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POWER BOUNDED AND EXPONENTIALLY BOUNDED MATRICES

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Abstract. The paper gives a new characterization of eigenprojections, which is then used to obtain a spectral decomposition for the power bounded and exponentially bounded matrices. The applications include series and integral representations of the Drazin inverse, and investigation of the asymptotic behaviour of the solutions of singular and singularly perturbed differential equations. An example is given of localized travelling waves for a system of conservation laws.

Keywords: power and exponentially bounded matrices, spectral decomposition, Drazin inverse, singularly perturbed differential equations, asymptotic behaviour

MSC 2000: 15A09, 34D05

1. Preliminaries

For any matrix $A \in \mathbb{C}^{p\times p}$ we denote its nullspace, range, spectrum and spectral radius by $N(A)$, $R(A)$, $\sigma(A)$ and $r(A)$, respectively. The set of all eigenvalues of A with $|\lambda| = r(A)$ is called the *peripheral spectrum* of A, written $\sigma_{\rm per}(A)$. We define the *index* of A, written $\operatorname{ind}(A)$, to be the least nonnegative integer q for which $N(A^q) = N(A^{q+1})$. Let μ be a complex number with $\operatorname{ind}(\mu I - A) = q$; then

$$\mathbb{C}^p = N((\mu I - A)^q) \oplus R((\mu I - A)^q) \tag{1.1}$$

(see [2, Chapter 7]). In particular, μ is an eigenvalue of A (of index q) if and only if $q > 0$. Conversely, $\operatorname{ind}(A - \mu I)$ is the smallest nonnegative integer q for which (1.1) holds.

A matrix B is a *Drazin inverse* of a matrix $A \in \mathbb{C}^{p\times p}$ if

$$BA = AB, \quad B^2A = B, \quad A^{q+1}B = A^q \quad \text{for some } q \geqslant 0. \tag{1.2}$$

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We write $A^D := B$ for the Drazin inverse of A. The condition $A^{q+1}B = A^q$ can be replaced by requiring that $A - A^2B$ is nilpotent. The smallest nonnegative integer q for which (1.2) holds is equal to the index of A (see [2]). The order of nilpotency of $A - A^2B$ is then also equal to the index of A. Every matrix A has a unique Drazin inverse A^D; if A is nonsingular, then $A^D = A^{-1}$.

Throughout the paper, $x \mapsto \|x\|$ denotes a vector norm on $\mathbb{C}^p$, and $A \mapsto \|A\|$ a matrix norm on $\mathbb{C}^{p\times p}$ consistent with the selected vector norm in the sense of Householder [3, Section 2.2]; we recall that

$$\|Ax\| \leqslant \|A\|\|x\| \quad \text{for all } A \in \mathbb{C}^{p\times p} \text{ and all } x \in \mathbb{C}^p.$$

If S is a set of complex numbers, $\operatorname{Re} S$ denotes the set of all real numbers $\operatorname{Re}\lambda$, where $\lambda \in S$. In this paper we use the abbreviation $\operatorname{Re} S < \alpha$ to mean that $\operatorname{Re}\lambda < \alpha$ for all $\lambda \in S$.

A matrix A satisfying $\operatorname{Re}\sigma(A) < 0$ is called *stable*.

We recall the necessary and sufficient conditions for $A^n \to 0$ as $n \to \infty$ (respectively $\exp(tA) \to 0$ as $t \to \infty$).

Lemma 1.1. [5, 6] *Let* $A \in \mathbb{C}^{p\times p}$.

(i) $A^n \to 0$ *as* $n \to \infty$ *if and only if* $r(A) < 1$.

(ii) $\exp(tA) \to 0$ *as* $t \to \infty$ *if and only if* A *is stable.*

The following known result is needed later in the paper; we include a proof for completeness.

Lemma 1.2. *Suppose* $M, N \in \mathbb{C}^{p\times p}$ *are commuting matrices with* M *nonsingular and* N *nilpotent. Then the matrix* $M + N$ *is nonsingular.*

Proof. Since $MN = NM$, we have also $M^{-1}N = NM^{-1}$. The matrix $M^{-1}N$ is nilpotent since $(M^{-1}N)^k = M^{-k}N^k$ for all integers k. Hence $\sigma(M^{-1}N) = \{0\}$, and $M + N = M(I + M^{-1}N)$ is nonsingular being the product of two nonsingular matrices. □

2. Resolutions of I and eigenprojections

We introduce the concept of a resolution of the unit matrix that is crucial for the spectral decompositions discussed in this paper followed by a theorem that sets out the basic facts about such resolutions.

Definition 2.1. Let $m \geqslant 2$ be an integer. An m-tuple $(E_1, \ldots, E_m)$ of $p \times p$ matrices is called a *resolution of* I if

$$E_k \neq 0 \text{ for all } k, \qquad E_j E_k = \delta_{jk} E_j, \qquad E_1 + \ldots + E_m = I.$$

Theorem 2.2. *Let $(E_1, \ldots, E_m)$ be a resolution of I and let $A_1, \ldots, A_m$ be matrices that commute with each E_i. The matrix $A = A_1E_1 + \ldots + A_mE_m$ has the following properties.*

(i) *If each A_k is nonsingular, then so is A, and $A^{-1} = A_1^{-1}E_1 + \ldots + A_m^{-1}E_m$.*
(ii) $\sigma(A) \subset \sigma(A_1) \cup \ldots \cup \sigma(A_m)$.
(iii) *If f is a function holomorphic in an open set $\Omega \supset \sigma(A_1) \cup \ldots \cup \sigma(A_m)$, then*

$$f(A) = \sum_{k=1}^{m} f(A_k)E_k. \tag{2.1}$$

Proof. Write $\Sigma = \sigma(A_1) \cup \ldots \cup \sigma(A_m)$.

(i) follows by a direct verification.

(ii) We observe that, for every complex λ, $\lambda I - A = \sum\limits_{k=1}^{m} (\lambda I - A_k)E_k$. If each $\lambda I - A_k$ is nonsingular, then so is $\lambda I - A$ by the preceding result. This proves the spectral inclusion.

(iii) By the preceding argument, $(\lambda I - A)^{-1} = \sum\limits_{k=1}^{m} (\lambda I - A_k)^{-1}E_k$ whenever $\lambda \notin \Sigma$. Integrating over a cycle γ consisting of a finite sum of Jordan contours surrounding Σ in Ω, we get

$$\begin{aligned} f(A) &= \frac{1}{2\pi \mathrm{i}} \int_\gamma f(\lambda)(\lambda I - A)^{-1}\,\mathrm{d}\lambda = \sum_{k=1}^{m} \left(\frac{1}{2\pi \mathrm{i}} \int_\gamma f(\lambda)(\lambda I - A_k)^{-1}\,\mathrm{d}\lambda \right) E_k \\ &= \sum_{k=1}^{m} f(A_k)E_k. \end{aligned}$$

□

We define the *eigenprojection* of A at a point μ to be the idempotent matrix E with $R(E) = N((\mu I - A)^q)$ and $N(E) = R((\mu I - A)^q)$, where $q = \operatorname{ind}(A - \mu I)$ (see (1.1)). We note that μ is an eigenvalue of A if and only if $E \neq 0$.

There is a close relation between the eigenprojection E of A at 0 and the Drazin inverse A^D (see, for instance, [7, 8]):

$$E = I - A^D A \tag{2.2}$$

and

$$A^D = (A+\xi E)^{-1}(I-E) \quad \text{for any } \xi \neq 0. \tag{2.3}$$

The following theorem gives a new characterization of eigenprojections; condition (iii) requires the verification of nonsingularity of $A+\xi E$ for only one value of ξ, a fact that can be often established directly.

Theorem 2.3. Characterization of eigenprojections *Let E be an idempotent matrix commuting with A. The following conditions are equivalent.*

(i) *E is the eigenprojection of A at 0.*
(ii) *$A+\lambda E$ is nonsingular for all $\lambda \neq 0$.*
(iii) *$A^q E = 0$ for some nonnegative integer q, and $A+\xi E$ is nonsingular for some $\xi \neq 0$. (In this case $\operatorname{ind}(A) \leqslant q$.)*

Proof. (i)$\Longrightarrow$(ii). This was proved in Rothblum [8, Theorem 4.2].

(ii) $\Longrightarrow$ (iii). For any $\lambda \neq 0$,

$$\lambda I - AE = (\lambda E - A)E + \lambda(I-E).$$

Then $\lambda I - AE$ is nonsingular by Theorem 2.2 (i) for every $\lambda \neq 0$. Hence $\sigma(AE) = \{0\}$, and $A^qE = (AE)^q = 0$ for some $q > 0$.

(iii)$\Longrightarrow$(i). If $q=0$, then A is nonsingular, and $E=0$ is the eigenprojection of A at 0. Let $q \neq 0$. Then $(A^kE)^q = (A^qE)^k = 0$ for any $k>0$, the matrix $B = \sum\limits_{k=1}^{q-1} \binom{q}{k} A^k \xi^{q-k} E$ is nilpotent, and

$$A^q + \xi^q E = (A+\xi E)^q - B.$$

$(A+\xi E)^q - B$ is nonsingular by Lemma 1.2, and hence so is $A^q + \xi^q E$. Set $S = (A^q + \xi^q E)^{-1}$. From $\xi^q ES + SA^q = I = A^qS + \xi^q SE$ it follows that

$$N(A^q) \subset R(E), \qquad N(E) \subset R(A^q).$$

From $A^qE = 0 = EA^q$ we obtain

$$R(E) \subset N(A^q), \qquad R(A^q) \subset N(E).$$

So $R(E) = N(A^q)$ and $N(E) = R(A^q)$, and the result follows. □

3. Power bounded matrices

Many properties of power bounded matrices are well known and can be found in standard matrix texts (e.g. [6]).

Definition 3.1. A matrix $A \in \mathbb{C}^{p\times p}$ is said to be *power bounded* if the elements of the matrix powers $A^n = [a_{ij}^{(n)}]$ are bounded for $n \in \mathbb{N}$. We say that A is *convergent* if the limit $P = \lim_{n\to\infty} A^n$ exists, and *zero convergent* if it is convergent with the limit $P = 0$.

We note that by Lemma 1.1, A is zero convergent if and only if $r(A) < 1$. The following theorem essentially says that a power bounded matrix is a zero convergent matrix plus a linear combination of mutually orthogonal idempotent matrices.

Theorem 3.2. Spectral decomposition for power bounded matrices *A matrix $A \in \mathbb{C}^{p\times p}$ is power bounded if and only if either $r(A) < 1$ or*

$$A = \mu_1 E_1 + \ldots + \mu_s E_s + C, \tag{3.1}$$

where $|\mu_k| = 1$ for $k = 1, \ldots, s$, and

$$E_j E_k = \delta_{jk} E_j, \quad E_k C = C E_k = 0, \quad r(C) < 1. \tag{3.2}$$

The decomposition (3.1) satisfying (3.2) is unique, and for each k, E_k is the eigenprojection of A at μ_k.

Proof. Suppose that A is power bounded. It is then known [6] that the spectrum of A lies in the closed unit disc, and that the eigenvalues on the unit circle are of index 1. For the sake of completeness we give an alternative proof which is of independent interest, emphasizing the spectral approach adopted in [5].

For any eigenvalue λ of A, $|\lambda^n| \leqslant \|A^n\|$ for all n, so that $|\lambda| \leqslant 1$, and $r(A) \leqslant 1$. If $r(A) < 1$, we are finished. Suppose that μ is an eigenvalue of A with $|\mu| = 1$. Let $(A - \mu I)^2 x = 0$ for some $x \neq 0$. Then

$$A^n x = (\mu I + (A - \mu I))^n x = \mu^n x + n\mu^{n-1}(A - \mu I)x;$$

since $\|A^n x\|$ is bounded and $|\mu| = 1$, we have $(A - \mu I)x = 0$. This shows that $\operatorname{ind}(A - \mu I) = 1$ for any $\mu \in \sigma_{\mathrm{per}}(A)$. Let $\mu_1, \ldots, \mu_s$ be the eigenvalues of A with $|\mu_k| = 1$, and $E_1, \ldots, E_s$ the corresponding eigenprojections. From $\operatorname{ind}(A - \mu_k I) = 1$ we get $(A - \mu_k I)E_k = 0$ for all $k = 1, \ldots, s$. Set

$$C = A - \sum_{i=1}^{s} \mu_i E_i.$$

Then $CE_k = AE_k - \mu_k E_k = (A - \mu_k I)E_k = 0$. Write $E = E_1 + \ldots + E_s$. Then E is idempotent, and $E_k E = E_k$ for all k. For any complex λ we have

$$\lambda I - C = (\lambda I - C)E + (\lambda I - C)(I - E) = \lambda E + (\lambda I - A)(I - E). \tag{3.3}$$

If $\lambda \notin \sigma(A) \cup \{0\}$, then $\lambda I - C$ is nonsingular by Theorem 2.2. Hence

$$\sigma(C) \subset \sigma(A) \cup \{0\}, \tag{3.4}$$

and $r(C) \leqslant 1$. For any $k = 1, \ldots, s$, (3.3) gives

$$\mu_k I - C = \mu_k E + (\mu_k I - A + E_k)(I - E);$$

since $\mu_k I - A + E_k$ is nonsingular by Theorem 2.3, $\mu_k I - C$ is nonsingular by Theorem 2.2. Hence $r(C) < 1$, and (3.1), (3.2) are proved.

Conversely, if (3.1), (3.2) hold, then

$$A^n = \mu_1^n E_1 + \ldots + \mu_s^n E_s + C^n;$$

since $|\mu_k| = 1$ and C is power bounded, so is A.

To complete the proof, we have to show that, for each k, E_k is the eigenprojection of A at μ_k. From (3.1), (3.2) we obtain $(A - \mu_k I)E_k = CE_k = 0$ and

$$A - \mu_k I + E_k = \sum_{i \neq k}^{s} (\mu_i - \mu_k)E_i + E_k + (C - \mu_k I)(I - E).$$

Consequently $A - \mu_k I + E_k$ is nonsingular by Theorem 2.2 (i), and the result follows by Theorem 2.3 (iii). □

The following theorem describes how the functions of a power bounded matrix can be calculated in terms of the spectral decomposition given in Theorem 3.2.

Theorem 3.3. *Let A be a power bounded matrix with the spectral decomposition given by* (3.1) *and* (3.2). *If f is a complex valued function holomorphic in an open neighbourhood Ω of the set $\sigma(A) \cup \{0\}$, then we have*

$$f(A) = \sum_{k=1}^{s} f(\mu_k)E_k + f(C)E_{s+1} = \sum_{k=1}^{s} (f(\mu_k) - f(0))E_k + f(C), \tag{3.5}$$

where $E_{s+1} = I - (E_1 + \ldots + E_s)$.

P r o o f. Note that $(E_1, \ldots, E_{s+1})$ is a resolution of I, and $C = CE_{s+1}$. Since

$$A = \mu_1 E_1 + \ldots + \mu_s E_s + CE_{s+1},$$

Theorem 2.2 (ii) implies that $\sigma(\mu_1 I) \cup \ldots \cup \sigma(\mu_s I) \cup \sigma(C) \subset \sigma(A) \cup \{0\}$ by (3.4). The first equality in (3.5) follows from Theorem 2.2 (iii).

The proof will be finished when we show that $f(C)E = f(0)E$, where $E = E_1 + \ldots + E_s$. Indeed, for any $\lambda \notin \sigma(A) \cup \{0\}$, $(\lambda I - C)E = \lambda E$, so that $(\lambda I - C)^{-1}E = \lambda^{-1}E$. For any cycle ω surrounding $\sigma(A) \cup \{0\}$ in Ω,

$$f(C)E = \frac{1}{2\pi \mathrm{i}} \int_\omega f(\lambda)(\lambda I - C)^{-1} E \, \mathrm{d}\lambda = \frac{1}{2\pi \mathrm{i}} \int_\omega f(\lambda)\lambda^{-1} E \, \mathrm{d}\lambda = f(0)E.$$

□

We give an application to a discrete system evolution described by the matrix equations

$$x_{n+1} = Ax_n, \qquad n = 0, 1, 2, \ldots, \tag{3.6}$$

where x_0 is a given initial vector (see [6]). The spectral decomposition for power bounded matrices that we derived earlier enables us to give a description of the system evolution.

Theorem 3.4. *Let A be a power bounded matrix with the spectral decomposition given by* (3.1) *and* (3.2). *Then the state of the system* (3.6) *after n stages is given by*

$$x_n = A^n x_0 = \mu_1^n E_1 x_0 + \ldots + \mu_s^n E_s x_0 + C^n x_0. \tag{3.7}$$

P r o o f. Follows from Theorems 3.2 and 3.3. □

We note that, for all sufficiently large n,

$$x_n \approx \mu_1^n E_1 x_0 + \ldots + \mu_s^n E_s x_0, \tag{3.8}$$

where $|\mu_k| = 1$. In particular, if x_0 is in the range of the projection $E = E_1 + \ldots + E_s$, then (3.8) holds with equality for all n.

A systematic study of a class of matrices generalizing power bounded matrices is presented in Rothblum [9]. This class consists of matrices of spectral radius 1 whose peripheral eigenvalues have arbitrary index; the boundedness and convergence of powers is studied in the sense of Cesàro averaging convergence. Some of the results of [9] can be given alternative proofs using the spectral decomposition of Theorem 3.2.

4. Convergent matrices

Convergent matrices are power bounded, and hence they conform to the spectral descriptions of Theorem 3.2. Specializing, we get the following result:

Theorem 4.1. *A matrix A is convergent if and only if*

$$A = E + C, \tag{4.1}$$

where

$$E^2 = E, \quad CE = EC = 0, \quad C \quad \text{is zero convergent.} \tag{4.2}$$

The limit of (A^n) is E, the eigenprojection of A at 1.

Corollary 4.2. [6] *A matrix A is convergent if and only if $r(A) \leqslant 1$, and in the case that $r(A) = 1$, $\sigma_{\text{per}}(A) = \{1\}$ and $\operatorname{ind}(I - A) = 1$.*

If a matrix C is zero convergent, then

$$\sum_{n=0}^{\infty} C^n = (I - C)^{-1}. \tag{4.3}$$

We use the spectral decomposition to derive a representation of the Drazin inverse for a matrix $I - A$ when A is convergent.

Theorem 4.3. *If a matrix A is convergent with $A^n \to E$, then*

$$(I - A)^D = \sum_{n=0}^{\infty} A^n (I - E). \tag{4.4}$$

P r o o f. Since A is convergent, it has the decomposition (4.1) and (4.2). Then

$$\begin{aligned}\sum_{n=0}^{\infty} A^n (I - E) &= I - E + \sum_{n=1}^{\infty} (E + C^n)(I - E) \\ &= \sum_{n=0}^{\infty} C^n (I - E) = (I - C)^{-1}(I - E) \\ &= (I - A + E)^{-1}(I - E) = (I - A)^D\end{aligned}$$

by (4.3) and (2.3); the representation (4.4) is thus proved. □

R e m a r k 4.4. Observe that the series $\sum\limits_{n=1}^{\infty} A^n(I-A)$ may converge, even when the series $\sum\limits_{n=1}^{\infty} A^n$ does not.

R e m a r k 4.5. Expansion (4.4) is a special case of the result obtained by Rothblum in [9, Corollary 4.4 (4)], which gives (4.4) for matrices A satisfying $r(A) \leqslant 1$ and $\operatorname{ind}(A-I) \leqslant m$, in terms of Cesàro (C, m) averaging convergence.

5. Exponentially bounded matrices

Many properties of exponentially bounded matrices can be found in texts on linear algebra (for instance [6]).

Definition 5.1. A matrix $A \in \mathbb{C}^{p\times p}$ is said to be *exponentially bounded* if the elements of $\exp(tA)$ are bounded for $t \geqslant 0$. We say that A is *exponentially convergent* if the limit $\lim\limits_{t\to\infty} \exp(tA) = P$ exists; A is *exponentially zero convergent* if the limit P is zero.

We note that by Lemma 1.1, A is exponentially zero convergent if and only if it is stable (that is, $\operatorname{Re}\sigma(A) < 0$). The next theorem says that an exponentially bounded matrix is essentially a stable matrix plus a linear combination of mutually orthogonal idempotent matrices.

Theorem 5.2. Spectral decomposition for exponentially bounded matrices. *A matrix $A \in \mathbb{C}^{p\times p}$ is exponentially bounded if and only if either A is stable or*

$$A = (\mu_1 + 1)E_1 + \ldots + (\mu_s + 1)E_s + C, \tag{5.1}$$

where $\operatorname{Re}\mu_k = 0$ *and*

$$E_jE_k = \delta_{jk}E_j, \quad E_kC = CE_k = -E_k, \quad \operatorname{Re}\sigma(C) < 0. \tag{5.2}$$

The decomposition (5.1) *and* (5.2) *is unique and, for each k, E_k is the eigenprojection of A at μ_k.*

P r o o f. If A be exponentially bounded, it is known that the spectrum of A lies in the closed left half plane; the eigenvalues that lie on the imaginary axis have index 1 (see [6]). We prefer to give a proof based on the spectral approach of [5], which we believe is of independent interest.

For any eigenvalue λ of A,

$$e^{t\,\mathrm{Re}\,\lambda} = |e^{t\lambda}| \leqslant \|\exp(tA)\| \leqslant M \quad \text{for all } t \geqslant 0;$$

so $\mathrm{Re}\,\lambda \leqslant 0$. Let μ be an eigenvalue of A with $\mathrm{Re}\,\mu = 0$ and let $(A-\mu I)^2 x = 0$. Then

$$\begin{aligned}\exp(tA)x &= \exp(t\mu I + t(A-\mu I))x = e^{\mu t}\exp(t(A-\mu I))x\\ &= e^{\mu t}(x + t(A-\mu I)x);\end{aligned}$$

since $\|\exp(tA)\|$ is bounded and $|e^{\mu t}| = 1$, $(A-\mu I)x = 0$, and $\mathrm{ind}(A-\mu I) = 1$. Let $\mu_1, \dots, \mu_s$ be the eigenvalues of A with $\mathrm{Re}\,\mu_k = 0$, and $E_1, \dots, E_s$ the corresponding eigenprojections. From $\mathrm{ind}(A-\mu_k I) = 1$ we get $(A-\mu_k I)E_k = 0$ for all $k = 1, \dots, s$. Set

$$C = A - \sum_{i=1}^{s} (\mu_i + 1)E_i.$$

Then $CE_k = AE_k - (\mu_k+1)E_k = (A-\mu_k I)E_k - E_k = -E_k$. Write $E = E_1 + \ldots + E_s$. Then E is idempotent, and $E_k E = E_k$ for all k. For any complex λ we have

$$\lambda I - C = (\lambda I - C)E + (\lambda I - C)(I-E) = (\lambda+1)E + (\lambda I - A)(I-E). \tag{5.3}$$

If $\lambda \notin \sigma(A) \cup \{-1\}$, then $\lambda I - C$ is nonsingular by Theorem 2.2, and

$$\sigma(C) \subset \sigma(A) \cup \{-1\}. \tag{5.4}$$

This shows that $\mathrm{Re}\,\sigma(C) \leqslant 0$.

For any $k = 1, \dots, s$, (5.3) gives

$$\mu_k I - C = (\mu_k + 1)E + (\mu_k I - A + E_k)(I-E);$$

since $\mu_k I - A + E_k$ is nonsingular by Theorem 2.3, $\mu_k I - C$ is nonsingular by Theorem 2.2. Hence $\mathrm{Re}\,\sigma(C) < 0$, and (5.1), (5.2) are proved.

Conversely, assume that (5.1), (5.2) hold. Anticipating the part of the next theorem based only on (5.1), (5.2), we write $E_{s+1} = I - E$ and observe that $(E_1, \dots, E_{s+1})$ is a resolution of I. Then

$$\exp(tA) = \sum_{k=1}^{s} e^{t\mu_k} E_k + \exp(tC)E_{s+1}$$

(see (5.5)), which implies that A is exponentially bounded.

Finally, we show that each E_k is the eigenprojection of A at μ_k. By (5.1), (5.2),

$$A - \mu_k I + E_k = \sum_{i \neq k}^{s} (\mu_i - \mu_k) E_i + E_k + (C - \mu_k I) E_{s+1}.$$

Consequently $A - \mu_k I + E_k$ is nonsingular by Theorem 2.2 (i), and the result follows by Theorem 2.3 (iii). □

The following example shows that (5.1) cannot be replaced by a 'natural' decomposition $A = \mu_1 E_1 + \ldots + \mu_s E_s + C_0$.

Example 5.3. Let

$$\begin{aligned} A &= \begin{bmatrix} \mathrm{i} & 0 & 0 \\ 0 & -3\mathrm{i} & 0 \\ 0 & 0 & -2 \end{bmatrix} = \mathrm{i} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - 3\mathrm{i} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \\ &= \mathrm{i} E_1 - 3\mathrm{i} E_2 + C_0. \end{aligned}$$

Then A is exponentially bounded, E_1, E_2 are the eigenprojections at i, -3i, respectively, but C_0 is *not* stable. In contrast,

$$\begin{aligned} A &= (\mathrm{i}+1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (-3\mathrm{i}+1) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \\ &= (\mathrm{i}+1) E_1 + (-3\mathrm{i}+1) E_2 + C \end{aligned}$$

yields C stable.

We can calculate functions of an exponentially bounded matrix A in terms of the spectral decomposition of Theorem 3.2. This is described in the following theorem.

Theorem 5.4. *Let A be an exponentially bounded matrix with the spectral decomposition given by (5.1) and (5.2). If f is a complex valued function holomorphic in an open neighbourhood Δ of the set $\sigma(A) \cup \{-1\}$, then*

$$f(A) = \sum_{k=1}^{s} f(\mu_k) E_k + f(C) E_{s+1} = \sum_{k=1}^{s} (f(\mu_k) - f(-1)) E_k + f(C), \tag{5.5}$$

where $E_{s+1} = I - (E_1 + \ldots + E_s)$.

P r o o f. Note that $(E_1, \ldots, E_{s+1})$ is a resolution of I, and $CE_{s+1} = C + I - E_{s+1}$. Since

$$A = \sum_{k=1}^{s} (\mu_k + 1)E_k + C = \sum_{k=1}^{s} \mu_k E_k + CE_{s+1},$$

we apply Theorem 2.2 (ii) and (5.4) to conclude that $\sigma(\mu_1 I) \cup \ldots \cup \sigma(\mu_s I) \cup \sigma(C) \subset \sigma(A) \cup \{-1\}$. The first equality in (5.5) follows from Theorem 2.2 (iii).

Set $E = E_1 + \ldots + E_s$. If $\lambda \notin \sigma(A) \cup \{-1\}$, then $(\lambda I - C)E = \lambda E - CE = (\lambda + 1)E$, and $(\lambda I - C)^{-1}E = (\lambda + 1)^{-1}E$. If γ is a cycle that surrounds $\sigma(A) \cup \{-1\}$ in Δ, then

$$f(C)E = \frac{1}{2\pi \mathrm{i}} \int_\gamma f(\lambda)(\lambda I - C)^{-1}E \,\mathrm{d}\lambda = \frac{1}{2\pi \mathrm{i}} \int_\gamma f(\lambda)(\lambda + 1)^{-1}E \,\mathrm{d}\lambda = f(-1)E.$$

From this we deduce the second half of (5.5). □

We consider an application to a continuous system evolution [6] governed by the differential equation

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \tag{5.6}$$

where x_0 is an initial vector.

Theorem 5.5. *Let A be an exponentially bounded matrix with the spectral decomposition given by* (5.1) *and* (5.2). *Then the solution $x(t)$ to the continuous system evolution* (5.6) *is given by*

$$x(t) = \exp(tA)x_0 = \mathrm{e}^{t\mu_1}E_1x_0 + \ldots + \mathrm{e}^{t\mu_s}E_sx_0 + \exp(tC)(I - E)x_0, \tag{5.7}$$

where $E = E_1 + \ldots + E_s$.

P r o o f. Follows from Theorems 5.2 and 5.4. □

We note that, for any sufficiently large n,

$$x(t) \approx \mathrm{e}^{t\mu_1}E_1x_0 + \ldots + \mathrm{e}^{t\mu_s}E_sx_0, \tag{5.8}$$

where $\operatorname{Re}\mu_k = 0$. For any x_0 in the range of the projection E, (5.8) holds with equality for all n.

If the limit $\lim_{t\to\infty} \exp(tA)$ exists, then A is exponentially bounded, and we can apply Theorem 5.2. In particular, we have the following result.

Theorem 6.1. *A matrix is exponentially convergent if and only if*

$$A = E + C, \tag{6.1}$$

where

$$E^2 = E, \quad CE = EC = -E, \quad C \quad \text{is stable.} \tag{6.2}$$

The matrix $\lim_{t\to\infty} \exp(tA) = E$ *is the eigenprojection of* A *at* 0.

We say that a matrix A is *semistable* if $\operatorname{Re}\sigma(A) \leqslant 0$, and in the case that the spectrum meets the imaginary axis, the intersection is $\{0\}$, with 0 an eigenvalue of index 1.

Corollary 6.2. [6] *A matrix A is exponentially convergent if and only if it is semistable.*

If a matrix C is stable, then $\exp(tC) \to 0$ as $t \to \infty$ (Lemma 1.1). Hence

$$C \int_0^\infty \exp(tC)\,\mathrm{d}t = \int_0^\infty \left(\frac{\mathrm{d}}{\mathrm{d}t}\exp(tC)\right)\,\mathrm{d}t = -I,$$

and we obtain a representation for the inverse of C,

$$C^{-1} = -\int_0^\infty \exp(tC)\,\mathrm{d}t. \tag{6.3}$$

As an application of the spectral decomposition we derive a representation for the Drazin inverse A^D of a semistable matrix A; this representation is instrumental in the investigation of the asymptotic behaviour of solutions to differential equations presented in the next section.

Theorem 6.3. *If a matrix A is exponentially convergent with* $\lim_{t\to\infty} \exp(tA) = E$, *then*

$$A^D = -\int_0^\infty \exp(tA)(I - E)\,\mathrm{d}t. \tag{6.4}$$

P r o o f. Let $A = C + E$ be the decomposition of Theorem 6.1. By (5.5),

$$\exp(tA)(I - E) = \exp(tC)(I - E).$$

Then

$$\begin{aligned}\int_0^\infty \exp(tA)(I - E)\,\mathrm{d}t &= \left(\int_0^\infty \exp(tC)\,\mathrm{d}t\right)(I - E) = -C^{-1}(I - E)\\ &= -(A - E)^{-1}(I - E) = -A^D\end{aligned}$$

by (6.3) and (2.3). □

R e m a r k 6.4. Note that, for A semistable, the integral $\int_0^\infty \exp(tA)(I - E)\,\mathrm{d}t$ exists, but $\int_0^\infty \exp(tA)\,\mathrm{d}t$ may not.

7. Applications to differential equations

In this section we give applications of the preceding results to the asymptotic behaviour of the solutions of differential equations. For a systematic treatment of singular and singularly perturbed differential equations see Campbell [1]. As before, $x \mapsto \|x\|$ is a vector norm on $\mathbb{C}^p$, and $A \mapsto \|A\|$ a matrix norm on $\mathbb{C}^{p\times p}$ consistent with the selected vector norm (in the sense of Householder).

The following well known estimate may be obtained by expressing the matrix exponential as a contour integral: If $\operatorname{Re}\sigma(A) < 0$, then there is a positively oriented Jordan contour ω in the left half plane containing $\sigma(A)$ in its interior. Then $\operatorname{Re}\lambda \leqslant -\mu$ for some $\mu > 0$ and all $\lambda \in \omega$, and

$$\begin{aligned}\|\exp(tA)\| &= \left\|\frac{1}{2\pi\mathrm{i}}\int_\omega \mathrm{e}^{t\lambda}(\lambda I - A)^{-1}\,\mathrm{d}\lambda\right\|\\ &\leqslant \sup_{\lambda\in\omega}\left\|(\lambda I - A)^{-1}\right\|\frac{\ell(\omega)}{2\pi}\sup_{\lambda\in\omega}\mathrm{e}^{t\operatorname{Re}\lambda} \leqslant M\mathrm{e}^{-\mu t}.\end{aligned}$$

This yields the following.

Lemma 7.1. *If $C \in \mathbb{C}^{p\times p}$ is stable, then there are constants $M > 0$, $\mu > 0$ such that*

$$\|\exp(tC)\| \leqslant M\mathrm{e}^{-\mu t} \text{ for all } t \geqslant 0. \tag{7.1}$$

For semistable matrices we have the following result.

Lemma 7.2. *If $A \in \mathbb{C}^{p\times p}$ is semistable and E is the eigenprojection of A at 0, then there are constants $N > 0$, $\nu > 0$ such that*

$$\|\exp(tA) - E\| \leqslant N\mathrm{e}^{-\nu t} \quad \textit{for all } t \geqslant 0. \tag{7.2}$$

P r o o f. For A semistable, (5.5) reduces to

$$f(A) = f(0)E + f(C)(I - E).$$

In particular, $\exp(tA) = E + \exp(tC)(I - E)$, and the result follows from (7.1). For future reference we also note that

$$\exp(tA)E = E \text{ for all } t \geqslant 0. \tag{7.3}$$

□

Theorem 7.3. *Let A be a semistable matrix with the eigenprojection E at 0. Let a function $f\colon [0,\infty) \to \mathbb{C}^p$ be bounded, Lebesgue measurable, let $Ef(t)$ be Lebesgue integrable, and let $\lim\limits_{t\to\infty} f(t) = w$ exist. Then the solution $u(t)$ of the differential problem*

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au(t) + f(t), \quad u(0) = x,$$

satisfies

$$\lim_{t\to\infty} u(t) = Ex - A^D w + \int_0^\infty Ef(s)\,\mathrm{d}s.$$

P r o o f. The solution to the differential problem is given by

$$u(t) = \exp(tA)x + \int_0^t \exp((t-s)A)f(s)\,\mathrm{d}s.$$

The second integral can be transformed to

$$\begin{aligned}\int_0^t \exp((t-s)A)f(s)\,\mathrm{d}s &= \int_0^t \exp((t-s)A)Ef(s)\,\mathrm{d}s + \int_0^t \exp((t-s)A)(I-E)f(s)\,\mathrm{d}s\\ &= \int_0^t Ef(s)\,\mathrm{d}s + \int_0^t \exp((t-s)A)(I-E)f(s)\,\mathrm{d}s\\ &= I_1(t) + I_2(t)\end{aligned}$$

using (7.3). The integral $I_1(t)$ converges to $\int_0^\infty Ef(s)\,\mathrm{d}s$ as $t\to\infty$, while $I_2(t)$ can be expressed as

$$I_2(t) = \int_0^t \left[\exp((t-s)A) - E\right](f(s) - w)\,\mathrm{d}s + \int_0^t \exp(\tau A)(I-E)w\,\mathrm{d}\tau$$

using (7.3) and a substitution. Since

$$\|\exp((t-s)A)-E\| \leqslant N\mathrm{e}^{-\nu(t-s)} \text{ for all } t \geqslant 0$$

by (7.2), the first summand, $J(t)$, in $I_2(t)$ will converge to 0 as $t \to \infty$: Let $\eta > 0$ and choose $T > 0$ so that $\|f(s)-w\| < \frac{1}{2}\eta\nu/M$ for all $s \geqslant T$. Then, for all $t \geqslant T$,

$$\begin{aligned}
\|J(t)\| &\leqslant \int_0^t N\mathrm{e}^{-\nu(t-s)}\|f(s)-w\|\,\mathrm{d}s \\
&= \int_0^T N\mathrm{e}^{-\nu(t-s)}2\|f\|_\infty\,\mathrm{d}s + \int_T^t N\mathrm{e}^{-\nu(t-s)}\|f(s)-w\|\,\mathrm{d}s \\
&\leqslant 2N\|f\|_\infty\nu^{-1}\mathrm{e}^{-\nu(t-T)} + \tfrac{1}{2}\eta \text{ for all } t \geqslant T.
\end{aligned}$$

Choosing t sufficiently large to make the first term less than $\eta/2$, we get $\|J(t)\| < \eta$ for all sufficiently large t. Consequently $\lim\limits_{t\to\infty} J(t) = 0$ as required.

Combining all preceding results we get

$$\begin{aligned}
\lim_{t\to\infty} u(t) &= \lim_{t\to\infty} \exp(tA) + \lim_{t\to\infty} I_1(t) + \lim_{t\to\infty} I_2(t) \\
&= Ex + \int_0^\infty Ef(s)\,\mathrm{d}s + \int_0^\infty \exp(\tau A)(I-E)w\,\mathrm{d}\tau \\
&= Ex + \int_0^\infty Ef(s)\,\mathrm{d}s - A^D w
\end{aligned}$$

by (6.4). □

Next we turn our attention to a singularly perturbed differential equation involving a positive parameter ε, and investigate when the solution has a limit as $\varepsilon \to 0+$.

Theorem 7.4. *Let A be a semistable matrix with the eigenprojection E, and let a function $f\colon [0,\infty) \to \mathbb{C}^p$ be bounded and continuous. Then the solution $u_\varepsilon(t)$ of the singularly perturbed problem*

$$\varepsilon\frac{\mathrm{d}u_\varepsilon(t)}{\mathrm{d}t} = Au_\varepsilon(t) + f(t), \quad u_\varepsilon(0) = x, \quad \varepsilon > 0, \tag{7.4}$$

has a limit $u(t)$ as $\varepsilon \to 0+$ if and only if $Ef(t) = 0$ for all $t \geqslant 0$. In this case

$$u(t) = Ex - A^D f(t). \tag{7.5}$$

P r o o f. Problem (7.4) can be rewritten as

$$\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}t} = A_\varepsilon u_\varepsilon(t) + f_\varepsilon(t), \qquad u_\varepsilon(0) = x, \tag{7.6}$$

where $A_\varepsilon = ee^{-1}A$ and $f_\varepsilon(t) = ee^{-1}f(t)$. The solution to (7.6) (and to (7.4)) is given by

$$u_\varepsilon(t) = \exp(tA_\varepsilon)x + \int_0^t \exp((t-s)A_\varepsilon)f_\varepsilon(s)\,\mathrm{d}s. \tag{7.7}$$

We pick $t > 0$ and keep it fixed throughout the proof. Since $\lim_{\tau\to\infty} \exp(\tau A) = E$, we have $\lim_{\varepsilon\to 0+} \exp(tA_\varepsilon)x = \lim_{\varepsilon\to 0+} \exp(ee^{-1}tA)x = Ex$. The second integral in (7.7) can be expressed as the sum

$$\varepsilon^{-1}\int_0^t Ef(s)\,\mathrm{d}s + \varepsilon^{-1}\int_0^t \exp((t-s)A_\varepsilon)(I-E)f(s)\,\mathrm{d}s. \tag{7.8}$$

Then $\lim_{\varepsilon\to 0+} \varepsilon^{-1}\int_0^t Ef(s)\,\mathrm{d}s$ exists if and only if $\int_0^t Ef(s)\,\mathrm{d}s = 0$; this is true for all $t \geqslant 0$ if and only if $Ef(t) =$ for all $t \geqslant 0$. The second integral in (7.8) can be written as the sum $J(\varepsilon,t) + K(\varepsilon,t)$, where

$$\begin{aligned}
J(\varepsilon,t) &= \varepsilon^{-1}\int_0^t \exp((t-s)\varepsilon^{-1}A)(I-E)f(t)\,\mathrm{d}s,\\
K(\varepsilon,t) &= \varepsilon^{-1}\int_0^t \big[\exp((t-s)\varepsilon^{-1}A) - E\big]\,(f(s)-f(t))\,\mathrm{d}s.
\end{aligned}$$

Then

$$\begin{aligned}
\lim_{\varepsilon\to 0+} J(\varepsilon,t) &= \lim_{\varepsilon\to 0+}\int_0^{t/\varepsilon} \exp(\sigma A)(I-E)f(t)\,\mathrm{d}\sigma\\
&= \int_0^\infty \exp(\sigma A)(I-E)f(t)\,\mathrm{d}\sigma = -A^D f(t)
\end{aligned}$$

by (6.4). As is the proof of the preceding theorem we split the domain of integration and estimate the integrand using the inequality $\|\exp(tA_\varepsilon) - E)\| \leqslant N\mathrm{e}^{-\nu t/\varepsilon}$ for all $t \geqslant 0$:

$$\begin{aligned}
\|K(\varepsilon,t)\| &\leqslant \varepsilon^{-1}\int_0^t N\mathrm{e}^{-\nu(t-s)/\varepsilon}\|f(s)-f(t)\|\,\mathrm{d}s\\
&= \varepsilon^{-1}\int_0^T N\mathrm{e}^{-\nu(s-t)/\varepsilon}2\|f\|_\infty\,\mathrm{d}s + \varepsilon^{-1}\int_T^t N\mathrm{e}^{-\nu(s-t)/\varepsilon}\|f(s)-f(t)\|\,\mathrm{d}s\\
&\leqslant 2N\nu^{-1}\|f\|_\infty \mathrm{e}^{-\nu(t-T)/\varepsilon} + N\nu^{-1}\sup_{T\leqslant s\leqslant t}\|f(s)-f(t)\|.
\end{aligned}$$

Given $\eta > 0$, choose first T close enough to t so that the second term is less that $\frac{1}{2}\eta$ by the continuity of f, and then ε small enough so that the first term is less than

$\frac{1}{2}\eta$. Then $\lim\limits_{\varepsilon\to 0+} K(\varepsilon,t)=0$ (for any $t\geqslant 0$). Combining all the preceding results, we conclude that

$$u(t)=\lim_{\varepsilon\to 0+} u_\varepsilon(t)=Ex-A^D f(t) \text{ for all } t\geqslant 0$$

if and only if $Ef(t)=0$ for all $t\geqslant 0$. $\square$

R e m a r k 7.5. Since $AE=0$ for a semistable matrix A, we can easily verify that the limit $u(t)=\lim\limits_{\varepsilon\to 0+} u_\varepsilon(t)$ satisfies

$$Au(t)+f(t)=0, \qquad u(0)=Ex-A^D f(0).$$

E x a m p l e 7.6. (*system of conservation laws with a moving source*) It is well known that systems of conservation laws have broad applications in fluid mechanics and other applied areas. Assume that we are given a system of conservation laws with a moving source

$$u_t+F(u)_x=g(ct-x), \quad x\leqslant 0, \quad t>0, \tag{7.9}$$

where $u(x,t)=(u_1(x,t),\dots,u_n(x,t))$, is to be found, $F(u)=(F_1(u),\dots,F_n(u))$ is a given smooth vector function and c is the velocity of the source. As pointed out in [4] and the references given there, this problem is of physical interest (gas flow through a nozzle, MHD shock tube, etc.). Given a constant state $v_0\in\mathbb{R}^n$ we can consider a localized linear version of (7.9)

$$u_t+F'(v_0)u_x=g(ct-x), \tag{7.10}$$

where $A:=F'(v_0)$ is the Jacobian of F at u_0. It is natural to consider travelling waves for (7.10) with the speed c, that is, $u(x,t)$ in the form

$$u(x,t):=u(ct-x). \tag{7.11}$$

Also, let us make the following assumption about the matrix A:

$$\begin{aligned}&c \text{ is a simple pole of } A,\\ &\text{each eigenvalue } \lambda\neq c \text{ of } A \text{ satisfies } \operatorname{Re}\lambda>c\end{aligned} \tag{7.12}$$

(note that the resonance case is included).

In addition to (7.10) we investigate the system

$$u_t+Au_x-\varepsilon u_{xx}=g(ct-x), \quad t>0, \quad x\leqslant 0 \tag{7.13}$$

with artificial viscous term $-\varepsilon u_{xx}$, where ε is a positive constant. Viscous profiles are of special interest in the theory of conservation laws as regular approximations of discontinuous solutions. Inserting (7.11) into (7.13) we obtain

$$\varepsilon u_\varepsilon'' + (A - cI)u_\varepsilon' = -g(y), \tag{7.14}$$

where v' means the derivative of v with respect to $y = ct - x$; observe that $y \geqslant 0$. We assume that the following conditions on $g\colon [0,\infty) \to \mathbb{R}^n$ are satisfied:

$$Ag(y) = cg(y) \text{ for all } y \geqslant 0, \text{ and} \tag{7.15}$$

$$\text{the function } y \mapsto \int_0^y g(\sigma)\,\mathrm{d}\sigma \text{ is bounded.} \tag{7.16}$$

Consider for (7.14) profiles satisfying the boundary conditions

$$u_\varepsilon(0) = u_0, \quad u_\varepsilon'(0) = 0 \tag{7.17}$$

with a given constant vector u_0. Write $B = cI - A$ for brevity. Integration of (7.14) with regard to (7.17) yields

$$\varepsilon u_\varepsilon' = Bu_\varepsilon - Bu_0 - \int_0^y g(\sigma)\,\mathrm{d}\sigma. \tag{7.18}$$

Put $f(y) := -Bu_0 - \int_0^y g(\sigma)\,\mathrm{d}\sigma$. Then f is a continuous and bounded function on $[0,\infty)$. Equation (7.18) then becomes

$$\varepsilon u_\varepsilon' = Bu_\varepsilon + f(y), \quad u_\varepsilon(0) = u_0. \tag{7.19}$$

The assumption (7.12) on A ensures that B is semistable. Let E be the eigenprojection of B. Then

$$Ef(y) = -EBu_0 - \int_0^y Eg(\sigma)\,\mathrm{d}\sigma = 0; \tag{7.20}$$

this is true since $Bu_0 \in R(B)$ and $R(B) = N(E)$, and since (7.15) implies that $g(\sigma) \in N(B)$ for all $\sigma \geqslant 0$. Equation (7.20) together with (7.16) guarantees that the conditions of Theorem 7.4 are satisfied, and that

$$\lim_{\varepsilon\to 0+} u_\varepsilon(y) = Eu_0 + B^D Bu_0 + \int_0^y B^D g(\sigma)\,\mathrm{d}\sigma = u_0 + \int_0^y (cI - A)^D g(\sigma)\,\mathrm{d}\sigma$$

since $E = I - B^D B$.

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