} + \sqrt{C_K(\Omega_2)}\big)^2. \end{aligned}$$

Now, we will find $\hat{h}_i \in \mathrm{RBM}$ and a constant function $\hat{q}_i$ such that

$$\|u - \hat{h}_i\|^2_{[H^1(B_i)]^d} = \inf_{h_i \in \mathrm{RBM}} \|u - h_i\|^2_{[H^1(B_i)]^d} \quad \text{and} \quad \int_{B_i} (u - \hat{h}_i - \hat{q}_i) = 0.$$

Because $(\hat{h}_i + \hat{q}_i) \in \mathrm{RBM}$, we can write using the scaled Poincaré inequality, Korn's inequality, and Assumption 4.5 together with the above argument on C_K

$$\begin{aligned} \inf_{k_i \in \mathrm{RBM}} \|u - k_i\|^2_{[L^2(B_i)]^d} &\leqslant \|u - \hat{h}_i - \hat{q}_i\|^2_{[L^2(B_i)]^d} \leqslant CH^2 |u - \hat{h}_i|^2_{[H^1(B_i)]^d} \\ &\leqslant C(N_\omega, \gamma_\omega, c_\omega) H^2 |u|^2_{\varepsilon(B_i)}. \end{aligned}$$

□

Having Poincaré-Korn's inequality, we can find a mapping $\tilde{Q}_0 \colon \mathbb{R}^{n_1} \to \mathrm{Rng}(\tilde{P})$ for which we will verify the weak approximation property of the tentative coarse space $\mathrm{Rng}(\tilde{P})$.

Lemma 4.7. *Let $\mathcal{A}_i$ and B_i be as in Assumption* 4.5. *Then a mapping $\tilde{Q}_0 \colon \mathbb{R}^{n_1} \to \mathrm{Rng}(\tilde{P})$, defined locally on each $\mathcal{A}_i$ as an l^2-orthogonal projection onto* **RBM**, *i.e.*

$$(\mathbf{u} - \tilde{Q}_0\mathbf{u}, \mathbf{v})_{[l^2(\mathcal{A}_i)]^d} = 0, \quad \forall \mathbf{v} \in \mathbf{RBM},\ \forall \mathbf{u} \in \mathbb{R}^{n_1}, \quad i = 1, \dots, J, \tag{17}$$

satisfies

$$\|\mathbf{u} - \tilde{Q}_0\mathbf{u}\|^2_{[l^2]^d} \leqslant C\frac{H^2}{h^d}\|\mathbf{u}\|_A, \tag{18}$$

where C depends only on N_ω, γ_ω, c_ω, and $N_{\mathcal{A}}$ from Assumption 4.5. Here d is the dimension of the problem.

P r o o f. For every $\mathcal{A}_i$ let $\mathbf{k}_i \in \mathbf{RBM}$ be the minimizer of

$$\min_{\mathbf{m}\in\mathbf{RBM}} \|\mathbf{u} - \mathbf{m}\|^2_{[l^2(\mathcal{A}_i)]^d}.$$

Using the fact that $\tilde{Q}_0$ is an l^2-projection in local sense as in (17), and that $\mathbf{k} = \tilde{Q}_0\mathbf{k}$ for all $\mathbf{k} \in \mathbf{RBM}$ together with Poincaré-Korn's inequality (Lemma 4.6) we get

$$\begin{aligned}
\|\mathbf{u} - \tilde{Q}_0\mathbf{u}\|^2_{[l^2(\mathcal{A}_i)]^d} &= \|(\mathbf{u} - \mathbf{k}_i) - \tilde{Q}_0(\mathbf{u} - \mathbf{k}_i)\|^2_{[l^2(\mathcal{A}_i)]^d} \leqslant C\|\mathbf{u} - \mathbf{k}_i\|^2_{[l^2(\mathcal{A}_i)]^d} \\
&= C \min_{\mathbf{m}\in\mathbf{RBM}} \|\mathbf{u} - \mathbf{m}\|^2_{[l^2(\mathcal{A}_i)]^d} \leqslant \frac{C}{h^d} \min_{m\in\mathrm{RBM}} \|u - m\|^2_{[L^2(B_i)]^d} \\
&\leqslant \frac{H^2}{h^d} C(N_\omega, \gamma_\omega, c_\omega)|u|^2_{\varepsilon(B_i)}.
\end{aligned}$$

Now, summing over $i = 1, \dots, J$ and using Assumption 4.5 1) on the bounded overlaps of $\{B_i\}$ we get

$$\begin{aligned}
\|\mathbf{u} - \tilde{Q}_0\mathbf{u}\|^2_{[l^2]^d} &= \sum_{i=1}^{J} \|\mathbf{u} - \tilde{Q}_0\mathbf{u}\|^2_{[l^2(\mathcal{A}_i)]^d} \leqslant \frac{H^2}{h^d} C(N_\omega, \gamma_\omega, c_\omega) \sum_{i=1}^{J} |u|^2_{\varepsilon(B_i)} \\
&\leqslant \frac{H^2}{h^d} C(N_\omega, \gamma_\omega, c_\omega, N_{\mathcal{A}})|u|^2_{\varepsilon(\Omega)} \leqslant \frac{H^2}{h^d} C(N_\omega, \gamma_\omega, c_\omega, N_{\mathcal{A}})\|\mathbf{u}\|^2_A.
\end{aligned}$$

□

Lemma 4.7, as we will see later, verifies the weak approximation property (9) of the tentative coarse space.

4.2. Prolongator smoother.

This section is devoted to the specification of our prolongator smoother S.

As S is supposed to take care of the energy properties of the smoothed coarse space, it is clear that S must be a function of A. For convenience of construction, let us take S to be a polynomial in A. It follows from the assumption (8) by taking $\mathbf{u} \in \mathbf{RBM}$ that the absolute term of S must be the identity matrix.

In the next part we present the optimal form of the polynomial S and verify its smoothing properties required by Assumption 3.1.

A l g o r i t h m 4.8. Let us have an available upper bound $\bar{\varrho}_A$ of $\varrho(A)$, and let us denote by d_S the desired degree of the polynomial smoother. Also, let us have $n_K = \dim(\mathrm{RBM})$ and the number of subdomains J. We obtain the smoothed prolongation operator $P = S\tilde{P}$ by the following algorithm:

For $i = 1, \ldots, J \cdot n_K$

Set $\mathbf{w}$ to be the i-th column of $\tilde{P}$.

For $k = 1, \ldots, d_S$

Set $\mathbf{w} := \mathbf{w} - r_k^{-1} A\mathbf{w}$, where $r_k = \bar{\varrho}_A \sin^2(\frac{k\pi}{2d_S+1})$.

end.

Set the i-th column of P to $\mathbf{w}$.

end.

The particular choice of r_k is justified in the next lemma proposed by Mandel: out of the set of polynomials of a fixed degree d_S with an absolute term equal to one we choose the one with the best smoothing properties.

Lemma 4.9. *Let $\mathcal{P}_n$ be a set of polynomials of degree n such that $p(0) = 1$ for all $p(x) \in \mathcal{P}_n$. Then for any integer $n > 0$, there is a unique polynomial $s(x) \in \mathcal{P}_n$ such that*

$$\min_{p \in \mathcal{P}_n} \left(\max_{x \in [0,1]} p^2(x)x \right) = \max_{x \in [0,1]} s^2(x)x. \tag{19}$$

The polynomial $s(x)$ is given by

$$s(x) = \prod_{k=1}^{n} \left(1 - \frac{x}{r_k}\right), \; r_k = \frac{1}{2}\left(1 - \cos\frac{2k\pi}{2n+1}\right) = \sin^2\frac{k\pi}{2n+1}, \; k = 1, \ldots, r. \tag{20}$$

In addition, the polynomial $s(x)$ satisfies

$$\max_{x\in[0,1]} s^2(x)x = \frac{1}{(2n+1)^2}, \tag{21}$$

$$\max_{x\in[0,1]} |s(x)| = 1, \tag{22}$$

$$[1-s(x)]^2 \leqslant \frac{\pi^2}{12}(2n+1)^2 x, \qquad \forall x \in [0,1]. \tag{23}$$

P r o o f. It follows from the minimization properties of the Chebyshev polynomials that the polynomial

$$q(x) = s^2(x)x$$

must be a scaled and shifted Chebyshev polynomial of degree (2n+1) such that it satisfies (19) and the constraint $s(0) = 1$. Using this argument one arrives at

$$q(x) = \frac{c}{2}\left(1 - w_{2n+1}(1-2x)\right), \tag{24}$$

where w_{2n+1} is the Chebyshev polynomial and from $s(0) = q'(0)$ we have

$$c = \max_{x\in[0,1]} q(x) = \frac{1}{w'_{2n+1}(1)} = \frac{1}{(2n+1)^2}.$$

This proves (19) and (21).

The polynomial $q(x)$ vanishes at the points x where $w_{2n+1}(1-2x) = 1$, that is; where $1 - 2x = \cos(2k\pi/(2n+1))$. The value $k = 0$ gives the simple root of $q(x)$ $r_0 = 0$, while $k = 1, \ldots, n$ yield double roots given by (20).

Now we prove (22). This is equivalent to proving $s^2(x) \leqslant 1$ for all $x \in [0,1]$. Using (24) we will show that

$$0 \leqslant \frac{1 - w_{2n+1}(1-2x)}{2x w'_{2n+1}(1)} \leqslant 1, \qquad \forall x \in [0,1].$$

After a substitution $1 - 2x = \tilde{x}$ it becomes

$$w_{2n+1}(\tilde{x}) \geqslant 1 + w'_{2n+1}(1)(\tilde{x} - 1), \quad \forall \tilde{x} \in [-1,1],$$

which is the well-known fact that the graph of the Chebyshev polynomial lies above its tangent at $\tilde{x} = 1$.

It remains to show (23). We know that $s(x)$ is a polynomial of degree n with real roots only. Further, we know that $q(x)$ is bounded by the line $y = \frac{1}{(2n+1)^2}$ and that this bound is attained at all local maxima of $q(x)$. Also, $s(0) = 1$. From these facts one deduces that $s(x)$ is bounded within $y = \pm\frac{1}{(2n+1)\sqrt{x}}$ and that the bound

is attained (n+1) times. Also, it is clear that $s(x)$ is convex for $x \in [0, r_1]$, where r_1 is the smallest root. It follows that the graph of $s(x)$ lies for $x \in [0,1]$ above its tangent at $x = 0$, i.e.

$$1 + s'(0)x \leqslant s(x), \qquad x \in [0,1]. \tag{25}$$

Now, using (25), $s'(0) = -\sum_{k=1}^{n} k^2 r_k^{-1}$, the estimates $\sin x \geqslant \frac{2}{\pi}x$, $x \in [0, \frac{\pi}{2}]$, and $\sum_{k=1}^{\infty} = \frac{\pi^2}{6}$, we can prove (23):

$$\begin{aligned}[1 - s(x)]^2 &\leqslant 2[1 - s(x)] \leqslant -2s'(0)x = 2\Big(\sum_{k=1}^{n} \sin^{-2}\Big(\frac{k\pi}{2n+1}\Big)\Big)x \\ &\leqslant \Big(\sum_{k=1}^{n} \pi^2 \frac{(2n+1)^2}{2k^2\pi^2}\Big)x \leqslant \frac{1}{2}\Big(\sum_{k=1}^{n} \frac{1}{k^2}\Big)(2n+1)^2 x \leqslant \frac{\pi^2}{12}(2n+1)^2 x,\end{aligned}$$

which concludes the proof. □

Lemma 4.10. *The polynomial prolongator smoother $S = s(\frac{A}{\bar{\varrho}_A})$ as in Algorithm 4.8, where $\bar{\varrho}_A$ is an available upper bound of $\varrho(A)$, commutes with A and satisfies*

$$\begin{aligned}\varrho(S) &\leqslant 1, \\ \|(I - S)\mathbf{u}\| &\leqslant C \deg(S) \frac{1}{\sqrt{\bar{\varrho}_A}} \|\mathbf{u}\|_A\end{aligned}$$

and

$$\varrho(S^T A S) \leqslant C \deg^{-2}(S) \bar{\varrho}_A. \tag{26}$$

P r o o f. The proof follows directly from Lemma 4.9 by using the spectral mapping theorem. □

We will see later that for one particular choice of d_S Lemma 4.10 verifies the assumptions (7) and (8).

4.3. Schwarz subspaces.

It remains to specify the Schwarz subdomains and verify the assumption (10).

Assumption 4.11. Each computational subdomain

$$\Omega_i = \bigcup_{\mathbf{u}\in \mathrm{Rng}(N_i)} \mathrm{supp}\{\Pi_h \mathbf{u}\}$$

satisfies

1. $\mathrm{diam}(\Omega_i) \leqslant CH$,
2. for all $x \in \Omega$ there exists $\Omega_k \colon\ x \in \Omega_k$ and $\mathrm{dist}(x, \partial\Omega_k \setminus \partial\Omega) \geqslant cH$,
3. there exists an integer constant N_Ω such that every point $x \in \Omega$ belongs to at most N_Ω subdomains Ω_i (bounded overlaps).

R e m a r k 4.12. For example, the computational subdomains can be derived from the supports of coarse-space basis functions. Information about the coarse-space basis functions is contained in n_K-tuples of columns of $\tilde{P}$, $n_K = \dim(\mathrm{RBM})$. For this choice of the computational subdomains, one can show that Assumption 4.11 on the geometry of the computational subdomains follows from Assumption 4.5 on the aggregates $\{\mathcal{A}_i\}$ and from the choice of the degree of the prolongator smoother $\deg(S) \approx H/h$.

Now, we will construct the unity-decomposing mappings $\mathcal{Q}_i$ that satisfy the assumption (10). For this purpose we define a set of weight functions $\{\psi_k\}$ which will be applied as masks to any finite element function $w \in \mathrm{V}$ to decompose w to the subdomains.

Lemma 4.13. *From Assumption* 4.11 *it follows that there is a set of functions* $\{\psi_k\}_{k=1,\dots,J}$, $\psi_k \in W^{1,\infty}(\Omega)$ *such that*

1. $|\psi_k|_{W^{1,\infty}(\Omega)} \leqslant CH^{-1}$,
2. $\sum\limits_{k=1}^{J} \psi_k = 1$ *on* Ω,
3. $\mathrm{supp}(\psi_k) \subset \Omega_k$, $k = 1, \dots, J$.

P r o o f. We will provide just an outline of this proof, for the full version see [13] or [7]. For each Ω_k we define

$$\tilde{\psi}_k(x) = \begin{cases} H^{-1}\,\mathrm{dist}(x, \partial\Omega_k \setminus \partial\Omega) & \text{for } x \in \Omega_k, \\ 0 & \text{for } x \in \Omega \setminus \Omega_k. \end{cases}$$

By Assumption 4.11 1) and 3), and from Assumption 4.11 2) we have

$$c \leqslant \sum_{k=1}^{J} \tilde{\psi}_k \leqslant C. \tag{27}$$

Further, one can show that

$$|\tilde{\psi}_k|_{\mathrm{Lip}(\Omega)} := \sup\Big\{ \frac{|\tilde{\psi}_k(x) - \tilde{\psi}_k(y)|}{\mathrm{dist}(x,y)} :\ x, y \in \Omega;\ x \neq y \Big\} \leqslant H^{-1}. \tag{28}$$

Let us set

$$w(x) = \Big(\sum_{k=1}^{J} \tilde{\psi}_k(x) \Big)^{-1}, \quad x \in \overline{\Omega}.$$

For $w(x)$ one can show that

$$\|w\|_{L^\infty(\overline{\Omega})} \leqslant C, \tag{29}$$

and from (28) using the known equivalence $|\cdot|_{\mathrm{Lip}} \approx |\cdot|_{W^{1,\infty}}$ one also has

$$\|\nabla w(x)\| \leqslant CH^{-1} \qquad \text{almost everywhere on } \overline{\Omega}. \tag{30}$$

The function $\psi_k(x)$ is then given by $\psi_k(x) = w(x)\tilde{\psi}_k(x)$, which trivially satisfies statements 2) and 3) of the lemma. Finally, from (28), (29), and (30) it follows that

$$\|\nabla \psi_k\| = \|w\big(\nabla \tilde{\psi}_k\big) + \tilde{\psi}_k\big(\nabla w\big)\| \leqslant \|\nabla \tilde{\psi}_k\| \cdot |w| + |\tilde{\psi}_k| \cdot \|\nabla w\| \leqslant CH^{-1}$$

almost everywhere on $\overline{\Omega}$, which is statement 1) of the lemma. □

Now, we have to deal with the fact that for a P1 finite element function $w \in \mathrm{V}$ the product $\psi_k w$ is not a P1 finite element function: we define an operator $I_h \colon [H^1(\Omega)]^d \to \mathrm{V}$,

$$I_h v = \sum_k \varphi_k v(x_k), \tag{31}$$

where the sum is taken over all nodes k in the finite element mesh, x_k are their coordinates in Ω and $\{\varphi_k\}$ is a finite element P1 basis.

Lemma 4.14. *The operator I_h defined by* (31) *satisfies*

$$|I_h|_\varepsilon \leqslant C, \tag{32}$$

where C is independent of h and depends on the aspect ratio of the finite elements.

P r o o f. Let us denote by $\mathcal{T}$ the finite element P1 triangulation understood as a set of finite elements T_j. Then for every $v \in [H^1(\Omega)]^d$,

$$|I_h v|_\varepsilon^2 = \sum_{T_j \in \mathcal{T}} |I_h v|_{\varepsilon(T_j)}^2 = \sum_{T_j \in \mathcal{T}} |I_h v - k_j|_{\varepsilon(T_j)}^2, \tag{33}$$

where k_j is the minimizer of $\min\limits_{k \in \mathrm{RBM}(\Omega)} \|v - k\|_{[H^1(T_j)]^d}$.

Let us prescribe function values $(f_1, \dots, f_N)$ at all N vertices $(x_1, \dots, x_N)$ of one element T_j. Then it is obvious that the minimizer of

$$\min_{u \in [H^1(T_j)]^d} \{|u|_{[H^1(T_j)]^d}, u(x_1) = f_1, \dots, u(x_N) = f_N\}$$

is a linear function, i.e. for all $v \in [H^1(T_j)]^d$ we have $|I_h v|_{[H^1(T_j)]^d} \leqslant |v|_{[H^1(T_j)]^d}$.

Using this argument, the fact that rigid body modes are approximated exactly on P1 elements, and Korn's inequality we have for all $v \in [H^1(\Omega)]^d$ on all finite elements T_j the inequality

$$\begin{aligned} |I_h v - k_j|_{\varepsilon(T_j)}^2 &= |I_h(v - k_j)|_{\varepsilon(T_j)}^2 \leqslant |I_h(v - k_j)|_{[H^1(T_j)]^d}^2 \\ &\leqslant |v - k_j|_{[H^1(T_j)]^d}^2 \leqslant C|v - k_j|_{\varepsilon(T_j)}^2 = C|v|_{\varepsilon(T_j)}^2. \end{aligned} \tag{34}$$

Note that Korn's constant depends on the chunkiness parameter of each of the P1 finite elements T_j, i.e. on their aspect ratio.

Finally, when we substitute (34) into (33) we get

$$|I_h v|_\varepsilon^2 \leqslant C \sum_{T_j} |v|_{\varepsilon(T_j)}^2 = C|v|_{\varepsilon(\Omega)}^2,$$

which completes the proof. □

Having still in mind the verification of the assumption (10), we are now ready to give a constructive proof of the following lemma.

Lemma 4.15. *Under Assumption* 4.11 *there exist operators* $\mathcal{Q}_i^h \colon \mathrm{V} \to \mathrm{V}_i$, $i = 1, \dots, J$ *decomposing the unity in* V *such that*

$$\sum_{i=1}^{J} |\mathcal{Q}_i^h v|_{\varepsilon(\Omega)}^2 \leqslant C \left[\frac{1}{H^2} \|v\|_{[L^2(\Omega)]^d}^2 + |v|_{\varepsilon(\Omega)}^2 \right] \tag{35}$$

with a constant C *which depends only on the aspect ratio of the elements and on the overlap bound* N_Ω.

P r o o f. Let us set

$$\mathcal{Q}_k^h = I_h \psi_k\,, \quad k = 1, \dots, J\,,$$

where, $I_h\colon\ [H^1(\Omega)]^d \to \mathrm{V}$ is as in (31) and ψ_k is as in Lemma 4.13.

The decomposition of unity of Lemma 4.13 2) immediately yields the decomposition of unity by $\mathcal{Q}_k^h$.

Using the stability of I_h in $|\cdot|_\varepsilon$ proved in Lemma 4.14 we get

$$\begin{aligned} |\mathcal{Q}_k^h v|_{\varepsilon(\Omega)}^2 &= |I_h \psi_k v|_{\varepsilon(\Omega)}^2 \leqslant C |\psi_k v|_{\varepsilon(\Omega)}^2 \\ &= C |\psi_k v|_{\varepsilon(\Omega_k)}^2 = C \int\limits_{\Omega_k} \sum_{i,j=1}^3 \varepsilon_{ij}^2(\psi_k v)\,\mathrm{d}\Omega. \end{aligned} \tag{36}$$

Investigating only the integrand and applying Lemma 4.13, we obtain

$$\begin{aligned} \varepsilon_{ij}(\psi_k v) &= \frac{1}{2}\Big(\frac{\partial}{\partial x_j}(\psi_k v_i) + \frac{\partial}{\partial x_i}(\psi_k v_j)\Big) \\ &= \frac{1}{2}\Big(\frac{\partial \psi_k}{\partial x_j} v_i + \frac{\partial \psi_k}{\partial x_i} v_j + \frac{\partial v_i}{\partial x_j}\psi_k + \frac{\partial v_j}{\partial x_i}\psi_k\Big) \\ &\leqslant \frac{1}{H}\frac{C}{2}(v_i + v_j) + \frac{1}{2}\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big). \end{aligned}$$

Substituting into (36), we conclude

$$\begin{aligned} |\mathcal{Q}_k^h v|_{\varepsilon(\Omega)}^2 &= \int\limits_{\Omega_k} \sum_{i,j=1}^3 \left(\frac{1}{H}\frac{C}{2}(v_i + v_j) + \frac{1}{2}\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big)\right)^2 \mathrm{d}\Omega \\ &\leqslant \frac{C}{H^2} \int\limits_{\Omega_k} \Big(\sum_{i=1}^3 v_i^2\Big)\,\mathrm{d}\Omega + C \int\limits_{\Omega_k} \sum_{i,j=1}^3 \varepsilon_{ij}^2(v)\,\mathrm{d}\Omega \\ &\leqslant C\left(\frac{1}{H^2}\|v\|_{[L^2(\Omega_k)]^d}^2 + |v|_{\varepsilon(\Omega_k)}^2\right). \end{aligned} \tag{37}$$

This inequality together with bounded overlaps in Assumption 4.11 3) gives

$$\begin{aligned} \sum_{i=1}^J |\mathcal{Q}_i^h v|_{\varepsilon(\Omega)}^2 &\leqslant \sum_{i=1}^J C\left(\frac{1}{H^2}\|v\|_{[L^2(\Omega_k)]^d}^2 + |v|_{\varepsilon(\Omega_k)}^2\right) \\ &\leqslant C\left(\frac{1}{H^2}\|v\|_{[L^2(\Omega)]^d}^2 + |v|_{\varepsilon(\Omega)}^2\right), \end{aligned}$$

which proves (35). □

5. Convergence theorem

In this section we are summing up the properties of the three components of the algorithm (the tentative prolongator, the prolongator smoother and the Schwarz domain decomposition) to verify the abstract Assumption 3.1 and to state the convergence theorem for the Schwarz domain decomposition algorithm with a coarse space by smoothed aggregation.

Theorem 5.1. *Consider Algorithm* 2.1 *with a coarse space by smoothed aggregation using the tentative prolongator* $\tilde{P}$ *from Algorithm* 4.2 *satisfying Assumption* 4.5, *together with the prolongator smoother* S *constructed by Algorithm* 4.8, *for which* $cH/h \leqslant \deg(S) \leqslant CH/h$, *and the computational subdomains satisfying Assumption* 4.11. *Such an algorithm converges at the rate*

$$\|E\|_A^2 \leqslant 1 - \frac{1}{C_L[K+2]^2},$$

where E *is the error-propagation operator as in* (3), K *is the overlap bound* (2), *and* $C_L > 1$ *is a constant which depends only on* N_ω, γ_ω, c_ω, N_Ω, $N_\mathcal{A}$ *and the aspect ratio of the finite elements.*

Proof. By the Gershgorin estimate of $\varrho(A)$ we set $\bar{\varrho}_A \equiv Ch^{d-2}$. The choice of $\bar{\varrho}_{S^TAS}$ follows from (26).

$$\varrho(S^TAS) \leqslant C \deg^{-2}(S)\bar{\varrho}_A = C\frac{h^2}{H^2}h^{d-2} = C\frac{h^d}{H^2} \equiv \bar{\varrho}_{S^TAS}.$$

Then the assumptions (7) and (8) on the smoother S from the abstract convergence theory are verified by Lemma 4.10. The assumption (9) on the tentative prolongator $\tilde{P}$ is verified in Lemma 4.7 under Assumption 4.5 on the aggregates. The assumption (10) on the Schwarz subspaces follows from Lemma 4.15 under Assumption 4.11 on the computational subdomains. □

6. Computational complexity

Following Vaněk and Brezina in [5], we will now give an asymptotic bound on the amount of floating point operations needed to carry out the iteration to reduce the error to the truncation level. We will give estimates for the implementation on both serial and parallel architectures.

Let N_{es} denote the typical number of elements per subdomain, let d be the dimension of the space on which the continuous problem is cast, and n the number of all degrees of freedom in the whole system.

Let us first compute the amount of work needed for the setup. On a machine with a single CPU, we need $\mathcal{O}(\deg(\mathcal{S})n)$ operations to compute the prolongator $P = \mathcal{S}\tilde{P}$. Taking into account that $\deg(\mathcal{S}) \approx \frac{H}{h} \approx N_{es}^{1/d}$, this becomes $\mathcal{O}(N_{es}^{1/d}n)$. Further, we need $\mathcal{O}(\frac{n}{N_{es}} N_{es}^{\frac{3d-2}{d}})$ and $\mathcal{O}((\frac{n}{N_{es}})^{\frac{3d-2}{d}})$ operations to compute Cholesky factorizations of the local and coarse level matrices, respectively. We also need $\mathcal{O}(n)$ operations to compute the coarse level matrix, but this number can be neglected, as it is dominated by other expenditures.

Each step of the iteration requires $\mathcal{O}(\frac{n}{N_{es}} N_{es}^{\frac{2d-1}{d}})$ and $\mathcal{O}((\frac{n}{N_{es}})^{\frac{2d-1}{d}})$ operations to compute the back substitutions in the local and coarse spaces, respectively. The amount of work required to compute the defect, the corrections and the restriction is $\mathcal{O}(n)$ and hence negligible.

Taking into account all the above listed expenditures, we use trivial calculus to conclude that the optimal value of the number of elements per subdomain is $N_{es} = n^{\frac{2d-2}{5d-4}}$. That is, $N_{es}^{opt} = n^{\frac{1}{3}}$ for 2D problems and $N_{es}^{opt} = n^{\frac{4}{11}}$ for 3D problems. The total amount of work involved in the setup and the iterations for these optimal values is $\mathcal{O}(n^{\frac{4}{3}})$ and $\mathcal{O}(n^{\frac{49}{33}})$ in 2D and 3D, respectively.

The reason why we have introduced the coloring classes $\mathcal{C}_i$ in the algorithm was to facilitate the use of modern parallel architecture computers. For simplicity, we assume that we have at least $n^{\frac{1}{2}}$ processors. Then most of the procedures can take advantage of parallel implementation. In the evaluations of computational work we omit all operations costing $\mathcal{O}(n)$ operations.

The setup will then require $\mathcal{O}(\deg(\mathcal{S})n^{\frac{1}{2}})$ operations to compute the prolongator $P = \mathcal{S}\tilde{P}$. If we assume that the local Cholesky decompositions are performed in parallel, we need $\mathcal{O}(N_{es}^{\frac{3d-2}{d}})$ and $\mathcal{O}((\frac{n}{N_{es}})^{\frac{3d-2}{d}})$ operations to compute Cholesky factorizations of the local and coarse level matrices, respectively.

Each step of the iteration will require $\mathcal{O}(N_{es}^{\frac{2d-1}{d}})$ and $\mathcal{O}((\frac{n}{N_{es}})^{\frac{2d-1}{d}})$ operations to compute the back substitutions in the local and coarse spaces, respectively.

Balancing these values, we obtain that the optimal size of a subdomain is about $n^{\frac{1}{2}}$ in both 2D and 3D. The resulting computational complexity can be then bounded by $\mathcal{O}(n)$ in 2D and by $\mathcal{O}(n^{\frac{7}{6}})$ in 3D.

The above estimates show that the amount of work required to complete the whole iterative process (including its setup) is asymptotically lower even than the back substitution step of direct methods based on matrix factorization, which would be $\mathcal{O}(n^{\frac{3}{2}})$ and $\mathcal{O}(n^{\frac{5}{3}})$ in 2D and 3D, respectively.

7.1. Cube-test.

We have tested numerical stability on the following example: let us pose a linear elasticity problem on a geometry as in Fig. 1 consisting of a cube with a truss attached to one of its sides. On one end of the truss we set the displacement to be zero and on the opposite face of the cube we place a load-force pointing downwards. Let us consider a variable width of the truss. The model is discretized by P1 finite elements and aggregation is performed to create aggregates of radius 2. The degree of the prolongator smoother is also 2. All subspace problems (on Schwarz subdomains and on the coarse space) are solved by LU decomposition. Table 1 compares the numerical performance of a non-preconditioned conjugate gradient method to the conjugate gradient method preconditioned by our algorithm. We stop the CGM iteration loop once the relative preconditioned residual satisfies

$$\frac{\langle CAe_i, Ae_i\rangle}{\langle CAe_0, Ae_0\rangle}\,\mathrm{cond}(C,A) \leqslant \varepsilon^2, \tag{38}$$

where C denotes the domain decomposition preconditioner, $\mathrm{cond}(C,A)$ is the condition number estimate computed at runtime and e_i is the error after the i-th iteration. In this case we have used $\varepsilon = 1.0 \cdot 10^{-6}$.

width a	cond(A)	iters. CGM	cond(C,A)	iters. PCGM	no. of nodes
no truss	$4.96 \cdot 10^4$	588	9.57	18	65025
12	$2.62 \cdot 10^7$	1215	7.99	16	66621
6	$2.98 \cdot 10^8$	1118	7.99	16	65557
4	$1.40 \cdot 10^9$	1509	8.16	19	65310
2	$1.69 \cdot 10^{10}$	2281	8.01	18	65139

Table 1. Comparison of CGM vs. PCGM in cube-test.

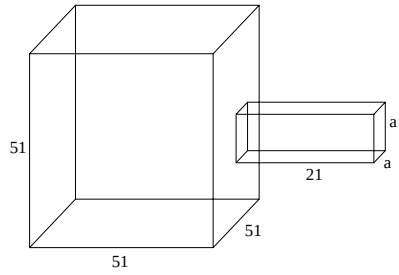

Figure 1. Geometry of cube-test.

Observe that as the truss is made thinner we loose control over the condition number of the non-preconditioned problem.

7.2. Industrial models.

In this section we give results of numerical experiments on some industrial models obtained by courtesy of the University of Colorado in Denver. Again, we have used the symmetric Schwarz overlapping domain decomposition algorithm as a preconditioner of the conjugate gradients method with the same stopping condition (38), in this case with $\varepsilon = 1.0 \cdot 10^{-5}$. All experiments have been performed by the courtesy of the Supercomputer Centre at the University of West Bohemia in Plzeň, Czech Republic, on Digital AlphaServer 8400, `kirke.zcu.cz`, at 3330–4800 MFlops/s with 2GB of RAM, using all of its 8 processors.

name	no. of nodes	dofs on node	$\|A\|_\infty$	input data	mesh
test1	12,125	3	$4.35 \cdot 10^8$	16.5 MB	n.a.
solid1	25,058	3	$2.88 \cdot 10^8$	27.7 MB	Fig. 2
solid2	40,329	3	$1.49 \cdot 10^7$	75.9 MB	n.a.
shell	9,915	6	$3.18 \cdot 10^5$	20.0 MB	n.a.

Table 2. Description of experimental input data

aggregates	domains				CPU/wall time [s]		RAM
radius=deg(S)	no.	colors	iter.	cond	setup	iteration	[MB]
1	620	18	13	8.08	193.4/302	26.9/133	47.6
2	206	14	12	7.56	45.27/93	36.8/166	83.0

Table 3. test1

aggregates	domains				CPU/wall time [s]		RAM
radius=deg(S)	no.	colors	iter.	cond	setup	iteration	[MB]
1	1,433	25	6	1.83	155/178	25/36	212
2	346	21	6	2.28	108/143	44/79	445
3	140	18	7	3.04	210/378	68/156	765

Table 4. solid1

aggregates	domains				CPU/wall time [s]		RAM
radius=deg(S)	no.	colors	iter.	cond	setup	iteration	[MB]
1	1,082	20	13	7.04	136/250	120/193	465
2	295	20	13	8.80	850/3297	410/493	1,300

Table 5. solid2

aggregates	domains				CPU/wall time [s]		RAM
radius=deg(S)	no.	colors	iter.	cond	setup	iteration	[MB]
1	1,215	9	9	3.47	652/793	25/84	78.6
2	450	12	9	4.02	108/393	22/161	84.8
3	235	11	9	4.57	49/95	28/117	125

Table 6. shell

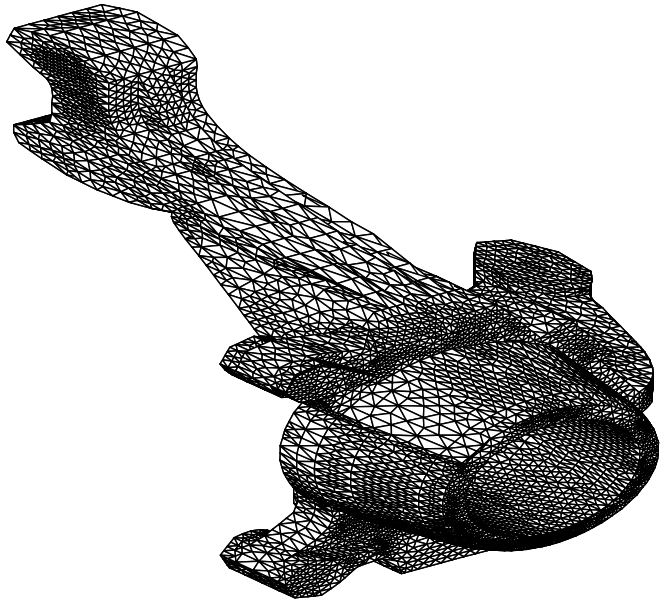

Figure 2. solid1, courtesy of prof. Charbel Farhat, University of Colorado at Boulder

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