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COUPLED STRING-BEAM EQUATIONS AS A MODEL OF SUSPENSION BRIDGES

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Abstract. We consider nonlinearly coupled string-beam equations modelling time-periodic oscillations in suspension bridges. We prove the existence of a unique solution under suitable assumptions on certain parameters of the bridge.

Keywords: Nonlinearly coupled string-beam equation, periodic oscillations, jumping nonlinearities, degree theory

MSC 2000: 35B10, 70K30, 73K03, 73K05

1. Introduction

In our previous paper [BDL] we studied time-periodic oscillations in suspension bridges and we proved the existence of a unique solution 'near equilibrium.' The bridge was considered as a vibrating beam, supported from above by cables behaving as nonlinear springs. The underlying mathematical model was the one-dimensional beam equation with time-periodic boundary conditions describing the periodic motion of the roadbed subject to periodic perturbations (refered to as a one-dimensional model).

In the present paper we try to explain the same phenomenon, but using now a more accurate model. Indeed, we no more consider the mechanical construction holding the cable stays as an immovable object, but we treat it as a vibrating string, coupled with the beam of the roadbed by nonlinear cable stays (see Figure 1) (refered to as

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a two-dimensional model).

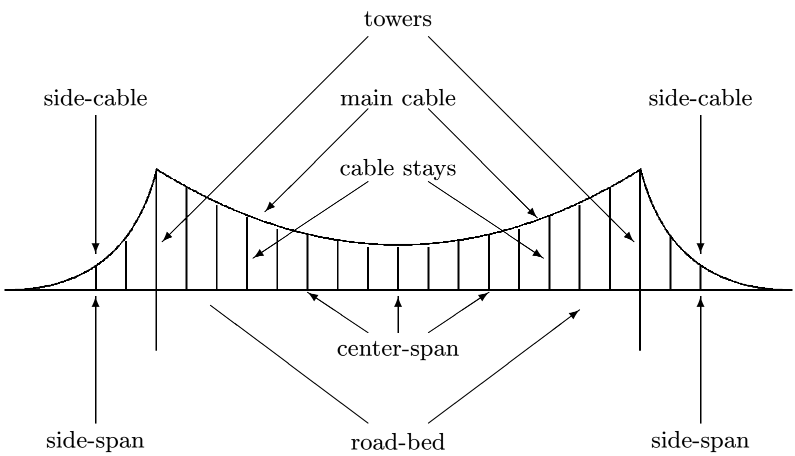

Figure 1. The model of suspension bridge.

This more accurate model of a suspension bridge can be described mathematically by the following boundary value problem for a system of 'string-beam' equations:

$$
\text{(SB)} \qquad \begin{cases} m_1 v_{tt} - T v_{xx} + b_1 v_t - \kappa(u-v)^+ = W_1(x) + f_1(x,t), \\ m_2 u_{tt} + EI u_{xxxx} + b_2 u_t + \kappa(u-v)^+ = W_2(x) + f_2(x,t), \\ u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \\ v(0,t) = v(L,t) = 0, \\ u(x,t+\tau) = u(x,t), \quad v(x,t+\tau) = v(x,t), \\ x \in \,]0,L[\,, t \in \mathbb{R}. \end{cases}
$$

Here $v(x,t)$ measures the displacement of the vibrating string representing the main cable and $u(x,t)$ means the displacement of the bending beam standing for the roadbed of the bridge. Both functions are considered to be τ-periodic with respect to the time variable t. The nonlinear stays connecting the beam and the string pull the main cable down, hence we have the minus sign in front of $\kappa(u-v)^+$ in the string equation, and hold the roadbed up leading to the plus sign in front of the same term in the beam equation. The meaning of the constants and functions used in (SB) is as follows:

m_1, m_2	masses per unit length of the main cable and the roadbed, respectively,
b_1, b_2	damping coefficients,
κ	stiffness of the cable stays,
T	tension of the main cable,
$W_1(x)$, $W_2(x)$	weight per unit length of the main cable and the roadbed, respectively,
$f_1(x,t)$, $f_2(x,t)$	external, in time τ-periodic, forcing terms,
L	length of the center span of the bridge,
E	Young's modulus,
I	moment of inertia of the cross section of the roadbed.

We would like to point out that the model just mentioned was introduced first in the work of Lazer and McKenna [LK_4] but has been studied under rather restrictive assumptions. As far as we know system (SB) was treated in its full generality for the first time in Tajčová [T], where the existence of a unique solution was proved by using the Banach contraction principle. The disadvantage of this powerful and general principle consists in the fact that its application requires a rather restrictive assumption on the parameters κ, m_i, b_i, E, I and T. In the present paper we focus on unique solvability of (SB), too. However, using a completely different approach than that in [T], we prove the existence of a unique time-periodic solution near stationary equilibrium under rather general assumptions on the above mentioned parameters, provided the external time-periodic forcing terms are small in a certain sense. As a consequence, our result and that of [T] provide rather general sufficient conditions for unique solvability of (SB).

Let us point out that a lot of papers have been devoted to the study of one-dimensional models of suspension bridges. See, e.g., Alonso, Ortega [AO], Berkovits et al. [BDL], Choi Q., Choi K. and Jung [CCJ], Drábek [D_1, D_2], Fonda, Schneider and Zanolin [FSZ], Glover, Lazer and McKenna [GLK], Lazer and McKenna [LK_{1-5}], McKenna and Walter [KW]. On the other hand, more complex models are rather rare in literature and the present paper should be understood as a contribution to this problem. Of course, in spite of its relative complexity problem (SB) does not describe the complete behaviour of a suspension bridge. Several partly restrictive simplifications are still made: the motions of the towers as well as the influence of the side spans are ignored, the torsional oscillations of the roadbed are neglected, no pretension of the roadbed is considered, the main cable is modelled by the straight string instead of a loaded catenary.

It is convenient to rescale u, v, x and t and write problem (SB) in an equivalent form as

$$(\mathscr{S}\mathscr{B}) \qquad \begin{cases} v_{tt} - \alpha_1^2 v_{xx} + \beta_1 v_t - k_1(u-v)^+ = h_1(x,t), \\ u_{tt} + \alpha_2^2 u_{xxxx} + \beta_2 u_t + k_2(u-v)^+ = h_2(x,t), \\ u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0, \\ v(0,t) = v(\pi,t) = 0, \\ u(x,t+2\pi) = u(x,t), \quad v(x,t+2\pi) = v(x,t), \\ x \in \,]0,\pi[\,, \ t \in \mathbb{R} \end{cases}$$

with

$$h_i(x,t) = W_i(x) + f_i(x,t) \quad (i = 1,2)$$

and

$$\text{(1.1)} \qquad \alpha_1^2 = \frac{T\tau^2}{4m_1L^2}; \quad \alpha_2^2 = \frac{\pi^2 EI\tau^2}{4m_2L^4}; \qquad \beta_i = \frac{b_i\tau}{2\pi m_i}; \quad k_i = \frac{\kappa\tau^2}{4\pi^2 m_i} \quad (i = 1,2).$$

Notice that we write again W_i, f_i, h_i for rescaled $\widetilde{W}_i$, $\widetilde{f}_i$, $\widetilde{h}_i$.

We tacitly assume that $h_i\colon \,]0,\pi[\, \times \,]0,2\pi[\, \to \mathbb{R}$ $(i = 1,2)$ is square integrable with respect to the Lebesgue measure. A couple (v,u), where $v,u\colon \,]0,\pi[\, \times \,]0,2\pi[\, \to \mathbb{R}$ are square integrable, is called a weak solution of $(\mathscr{S}\mathscr{B})$ if and only if the integral identities

$$\text{(1.2)} \qquad \int_0^{2\pi}\!\!\int_0^{\pi} v(\varphi_{tt} - \alpha_1^2\varphi_{xx} - \beta_1\varphi_t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^{2\pi}\!\!\int_0^{\pi} (h_1 + k_1(u-v)^+)\varphi\,\mathrm{d}x\,\mathrm{d}t,$$

$$\text{(1.3)} \qquad \int_0^{2\pi}\!\!\int_0^{\pi} u(\psi_{tt} + \alpha_2^2\psi_{xxxx} - \beta_2\psi_t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^{2\pi}\!\!\int_0^{\pi} (h_2 - k_2(u-v)^+)\psi\,\mathrm{d}x\,\mathrm{d}t$$

hold for all $\varphi,\psi \in C^\infty(\mathbb{R}^2;\mathbb{R})$ being odd functions in x and 2π-periodic in x and t. For $p,r \in \mathbb{N}\cup\{0\}$ define $\widehat{H}^{p,r}$ to be the space of all distributions $u \in \mathscr{D}'(\mathbb{R}^2)$ being odd in x and 2π-periodic in x and t such that the distributional derivatives $\partial_x^\alpha u$ and $\partial_t^\beta u$ belong to $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ for all $\alpha,\beta \in \mathbb{N}\cup\{0\}$ satisfying $0 \leqslant \alpha \leqslant p$ and $0 \leqslant \beta \leqslant r$. The space $\widehat{H}^{p,r}$ equipped with the norm

$$\|u\|_{p,r} = \Bigl(\sum_{0\leqslant\alpha\leqslant p}\int_Q |\partial_x^\alpha u|^2 + \sum_{0\leqslant\beta\leqslant r}\int_Q |\partial_t^\beta u|^2\Bigr)^{1/2},$$

where $Q = \,]0,\pi[\, \times \,]0,2\pi[$, is a standard anisotropic Sobolev space (see, e.g., [V]) of 2π-periodic functions in x and t that are in addition odd in x. Notice that any square integrable function $u\colon Q \to \mathbb{R}$ can be extended in a unique way to a 2π-periodic (in x and t) and odd (in x) function $\widehat{u} \in \widehat{H}^{0,0}$. Finally, we call a couple $(v,u) \in C^1(\overline{Q}) \times C^1(\overline{Q})$ *strictly positive* provided $u - v > 0$ on $]0,\pi[\times [0,2\pi]$ and $\partial_x(u-v)(0,t) > 0$ as well as $\partial_x(u-v)(\pi,t) < 0$ for all $t \in [0,2\pi]$.

Using this definition of strict positivity we are now able to formulate the basic conclusions of our paper.

Result 1. *Problem ($\mathscr{SB}$) has at least one weak solution $(\widehat{v},\widehat{u}) \in \widehat{H}^{1,1} \times \widehat{H}^{2,1}$ for any right hand side $\mathbf{h} = (h_1,h_2) \in L^2(Q) \times L^2(Q)$.*

Result 2. *Suppose problem ($\mathscr{SB}$) with $f_1 \equiv 0 \equiv f_2$ admits a strictly positive weak solution (v_0,u_0). Then there exists an $\varepsilon > 0$ such that for any $\widehat{f}_1, \widehat{f}_2 \in \widehat{H}^{2,2}$ satisfying $\|\widehat{f}_i\|_{2,2} < \varepsilon$ $(i = 1,2)$ problem ($\mathscr{SB}$) has a unique weak solution (v,u). Moreover, $(\widehat{v},\widehat{u}) \in \widehat{H}^{2,2} \times \widehat{H}^{4,2}$, $(\widehat{v},\widehat{u})$ is strictly positive and close to $(\widehat{v}_0,\widehat{u}_0)$ in the norm of $\widehat{H}^{3,3} \times \widehat{H}^{3,3}$.*

Result 3. *Suppose $W_1(x) \equiv W_1$ and $W_2(x) \equiv W_2$ are positive constants in problem* (SB) *and assume $\frac{W_1}{W_2} \ll 1$ (for detailed statements see Theorem* 4.2*). Then* (SB) *with $f_1 \equiv 0 \equiv f_2$ admits a strictly positive weak solution and thus Result* 2 *applies.*

For detailed and more general statements of these results we refer the reader to Theorems 3.1, 4.1 and 4.2. Let us point out that a very important part of our paper is devoted to the investigation of the time-independent case of problem ($\mathscr{SB}$), i.e., the case $f_1 \equiv 0 \equiv f_2$, and to the search for sufficient conditions which guarantee the existence of a strictly positive solution (v_0,u_0).

The present paper should be understood as a generalization of our contribution [BDL] where one-dimensional models were studied. Though the character of our results presented here is similar to that of [BDL] we would like to mention that the two-dimensional model now considered in this paper is more complex and may be regarded as a better approximation of the real behaviour of a suspension bridge.

Our paper is organized as follows. In Section 2 we introduce the underlying function spaces and the differential operators S, T, W, B, $\mathbb{W}$, $\mathbb{B}$ and $\mathbf{L}$. We recall basic facts of spectral theory concerning these operators and we collect less known properties of the beam operator $\mathbb{B}$ and the wave operator $\mathbb{W}$ and of the underlying function spaces. We introduce the precise setting of the system ($\mathscr{SB}'$) connecting it with the differential operator $\mathbf{L}$. In Section 3 we study in detail time-independent solutions and prove uniqueness and stability of weak solutions in that case. A similar result holds for time-dependent solutions of the linearly coupled string-beam equation. Moreover, we prove the existence of at least one weak solution of ($\mathscr{SB}'$) for rather general nonlinear coupling functions $g(\xi)$ satisfying a growth condition $(\mathscr{G})$. In the final section we will present existence and uniqueness of time-dependent solutions of $(\mathscr{S}\mathscr{B}')$ as far as the corresponding time-independent system (i.e., $f_1 \equiv 0 \equiv f_2$) admits a strictly positive solution. This result is then applied to a suspension bridge modelled by (SB). The paper ends with a technical two-piece appendix. Appendix A presents those very special regularity results concerning the damped wave equation that are not covered by the standard L^2-regularity. Finally, Appendix B provides a criterion on $(\mathscr{S}\mathscr{B})$ which guarantees the existence of a strictly positive solution of $(\mathscr{S}\mathscr{B})$.

2. Functional setting of the problem

We start by giving the precise setting of the differential operators related to problem $(\mathscr{S}\mathscr{B})$. Denote $Q = \,]0,\pi[\, \times \,]0,2\pi[$ and let $H = L^2(Q) = L^2(Q;\mathbb{R})$ be the real Hilbert space of square integrable functions $u\colon Q \to \mathbb{R}$. The complexification $H_{\mathbb{C}} = H + \mathrm{i}H = L^2(Q,\mathbb{C})$ of H is equipped with the scalar product

$$\langle u, v\rangle = \int_Q u\bar{v} \qquad (u, v \in H_{\mathbb{C}})$$

and norm $\|u\| = \langle u, u\rangle^{1/2}$. Denoting $\mathbb{N} = \{1, 2, 3, \dots\}$ and

$$\varphi_{n,m}(x,t) = \frac{1}{\pi}\mathrm{e}^{\mathrm{i}nt}\sin mx \qquad (n \in \mathbb{Z}, m \in \mathbb{N}, (x,t) \in Q)$$

the family $\{\varphi_{n,m}\}_{n\in\mathbb{Z},m\in\mathbb{N}}$ forms an orthonormal basis in the Hilbert space $H_{\mathbb{C}}$. Each $u \in H_{\mathbb{C}}$ has a unique representation

$$u = \sum_{n\in\mathbb{Z},m\in\mathbb{N}} \langle u, \varphi_{n,m}\rangle\, \varphi_{n,m}.$$

Notice that $\overline{\langle u, \varphi_{n,m}\rangle} = \langle u, \varphi_{-n,m}\rangle$ if u belongs to the real Hilbert space H.

In the following text we define some operators and state their properties. For the notions like the maximal, selfadjoint and normal operator see e.g. the book of Weidmann [W].

For any $p \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}^+$ the abstract realizations T^p, W, B, $\mathbb{W}$, $\mathbb{B}$ of the operators ∂_t^p, $\partial_t^2 - \alpha^2\partial_x^2$, $\partial_t^2 + \alpha^2\partial_x^4$, $\partial_t^2 - \alpha^2\partial_x^2 + \beta\partial_t$, $\partial_t^2 + \alpha^2\partial_x^4 + \beta\partial_t$ are the *maximal* operators in $H_{\mathbb{C}}$ defined as follows:

$$T^p u = \sum_{n\in\mathbb{Z},m\in\mathbb{N}} (\mathrm{i}n)^p \langle u, \varphi_{n,m}\rangle \varphi_{n,m}, \tag{2.1}$$

$$Wu = \sum_{n\in\mathbb{Z},m\in\mathbb{N}} (\alpha^2 m^2 - n^2) \langle u, \varphi_{n,m}\rangle \varphi_{n,m}, \tag{2.2}$$

$$Bu = \sum_{n\in\mathbb{Z},m\in\mathbb{N}} (\alpha^2 m^4 - n^2) \langle u, \varphi_{n,m}\rangle \varphi_{n,m}, \tag{2.3}$$

$$\mathbb{W} = W + \beta T, \tag{2.4}$$

$$\mathbb{B} = B + \beta T. \tag{2.5}$$

Unfortunately the operator ∂_x has no equivalent to (2.1) since $\partial_x\varphi_{n,m} \notin H$. Thus we introduce the closely related operator S or more generally the powers S^p of S for $p \in \mathbb{N}$ by setting

$$S^p u = \sum_{n\in\mathbb{Z},m\in\mathbb{N}} m^p \langle u, \varphi_{n,m}\rangle \varphi_{n,m}. \tag{2.6}$$

Notice that $W = W(\alpha)$, $B = B(\alpha)$ depend on the parameter $\alpha \in \mathbb{R}^+$ and $\mathbb{W} = \mathbb{W}(\alpha, \beta)$, $\mathbb{B} = \mathbb{B}(\alpha, \beta)$ on the parameters $\alpha, \beta \in \mathbb{R}^+$. For any $n \in \mathbb{Z}$, $m, p \in \mathbb{N}$, using the abbreviations

$$\gamma_{n,m} = \alpha^2 m^2 - n^2 \quad \text{and} \quad \nu_{n,m} = \gamma_{n,m} + \mathrm{i}\beta n, \tag{2.7}$$

$$\lambda_{n,m} = \alpha^2 m^4 - n^2 \quad \text{and} \quad \mu_{n,m} = \lambda_{n,m} + \mathrm{i}\beta n, \tag{2.8}$$

we have

$$S^p\varphi_{n,m} = m^p\varphi_{n,m}; \quad T^p\varphi_{n,m} = (\mathrm{i}n)^p\varphi_{n,m}; \tag{2.9}$$

$$W\varphi_{n,m} = \gamma_{n,m}\varphi_{n,m}; \quad \mathbb{W}\varphi_{n,m} = \nu_{n,m}\varphi_{n,m}; \tag{2.10}$$

$$B\varphi_{n,m} = \lambda_{n,m}\varphi_{n,m}; \quad \mathbb{B}\varphi_{n,m} = \mu_{n,m}\varphi_{n,m}. \tag{2.11}$$

Let A denote any one of the operators S^p, T^p, W, $\mathbb{W}$, B, $\mathbb{B}$. Then A is a real operator in the sense that $u \in \mathscr{D}(A) := \mathscr{D}_{\mathbb{C}}(A)\cap H$ implies $Au \in H$. Notice that we do not distinguish in notation $A\colon \mathscr{D}_{\mathbb{C}}(A) \to H_{\mathbb{C}}$ from its real part $A|_{\mathscr{D}(A)}\colon \mathscr{D}(A) \to H$. In each particular situation it will be clear which operator we actually mean. For any closed operator $A\colon \mathscr{D}_{\mathbb{C}}(A) \to H_{\mathbb{C}}$ we denote by $\sigma(A)$ the spectrum of A and by $\sigma_p(A)$ the point spectrum of A, i.e., the set of all eigenvalues of A.

We now collect basic properties of the operators S^q, T^q, W, B, $\mathbb{W}$ and $\mathbb{B}$ for any $q \in \mathbb{N}$.

Lemma 2.1.

(i) S^q, W, B *are selfadjoint and* T^q, $\mathbb{W}$, $\mathbb{B}$ *normal operators satisfying*

$$(2.12)\qquad (T^q)^* = (-1)^q T^q; \qquad \mathbb{W}^* = W - \beta T; \qquad \mathbb{B}^* = B - \beta T.$$

(ii) S^q *and* T^q *both commute with* $(\mathbb{W}-\lambda I)^{-1}$ *and* $(\mathbb{B}-\lambda I)^{-1}$ *for all* $\lambda \notin \sigma(\mathbb{W}) \cup \sigma(\mathbb{B})$.

(iii) *Concerning the spectra of* S^q, T^q, W, $\mathbb{W}$, B *and* $\mathbb{B}$ *the following formulas hold true:*

$$(2.13)\quad \sigma(S^q) = \sigma_p(S^q) = \{\lambda^q \mid \lambda \in \mathbb{N}\}; \quad \sigma(T^q) = \sigma_p(T^q) = \{\mathrm{i}^q\lambda^q \mid \lambda \in \mathbb{Z}\};$$

$$(2.14)\quad \sigma(W) = \overline{\sigma_p(W)}, \quad \sigma_p(W) = \{\gamma_{n,m} \mid n \in \mathbb{Z}, m \in \mathbb{N}\};$$
$$\sigma(\mathbb{W}) = \sigma_p(\mathbb{W}) = \{\nu_{n,m} \mid n \in \mathbb{Z}, m \in \mathbb{N}\};$$

$$(2.15)\quad \sigma(B) = \overline{\sigma_p(B)}, \quad \sigma_p(B) = \{\lambda_{n,m} \mid n \in \mathbb{Z}, m \in \mathbb{N}\};$$
$$\sigma(\mathbb{B}) = \sigma_p(\mathbb{B}) = \{\mu_{n,m} \mid n \in \mathbb{Z}, m \in \mathbb{N}\}.$$

P r o o f. Concerning the operators S^q, T^q, B and $\mathbb{B}$ see Lemma 2.1 of [BDL] and its proof. Moreover, a minor change in that proof shows the above results for W and $\mathbb{W}$, too. Thus we do not repeat those arguments. □

We now introduce the function spaces needed in this paper and collect the properties of these spaces used henceforth.

For $p, r \in \mathbb{N} \cup \{0\}$ we put

$$(2.16)\quad H^{p,r} = \mathscr{D}(S^p) \cap \mathscr{D}(T^r) = \left\{ u \in H \mid \sum_{n\in\mathbb{Z}, m\in\mathbb{N}} (m^{2p} + n^{2r}) |\langle u, \varphi_{n,m}\rangle|^2 < \infty \right\}$$

with the norm

$$(2.17)\quad \|u\|_{p,r} = \left(\|S^p u\|^2 + \|T^r u\|^2\right)^{1/2} = \left(\sum_{n\in\mathbb{Z}, m\in\mathbb{N}} (m^{2p} + n^{2r}) |\langle u, \varphi_{n,m}\rangle|^2 \right)^{1/2}.$$

It is not hard to see that $H^{p,r}$ is actually a Hilbert space with the norm $\|\cdot\|_{p,r}$. It is useful to interpret the elements $u \in H^{p,r}$ in a different way by extending them uniquely to real valued distributions $\widehat{u} \in \mathscr{D}'(\mathbb{R}^2) \cap L^2_{\mathrm{loc}}(\mathbb{R}^2)$ being odd in x and 2π-periodic in x and t. If we denote

$$(2.18)\qquad \widehat{H} = \left\{ v \in \mathscr{D}'(\mathbb{R}^2) \cap L^2_{\mathrm{loc}}(\mathbb{R}^2) \mid v \text{ odd in } x \text{ and } 2\pi\text{-periodic in } x \text{ and } t \right\}$$

with the norm $\|v\| = \left(\int_Q |v|^2\right)^{1/2}$, then the extension map $\widehat{\ }\colon H \to \widehat{H}$, $u \mapsto \widehat{u}$, generates a topological isomorphism.

To state the properties of the function spaces needed in this paper we have to introduce further notation. Denote as in the Introduction

$$\widehat{H}^{p,r} = \Big\{v \in \widehat{H} \mid \partial_x^p v, \partial_t^r v \in L^2_{\mathrm{loc}}(\mathbb{R}^2)\Big\} \tag{2.19}$$

with the norm $\|v\|_{p,r} = \Big(\sum\limits_{0\leqslant\alpha\leqslant p} \|\partial_x^\alpha v\|^2 + \sum\limits_{0\leqslant\beta\leqslant r} \|\partial_t^\beta v\|^2\Big)^{1/2}$ (beeing equivalent to $\|v\|_{p,r} = \big(\|\partial_x^p v\|^2 + \|\partial_t^r v\|^2\big)^{1/2}$) and introduce operators $\mathscr{B}_0 = \partial_t^2 + \alpha^2\partial_x^4$, $\mathscr{B} = \mathscr{B}_0 + \beta\partial_t$, $\mathscr{W}_0 = \partial_t^2 - \alpha^2\partial_x^2$ and $\mathscr{W} = \mathscr{W}_0 + \beta\partial_t$, all operating on $\mathscr{D}'(\mathbb{R}^2)$. Finally, let $\widehat{C}^{p,r}$ be the space of all functions $v \in \widehat{H}$ that have continuous derivatives up to order p in x and up to order r in t. We endow this space with the norm

$$\|v\|_{\widehat{C}^{p,r}} = \sum_{0\leqslant\alpha\leqslant p} \sup_{(x,t)\in\overline{Q}} |\partial_x^\alpha v(x,t)| + \sum_{0\leqslant\beta\leqslant r} \sup_{(x,t)\in\overline{Q}} |\partial_t^\beta v(x,t)|. \tag{2.20}$$

Lemma 2.2.

(i) *The operator* $\widehat{\ }\colon H \to \widehat{H}$ *maps* $H^{p,r}$ *topologically onto* $\widehat{H}^{p,r}$. *Moreover,*

$$\widehat{T^p u} = \partial_t^p \widehat{u} \qquad (u \in \mathscr{D}(T^p),\ p \in \mathbb{N});$$
$$\widehat{S^p u} = (-1)^{\frac{p}{2}}\partial_x^p \widehat{u} \quad (u \in \mathscr{D}(S^p),\ p \text{ even}); \qquad \|\widehat{S^p u}\| = \|\partial_x^p \widehat{u}\| \quad (u \in \mathscr{D}(S^p),\ p \text{ odd});$$
$$\widehat{Bu} = \mathscr{B}_0\widehat{u} \quad (u \in \mathscr{D}(B)); \qquad \widehat{\mathbb{B}u} = \mathscr{B}\widehat{u} \quad (u \in \mathscr{D}(\mathbb{B}));$$
$$\widehat{Wu} = \mathscr{W}_0\widehat{u} \quad (u \in \mathscr{D}(W)); \qquad \widehat{\mathbb{W}u} = \mathscr{W}\widehat{u} \quad (u \in \mathscr{D}(\mathbb{W})).$$

(ii) $\widehat{H}^{p,r}$ *is continuously embedded into* $\widehat{C}^{\alpha,\beta}$ *provided*

$$\max\Big\{\frac{\alpha+\frac12}{p} + \frac{1}{2r}, \frac{\beta+\frac12}{r} + \frac{1}{2p}\Big\} < 1.$$

P r o o f. See the proof of Lemma 2.2 in [BDL]. To include the operators W and $\mathbb{W}$ a simple modification is needed. □

Concerning the regularity of solutions of $(\mathscr{S}\mathscr{B})$ we have the following result.

Lemma 2.3. *Let* $\mathbb{B} = B + \beta T$ *and* $\mathbb{W} = W + \beta T$ *with* $\beta \in \mathbb{R}^+$. *Then the following assertions* (i)–(iv) *hold true:*

(i) $\mathscr{D}(\mathbb{B}) \subset \mathscr{D}(S^2) \cap \mathscr{D}(T) = H^{2,1}$; $\mathscr{D}(\mathbb{W}) \subset \mathscr{D}(S) \cap \mathscr{D}(T) = H^{1,1}$.

(ii) $\mathscr{D}(T\mathbb{B}) \subset \mathscr{D}(S^4) \cap \mathscr{D}(T^2) = H^{4,2}$; $\mathscr{D}(T\mathbb{W}) \subset \mathscr{D}(S^2) \cap \mathscr{D}(T^2) = H^{2,2}$.

(iii) $(\mathbb{B} - \lambda I)^{-1}$ *and* $(\mathbb{W} - \lambda I)^{-1}$ *map* $H^{p,r}$ *continuously into* $H^{p+2,r+1}$ *and* $H^{p+1,r+1}$, *respectively, for all* $p, r \in \mathbb{N} \cup \{0\}$ *and all* $\lambda \in \mathbb{R} \setminus (\sigma(\mathbb{B}) \cup \sigma(\mathbb{W}))$.

(iv) $(\mathbb{B} - \lambda I)^{-1}$ *and* $(\mathbb{W} - \lambda I)^{-1}$ *are compact operators for each* $\lambda \notin \sigma(\mathbb{B}) \cup \sigma(\mathbb{W})$.

P r o o f. See the proof of Lemma 2.3 in [BDL] and modify it to include the operator $\mathbb{W}$. $\square$

In order to translate the problem $(\mathscr{S}\mathscr{B})$ in the operator theoretic language we denote

$$\mathbf{H} = H \times H, \quad \mathbf{H}_{\mathbb{C}} = H_{\mathbb{C}} \times H_{\mathbb{C}}$$

endowed with the scalar product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \langle f_1, g_1 \rangle + \langle f_2, g_2 \rangle \qquad (\mathbf{f} = (f_1, f_2)^t, \mathbf{g} = (g_1, g_2)^t \in \mathbf{H}_{\mathbb{C}})$$

and the norm

$$\|\mathbf{f}\| = \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle},$$

$\mathbf{H}$ and $\mathbf{H}_{\mathbb{C}}$ being a real and complex Hilbert space, respectively. We shall use the notation $\mathbf{H}^{p,r} = H^{p,r} \times H^{p,r}$ with the norm $\|\mathbf{w}\|_{p,r} = (\|w_1\|_{p,r}^2 + \|w_2\|_{p,r}^2)^{\frac{1}{2}}$, $\mathbf{w} = (w_1, w_2)^t \in \mathbf{H}^{p,r}$. The linear part of $(\mathscr{S}\mathscr{B})$ leads to the definition of a real operator

$$\mathbf{L} = (\mathbb{W}, \mathbb{B})^t, \qquad \mathscr{D}_{\mathbb{C}}(\mathbf{L}) = \mathscr{D}_{\mathbb{C}}(\mathbb{W}) \times \mathscr{D}_{\mathbb{C}}(\mathbb{B}) \tag{2.21}$$

with

$$\begin{aligned} \mathbb{W} &= \mathbb{W}(\alpha_1, \beta_1) = W(\alpha_1) + \beta_1 T = W + \beta_1 T, \\ \mathbb{B} &= \mathbb{B}(\alpha_2, \beta_2) = B(\alpha_2) + \beta_2 T = B + \beta_2 T. \end{aligned} \tag{2.22}$$

We summarize the basic properties of $\mathbf{L}$ needed in this paper.

Lemma 2.4. *Let* $\mathbf{L}$ *be given by* (2.21). *Then the following assertions hold:*
(i) $\mathbf{L}^* = (W - \beta_1 T, B - \beta_2 T)^t$, *in particular* $\mathbf{L}$ *is normal.*
(ii) $\sigma(\mathbf{L}) = \sigma(\mathbb{W}) \cup \sigma(\mathbb{B})$.
(iii) $(\mathbf{L} - \lambda \mathbf{I})^{-1}$ *is compact for all* $\lambda \notin \sigma(\mathbf{L})$.

R e m a r k 2.1. As a consequence of Lemma 2.4 (iii) and Lemma 2.1 (iii) we have $0 \notin \sigma(\mathbf{L})$ and

$$\sigma(\mathbf{L}) \cap \mathbb{R} = \{\lambda \mid \lambda = \alpha_1^2 m^2 \text{ or } \lambda = \alpha_2^2 m^4,\ m \in \mathbb{N}\}.$$

P r o o f (of Lemma 2.4). (i) Using (2.12) of Lemma 2.1 we get

$$\mathbf{L}^* = (\mathbb{W}^*, \mathbb{B}^*)^t = (W - \beta_1 T, B - \beta_2 T)^t.$$

(ii) Suppose $\lambda \notin \sigma(\mathbb{W}) \cup \sigma(\mathbb{B})$, then both $\mathbb{W} - \lambda I$ and $\mathbb{B} - \lambda I$ are bijective operators with the range $H_{\mathbb{C}}$, hence $\mathbf{L} - \lambda \mathbf{I} = (\mathbb{W} - \lambda I, \mathbb{B} - \lambda I)^t$ is bijective with the range $\mathbf{H}_{\mathbb{C}}$

and thus $\lambda \notin \sigma(\mathbf{L})$. On the other hand, if $\lambda \notin \sigma(\mathbf{L})$, then $\mathbf{L} - \lambda\mathbf{I}$ is bijective with the range $\mathbf{H}_{\mathbb{C}}$, so $\mathbb{W} - \lambda I$ and $\mathbb{B} - \lambda I$ are both bijective with the range $H_{\mathbb{C}}$. This shows that $\lambda \notin \sigma(\mathbb{W})$ and $\lambda \notin \sigma(\mathbb{B})$.

(iii) We know by the proof of (ii) that for $\lambda \notin \sigma(\mathbf{L})$

$$(\mathbf{L} - \lambda\mathbf{I})^{-1} = \big((\mathbb{W} - \lambda I)^{-1}, (\mathbb{B} - \lambda I)^{-1}\big)^t .$$

Applying Lemma 2.3 (iv) yields the compactness of the operator $(\mathbf{L} - \lambda\mathbf{I})^{-1}$. $\square$

To introduce the concept of a weak solution of $(\mathscr{S}\mathscr{B})$ admitting in addition more general nonlinearities $g(\xi)$ instead of ξ^+ we need to define the space

$$\mathscr{C} = \{\varphi \in H \mid \widehat{\varphi} \in C^\infty(\mathbb{R}^2; \mathbb{R})\} \tag{2.23}$$

of test functions associated with $(\mathscr{S}\mathscr{B})$ and to consider the class of continuous functions $g\colon\ \mathbb{R} \to \mathbb{R}$ satisfying for all $\xi \in \mathbb{R}$ the growth condition

$$|g(\xi)| \leqslant c_1 + c_2|\xi| \quad \text{with some} \quad c_1, c_2 > 0. \tag{$\mathscr{G}$}$$

For a given $\mathbf{h} = (h_1, h_2)^t \in \mathbf{H}$ we are led to call a couple $\mathbf{w} = (v, u)^t \in \mathbf{H}$ a weak solution of

$$\begin{cases} v_{tt} - \alpha_1^2 v_{xx} + \beta_1 v_t - k_1 g \circ (u - v) = h_1(x, t), \\ u_{tt} + \alpha_2^2 u_{xxxx} + \beta_2 u_t + k_2 g \circ (u - v) = h_2(x, t), \\ v(0, t) = v(\pi, t) = 0, \\ u(0, t) = u(\pi, t) = u_{xx}(0, t) = u_{xx}(\pi, t) = 0, \\ u(x, t + 2\pi) = u(x, t), \quad v(x, t + 2\pi) = v(x, t), \\ x \in \left]0, \pi\right[, t \in \mathbb{R} \end{cases} \tag{$\mathscr{S}\mathscr{B}'$}$$

if and only if

$$\begin{aligned} \langle v, \mathbb{W}^* \varphi\rangle &= \langle k_1 g \circ (u - v) + h_1, \varphi\rangle , \\ \langle u, \mathbb{B}^* \psi\rangle &= \langle -k_2 g \circ (u - v) + h_2, \psi\rangle \end{aligned} \tag{2.24}$$

for all $\varphi, \psi \in \mathscr{C}$. Notice that any $\varphi \in \mathscr{C}$ satisfies the boundary conditions of $(\mathscr{S}\mathscr{B}')$. Let us introduce a nonlinear operator $\mathbf{N}\colon\ \mathbf{H} \to \mathbf{H}$ by setting

$$\mathbf{N}(\mathbf{w}) = (-k_1 g \circ (u - v), k_2 g \circ (u - v))^t \qquad (\mathbf{w} = (v, u)^t \in \mathbf{H}). \tag{2.25}$$

Then $\mathbf{w} = (v, u)^t \in \mathbf{H}$ is a weak solution of $(\mathscr{S}\mathscr{B}')$ if and only if

$$\langle \mathbf{w}, \mathbf{L}^* \boldsymbol{\varphi}\rangle = \langle \mathbf{h} - \mathbf{N}(\mathbf{w}), \boldsymbol{\varphi}\rangle \qquad (\boldsymbol{\varphi} \in \mathscr{C} \times \mathscr{C}). \tag{2.26}$$

For our next result we recall the definition of the operators $\mathscr{T} = \partial_t$, $\mathscr{W} = \partial_t^2 - \alpha_1^2\partial_x^2 + \beta_1\partial_t$ and $\mathscr{B} = \partial_t^2 + \alpha_2^2\partial_x^4 + \beta_2\partial_t$, all operating on $\mathscr{D}'(\mathbb{R}^2)$.

Proposition 2.1. *Suppose* $\mathbf{w}, \mathbf{h} \in \mathbf{H}$ *and write* $\mathbf{w} = (v, u)^t$, $\mathbf{l} = \mathbf{h} - \mathbf{N}(\mathbf{w})$. *Then the following assertions are equivalent:*

(i) $\mathbf{w}$ *is a weak solution of* $(\mathscr{S}\mathscr{B}')$,

(ii) $\mathbf{w} \in \mathscr{D}(\mathbf{L})$ *and* $\mathbf{L}\mathbf{w} = \mathbf{l}$,

(iii) $\mathscr{T}\widehat{u}, \mathscr{T}\widehat{v}, \mathscr{B}\widehat{u}, \mathscr{W}\widehat{v} \in H$ *and* $\mathscr{W}\widehat{v} = \widehat{l_1}$, $\mathscr{B}\widehat{u} = \widehat{l_2}$.

P r o o f. Using formula (2.24) it is clear that $\mathbf{w}$ is a weak solution of $(\mathscr{S}\mathscr{B}')$ with the right hand side $\mathbf{h}$ if and only if u is a weak solution of the beam equation and v is a weak solution of the string equation with the corresponding boundary conditions from $(\mathscr{S}\mathscr{B}')$. We apply Lemma 3.1 of [BDL] and the corresponding result for the string equation (i.e., the operator $\mathbb{W}$) to see that

$$\begin{aligned}
\text{(i)} &\iff v \in \mathscr{D}(\mathbb{W}), \quad \mathbb{W}v = l_1 \quad \text{and} \quad u \in \mathscr{D}(\mathbb{B}), \quad \mathbb{B}u = l_2 \\
&\iff \mathbf{w} \in \mathscr{D}(\mathbf{L}), \quad \mathbf{L}\mathbf{w} = \mathbf{l} = \mathbf{h} - \mathbf{N}(\mathbf{w}) \\
&\iff \text{(ii)}
\end{aligned}$$

The equivalence of (ii) and (iii) follows from Lemma 3.1 of [BDL] (dealing with $\mathbb{B}$) and the corresponding result for the operator $\mathbb{W}$. □

R e m a r k 2.2. The corresponding result of Lemma 3.1 from [BDL] concerning the string equation is obtained by a literal translation from $\mathbb{B}$ to $\mathbb{W}$. We thus omit the proof.

3. General existence results and properties of coupled string-beam equations

According to Proposition 2.1 the investigation of the coupled string-beam equation $(\mathscr{S}\mathscr{B}')$ is reduced to the study of the nonlinear operator equation

$$\mathbf{L}\mathbf{w} + \mathbf{N}(\mathbf{w}) = \mathbf{h}. \tag{3.1}$$

It is worth mentioning that equation (3.1) can be transformed to an equivalent equation

$$\widetilde{\mathbf{L}}\mathbf{w} + \widetilde{\mathbf{N}}(\mathbf{w}) = \widetilde{\mathbf{h}} \tag{3.2}$$

by dividing the first component of (3.1) by k_1 and the second by k_2. Notice that in (3.2)

$$\widetilde{\mathbf{L}} = \Big(\frac{1}{k_1}\mathbb{W}, \frac{1}{k_2}\mathbb{B}\Big)^t, \qquad \widetilde{\mathbf{h}} = \Big(\frac{1}{k_1}h_1, \frac{1}{k_2}h_2\Big)^t, \tag{3.3}$$

$$\widetilde{\mathbf{N}}(\mathbf{w}) = (-g\circ(u-v), g\circ(u-v))^t \qquad (\mathbf{w} = (v,u)^t \in \mathbf{H}). \tag{3.4}$$

The advantage of considering (3.2) instead of (3.1) consists in the fact that under suitable assumptions on g the operator $\widetilde{\mathbf{N}}$ becomes monotone whereas $\mathbf{N}$ does not.

Lemma 3.1. *Suppose $g\colon \mathbb{R}\to\mathbb{R}$ is continuous and satisfies the growth condition $(\mathscr{G})$. Assume $N\colon H\to H$ given by $N(u) = g\circ u$ is monotone. Then the operator $\widetilde{\mathbf{N}}\colon \mathbf{H}\to\mathbf{H}$ defined by (3.4) is monotone, too.*

Remark 3.1. Notice that the functions $g(\xi)=\xi$ as well as $g(\xi)=\xi^+$ lead to monotone operators N.

Proof. With the notation $\mathbf{w}_i = (v_i,u_i)^t$ $(i=1,2)$, $a = u_1 - v_1$ and $b = u_2 - v_2$ we get

$$\begin{aligned}
&\big\langle \widetilde{\mathbf{N}}(\mathbf{w}_1) - \widetilde{\mathbf{N}}(\mathbf{w}_2), \mathbf{w}_1 - \mathbf{w}_2\big\rangle \\
&\ = \langle -g\circ(u_1-v_1) + g\circ(u_2-v_2), v_1 - v_2\rangle + \langle g\circ(u_1-v_1) - g\circ(u_2-v_2), u_1-u_2\rangle \\
&\ = \langle g\circ a, v_2 - v_1 + u_1 - u_2\rangle + \langle g\circ b, v_1 - v_2 - u_1 + u_2\rangle \\
&\ = \langle g\circ a, a-b\rangle + \langle g\circ b, b-a\rangle \\
&\ = \langle N(a) - N(b), a-b\rangle\,.
\end{aligned}$$

This proves Lemma 3.1. □

Before we state the first result concerning time-independent solutions of $(\mathscr{S}\mathscr{B}')$ needed to show our general existence result for $(\mathscr{S}\mathscr{B}')$ we have to recall a basic lemma from [BDL].

Lemma 3.2. *Suppose $g\colon \mathbb{R}\to\mathbb{R}$ is continuous and satisfies growth condition $(\mathscr{G})$. Then*

(i) $\langle g\circ u, Tu\rangle = 0$ *for all* $u\in\mathscr{D}(T)$,
(ii) $\langle Bu, Tu\rangle = 0$ *for all* $u\in\mathscr{D}(B)\cap\mathscr{D}(T)$,
(iii) $\langle Wu, Tu\rangle = 0$ *for all* $u\in\mathscr{D}(W)\cap\mathscr{D}(T)$.

Proof. (i) + (ii) is exactly Lemma 3.2 from [BDL] and the proof of (iii) is analogous to (ii). We have simply to replace the operator B by W. □

Lemma 3.3. *Suppose $g\colon\ \mathbb{R}\to\mathbb{R}$ is continuous and satisfies the growth condition $(\mathscr{G})$. Assume in addition that the operator $N\colon\ H\to H$ given by $N(u)=g\circ u$ is monotone. Then the following assertions hold true:*

(i) *$\mathbf{w}\in\mathscr{D}(\mathbf{L})$ and $\mathbf{T}\mathbf{w}=\mathbf{o}$ imply $\mathbf{w}\in\mathscr{D}(S^2)\times\mathscr{D}(S^4)$ and $\mathbf{L}\mathbf{w}=(\alpha_1^2S^2w_1,\alpha_2^2S^4w_2)^t$.*

(ii) *Suppose $\mathbf{w}_i\in\mathscr{D}(\mathbf{L})$, $\mathbf{f}_i\in\mathscr{D}(\mathbf{T})$ satisfy*

$$\mathbf{L}\mathbf{w}_i+\mathbf{N}(\mathbf{w}_i)=\mathbf{f}_i,\qquad \mathbf{T}\mathbf{f}_i=\mathbf{0}\qquad (i=1,2).$$

Then there is a positive constant $c=c(\alpha_1,\alpha_2,k_1,k_2)$ such that

$$\|\mathbf{w}_1-\mathbf{w}_2\|\leqslant c\|\mathbf{f}_1-\mathbf{f}_2\|. \tag{3.5}$$

R e m a r k 3.2. Notice that $\mathbf{T}=(T,T)^t$ and $\mathbf{N}$ is defined by formula (2.25).

R e m a r k 3.3. A possible choice for the constant c appearing in (3.5) is

$$c=\left[\min\left\{\frac{\alpha_1^2}{k_1},\frac{\alpha_2^2}{k_2}\right\}\min\{k_1,k_2\}\right]^{-1}.$$

P r o o f (of Lemma 3.3). (i) Let $\mathbf{w}=(v,u)^t\in\mathscr{D}(\mathbf{L})$ and $\mathbf{T}\mathbf{w}=\mathbf{0}$, which means $v\in\mathscr{D}(\mathbb{W})$, $Tv=0$ as well as $u\in\mathscr{D}(\mathbb{B})$, $Tu=0$. Concerning u it follows from Lemma 3.3 (i) from [BDL] that $u\in\mathscr{D}(S^4)$ and $\mathbb{B}u=\alpha_2^2S^4u$. Performing the proof of Lemma 3.3 in [BDL] for $\mathbb{W}$ instead of $\mathbb{B}$ shows that $v\in\mathscr{D}(S^2)$ and $\mathbb{W}v=\alpha_1^2S^2v$.

(ii) Let us first show that with $\mathbf{w}=(v,u)^t$ and $\mathbf{f}=(h,l)^t$ the equations

$$\mathbf{L}\mathbf{w}+\mathbf{N}(\mathbf{w})=\mathbf{f};\qquad \mathbf{T}\mathbf{f}=\mathbf{0} \tag{3.6}$$

reduce to the time-independent system

$$\begin{aligned}\alpha_1^2S^2v-k_1g\circ(u-v)&=h,\\ \alpha_2^2S^4u+k_2g\circ(u-v)&=l.\end{aligned} \tag{3.7}$$

Indeed, $\mathbf{L}\mathbf{w}+\mathbf{N}(\mathbf{w})=\mathbf{f}$ means

$$\begin{aligned}\frac{1}{k_1}\mathbb{W}v-g\circ(u-v)&=\frac{1}{k_1}h,\\ \frac{1}{k_2}\mathbb{B}u+g\circ(u-v)&=\frac{1}{k_2}l.\end{aligned} \tag{3.8}$$

Multiplying the first equation of (3.8) by Tv and the second by Tu with respect to the scalar product $\langle\cdot,\cdot\rangle$ we obtain

$$\begin{aligned}\frac{1}{k_1}\langle\mathbb{W}v,Tv\rangle-\langle g\circ(u-v),Tv\rangle&=\frac{1}{k_1}\langle h,Tv\rangle,\\ \frac{1}{k_2}\langle\mathbb{B}u,Tu\rangle+\langle g\circ(u-v),Tu\rangle&=\frac{1}{k_2}\langle l,Tu\rangle.\end{aligned} \tag{3.9}$$

Using $\langle \mathbb{W}v, Tv\rangle = \beta_1\|Tv\|^2$, $\langle \mathbb{B}u, Tu\rangle = \beta_2\|Tu\|^2$ (see Lemma 3.2 (ii), (iii) and recall that $\mathbb{W} = W + \beta_1 T$, $\mathbb{B} = B + \beta_2 T$) and $\langle h, Tv\rangle = -\langle Th, v\rangle = -\langle 0, v\rangle = 0$ as well as $\langle l, Tu\rangle = 0$ we end up with the equations

$$\begin{aligned}\frac{\beta_1}{k_1}\|Tv\|^2 - \langle g\circ(u-v), Tv\rangle &= 0,\\ \frac{\beta_2}{k_2}\|Tu\|^2 + \langle g\circ(u-v), Tu\rangle &= 0.\end{aligned}\tag{3.10}$$

Now we add up the two equations in (3.10) to get

$$\frac{\beta_1}{k_1}\|Tv\|^2 + \frac{\beta_2}{k_2}\|Tu\|^2 + \langle g\circ(u-v), T(u-v)\rangle = 0.\tag{3.11}$$

By Lemma 3.2 (i), $\langle g\circ(u-v), T(u-v)\rangle = 0$ and thus $\|Tu\| = 0 = \|Tv\|$. Hence by (i) we conclude $\mathbb{W}v = \alpha_1^2 S^2 v$ and $\mathbb{B}u = \alpha_2^2 S^4 u$, which proves (3.7). Notice that we have to assume $\beta_i/k_i > 0$ for $i = 1, 2$.

To complete the proof of (ii) let $\mathbf{L}\mathbf{w}_i + \mathbf{N}(\mathbf{w}_i) = \mathbf{f}_i$ with $\mathbf{T}\mathbf{f}_i = \mathbf{0}$ for $i = 1, 2$. Since $\mathbf{L}\mathbf{w}_i + \mathbf{N}(\mathbf{w}_i) = \mathbf{f}_i$ is equivalent to $\widetilde{\mathbf{L}}\mathbf{w}_i + \widetilde{\mathbf{N}}(\mathbf{w}_i) = \widetilde{\mathbf{f}}_i$ (see (3.2)–(3.4)) we get

$$\underbrace{\left\langle \widetilde{\mathbf{L}}(\mathbf{w}_1 - \mathbf{w}_2), \mathbf{w}_1 - \mathbf{w}_2\right\rangle}_{I} + \underbrace{\left\langle \widetilde{\mathbf{N}}(\mathbf{w}_1) - \widetilde{\mathbf{N}}(\mathbf{w}_2), \mathbf{w}_1 - \mathbf{w}_2\right\rangle}_{II} = \left\langle \widetilde{\mathbf{f}}_1 - \widetilde{\mathbf{f}}_2, \mathbf{w}_1 - \mathbf{w}_2\right\rangle.$$

Using (i) from above and Lemma 2.1 (iii), formula (2.13), we get

$$I = \left\langle \left(\frac{\alpha_1^2}{k_1}S^2, \frac{\alpha_2^2}{k_2}S^4\right)^t (\mathbf{w}_1 - \mathbf{w}_2), \mathbf{w}_1 - \mathbf{w}_2\right\rangle \geqslant \frac{\alpha_1^2}{k_1}\|v_1 - v_2\|^2 + \frac{\alpha_2^2}{k_2}\|u_1 - u_2\|^2.$$

Here we have used the fact that $\sigma(S^2)$ and $\sigma(S^4)$ are both contained in $[1, \infty[$ (see (2.13)). Since $\widetilde{\mathbf{N}}$ is a monotone operator by Lemma 3.1 we have $II \geqslant 0$ and thus

$$\frac{\alpha_1^2}{k_1}\|v_1 - v_2\|^2 + \frac{\alpha_2^2}{k_2}\|u_1 - u_2\|^2 \leqslant \left\langle \widetilde{\mathbf{f}}_1 - \widetilde{\mathbf{f}}_2, \mathbf{w}_1 - \mathbf{w}_2\right\rangle \leqslant \|\widetilde{\mathbf{f}}_1 - \widetilde{\mathbf{f}}_2\|\ \|\mathbf{w}_1 - \mathbf{w}_2\|.$$

Putting $a = \min\{\frac{\alpha_1^2}{k_1}, \frac{\alpha_2^2}{k_2}\}$ we estimate

$$a\|\mathbf{w}_1 - \mathbf{w}_2\|^2 \leqslant \frac{1}{\min\{k_1, k_2\}}\|\mathbf{f}_1 - \mathbf{f}_2\|\ \|\mathbf{w}_1 - \mathbf{w}_2\|$$

and setting $c = [a\min\{k_1, k_2\}]^{-1}$ we finally obtain

$$\|\mathbf{w}_1 - \mathbf{w}_2\| \leqslant c\|\mathbf{f}_1 - \mathbf{f}_2\|.$$

□

R e m a r k 3.4. Actually the proof of Lemma 3.3 (ii) shows (see formula (3.7)) that any solution $\mathbf{w}$ of $\mathbf{L}\mathbf{w} + \mathbf{N}(\mathbf{w}) = \mathbf{f}$ with $\mathbf{T}\mathbf{f} = \mathbf{0}$ is time-independent.

Our basic existence result on string-beam equations uses the concept of homogeneous functions. We thus remind the reader that a function $g\colon \mathbb{R} \to \mathbb{R}$ is called homogeneous if $g(tx) = tg(x)$ for all $x \in \mathbb{R}$ and $t \in \mathbb{R}^+$.

Theorem 3.1. *Suppose $g\colon \mathbb{R} \to \mathbb{R}$ is continuous, homogeneous and satisfies the growth condition $(\mathscr{G})$. Then for any $\mathbf{h} \in \mathbf{H}$ the string-beam equation $(\mathscr{S}\mathscr{B}')$ has at least one weak solution $\mathbf{w} \in H^{1,1} \times H^{2,1}$.*

P r o o f. By Proposition 2.1 any solution of $(\mathscr{S}\mathscr{B}')$ is equivalent to a solution of the system $\mathbf{L}\mathbf{w} + \mathbf{N}(\mathbf{w}) = \mathbf{h}$. Since $0 \notin \sigma(\mathbf{L})$ this means to look for solutions $\mathbf{w}$ of

$$\mathbf{w} + \mathbf{L}^{-1}(\mathbf{N}(\mathbf{w}) - \mathbf{h}) = \mathbf{0}. \tag{3.12}$$

We will use the Leray-Schauder degree theory to find solutions of (3.12)—see e.g. Fučík, Kufner [FK] for basic properties of the degree used in the sequel.

To solve (3.12) it suffices to show that

$$\deg[\mathbf{G}; \mathbf{B}_R(\mathbf{0}), \mathbf{0}] \neq 0, \tag{3.13}$$

where *deg* denotes the Leray-Schauder degree and $\mathbf{G}\colon\ \mathbf{H} \to \mathbf{H}$ is defined by

$$\mathbf{G}(\mathbf{w}) = \mathbf{w} + \mathbf{L}^{-1}(\mathbf{N}(\mathbf{w}) - \mathbf{h})$$

and $\mathbf{B}_R(\mathbf{0})$ is the ball in $\mathbf{H}$ centered at the origin $\mathbf{0}$ with sufficiently large radius $R > 0$ (specified during the proof). To prove (3.13) consider the homotopy

$$\mathscr{H}(\tau, \mathbf{w}) = \mathbf{w} + \tau\mathbf{L}^{-1}(\mathbf{N}(\mathbf{w}) - \mathbf{h}) \qquad (\mathbf{w} \in \mathbf{H}, \tau \in [0,1]).$$

We prove that this homotopy is admissible. Assume to the contrary that there are $\mathbf{w}_n \in \mathbf{H}$, $\tau_n \in [0,1]$ such that $\|\mathbf{w}_n\| \to \infty$ as $n \to \infty$ and

$$\mathscr{H}(\tau_n, \mathbf{w}_n) = \mathbf{0}. \tag{3.14}$$

Passing to a suitable subsequence we may assume $\tau_n \to \tau \in [0,1]$, $\mathbf{x}_n := \frac{\mathbf{w}_n}{\|\mathbf{w}_n\|} \rightharpoonup \mathbf{x}$ as well as $\mathbf{N}(\mathbf{x}_n) \rightharpoonup \mathbf{f}$ (note that $\mathbf{N}$ maps bounded sequences in $\mathbf{H}$ to bounded sequences in $\mathbf{H}$). Then using Lemma 2.4 (iii) and the homogeneity of g we obtain

$$\mathbf{x}_n = -\tau_n\mathbf{L}^{-1}\Big(\mathbf{N}(\mathbf{x}_n) - \frac{\mathbf{h}}{\|\mathbf{w}_n\|}\Big) \longrightarrow -\tau\mathbf{L}^{-1}(\mathbf{f}) = \mathbf{x}.$$

Hence $\mathbf{N}(\mathbf{x}_n) \to \mathbf{N}(\mathbf{x})$ and thus $\mathscr{H}(\tau, \mathbf{x}) = \mathbf{0}$, which means

$$\mathbf{L}\mathbf{x} + \tau \mathbf{N}(\mathbf{x}) = \mathbf{0}. \tag{3.15}$$

Applying Lemma 3.3 (ii) to (3.15) we conclude that $\mathbf{x} = \mathbf{0}$ contradicting $\|\mathbf{x}\| = 1$. The homotopy invariance property and the basic property of the degree (see [FK]) imply the existence of a solution $\mathbf{w} \in \mathbf{H}$ of (3.12). Moreover, $\mathbf{w} \in H^{1,1} \times H^{2,1}$ by the properties of $\mathbf{L}$ (cf. Lemma 2.3 (i)). □

Before studying uniqueness questions of the string-beam equation ($\mathscr{S}\mathscr{B}$) we have to examine to some extent the linear system connected with ($\mathscr{S}\mathscr{B}$). We thus concentrate on the operator equation

$$\mathbf{L}\mathbf{w} + \mathbf{M}\mathbf{w} = \mathbf{f},$$

where $\mathbf{f}$ is any element in $\mathbf{H}$ and the linear operator $\mathbf{M}\colon\ \mathbf{H} \to \mathbf{H}$ is defined by

$$\mathbf{M}\mathbf{w} = (-k_1(u-v), k_2(u-v))^t \qquad (\mathbf{w} = (v,u)^t \in \mathbf{H}). \tag{3.16}$$

Proposition 3.1. *Suppose* $\mathbf{f} \in \mathbf{H}$ *and* $\mathbf{w}$ *is a solution of*

$$\mathbf{L}\mathbf{w} + \mathbf{M}\mathbf{w} = \mathbf{f}, \tag{3.17}$$

where $\mathbf{M}$ *is defined by* (3.16). *Then the apriori estimate*

$$\|\mathbf{w}\| \leqslant c\,\|\mathbf{f}\| \tag{3.18}$$

holds true with a positive constant $c = c(\alpha_1, \alpha_2, \beta_1, \beta_2, k_1, k_2)$ *independent of* $\mathbf{w}$ *and* $\mathbf{f}$. *In particular, system* (3.17) *has a unique solution for any right hand side* $\mathbf{f} \in \mathbf{H}$.

Remark 3.5. Notice that both β_1, β_2 have to be positive.

Proof (of Proposition 3.1). Suppose $\mathbf{w} = (v,u)^t$ is a solution of (3.17) with the right hand side $\mathbf{f} = (f_1, f_2)^t \in \mathbf{H}$. Then

$$\begin{aligned} \frac{1}{k_1}\mathbb{W}v - (u-v) &= \frac{1}{k_1} f_1, \\ \frac{1}{k_2}\mathbb{B}u + (u-v) &= \frac{1}{k_2} f_2. \end{aligned} \tag{3.19}$$

Multiplying $(3.19)_1$ by Tv and $(3.19)_2$ by Tu with respect to the scalar product $\langle\cdot,\cdot\rangle$ we get

$$\begin{aligned} \frac{1}{k_1}\langle \mathbb{W}v, Tv\rangle - \langle u-v, Tv\rangle &= \frac{1}{k_1}\langle f_1, Tv\rangle, \\ \frac{1}{k_2}\langle \mathbb{B}u, Tu\rangle + \langle u-v, Tu\rangle &= \frac{1}{k_2}\langle f_2, Tu\rangle. \end{aligned}$$

If we add up these two equations using (see Lemma 3.2)

$$\langle \mathbb{W}v, Tv\rangle = \beta_1\|Tv\|^2, \qquad \langle \mathbb{B}u, Tu\rangle = \beta_2\|Tu\|^2, \qquad \langle u-v, T(u-v)\rangle = 0$$

we obtain

$$\begin{aligned}\frac{\beta_1}{k_1}\|Tv\|^2 + \frac{\beta_2}{k_2}\|Tu\|^2 &= \frac{1}{k_1}\langle f_1, Tv\rangle + \frac{1}{k_2}\langle f_2, Tu\rangle \leqslant \frac{1}{k_1}\|f_1\|\,\|Tv\| + \frac{1}{k_2}\|f_2\|\,\|Tu\| \\ &\leqslant \frac{1}{2k_1}\left(\frac{\|f_1\|^2}{\beta_1} + \beta_1\|Tv\|^2\right) + \frac{1}{2k_2}\left(\frac{\|f_2\|^2}{\beta_2} + \beta_2\|Tu\|^2\right)\end{aligned}$$

and thus

$$\frac{\beta_1}{2k_1}\|Tv\|^2 + \frac{\beta_2}{2k_2}\|Tu\|^2 \leqslant \frac{1}{2k_1\beta_1}\|f_1\|^2 + \frac{1}{2k_2\beta_2}\|f_2\|^2,$$

which gives, with a suitable constant $c_1 = c_1(\beta_1, \beta_2, k_1, k_2)$,

$$\|Tv\|^2 + \|Tu\|^2 \leqslant c_1^2\,\|\mathbf{f}\|^2. \tag{3.20}$$

Let $P = \sum\limits_{m\in\mathbb{N}} P_{0,m}$, where $P_{0,m}$ is the projection to the one-dimensional space generated by $\varphi_{0,m}$. For any $u \in H$ we have representations

$$\begin{aligned}u &= \sum_{n\in\mathbb{Z},m\in\mathbb{N}} \langle u, \varphi_{n,m}\rangle\,\varphi_{n,m} \\ &= \sum_{n\in\mathbb{Z}\setminus\{0\},m\in\mathbb{N}} \langle u, \varphi_{n,m}\rangle\,\varphi_{n,m} + \sum_{m\in\mathbb{N}} \langle u, \varphi_{0,m}\rangle\,\varphi_{0,m} = (I-P)u + Pu.\end{aligned}$$

Moreover, if $u \in \mathscr{D}(T)$ then

$$Tu = \sum_{n\in\mathbb{Z},m\in\mathbb{N}} (\mathrm{i}n)\,\langle u, \varphi_{n,m}\rangle\,\varphi_{n,m} = \sum_{n\in\mathbb{Z}\setminus\{0\},m\in\mathbb{N}} (\mathrm{i}n)\,\langle u, \varphi_{n,m}\rangle\,\varphi_{n,m}$$

with

$$\|Tu\|^2 = \sum_{n\in\mathbb{Z}\setminus\{0\},m\in\mathbb{N}} n^2\,|\langle u, \varphi_{n,m}\rangle|^2 \geqslant \sum_{n\in\mathbb{Z}\setminus\{0\},m\in\mathbb{N}} |\langle u, \varphi_{n,m}\rangle|^2 = \|(I-P)u\|^2.$$

Using the last estimate, inequality (3.20) implies

$$\|(I-P)v\|^2 + \|(I-P)u\|^2 \leqslant c_1^2\,\|\mathbf{f}\|^2. \tag{3.21}$$

Since P commutes with $\mathbb{W}$ as well as with $\mathbb{B}$ we obtain from (3.19) by applying P to each side of both equations

$$\begin{aligned}\frac{1}{k_1}\mathbb{W}Pv - (Pu - Pv) &= \frac{1}{k_1}Pf_1, \\ \frac{1}{k_2}\mathbb{B}Pu + (Pu - Pv) &= \frac{1}{k_2}Pf_2.\end{aligned} \tag{3.22}$$

Put $\bar{u} = Pu$, $\bar{v} = Pv$ and $\bar{f}_i = Pf_i$ $(i = 1, 2)$, then

$$\begin{aligned} &\frac{1}{k_1}\mathbb{W}\bar{v} - (\bar{u} - \bar{v}) = \frac{1}{k_1}\bar{f}_1; \qquad T\bar{f}_1 = 0, \\ &\frac{1}{k_2}\mathbb{B}\bar{u} + (\bar{u} - \bar{v}) = \frac{1}{k_2}\bar{f}_2; \qquad T\bar{f}_2 = 0. \end{aligned} \tag{3.23}$$

Now we apply Lemma 3.3 (ii) with $\mathbf{w}_1 = (\bar{v}, \bar{u})^t$ and $\mathbf{w}_2 = (0, 0)^t$ to obtain, with a suitable constant $c_2 = c_2(\alpha_1, \alpha_2, k_1, k_2)$, the estimate

$$\|Pv\|^2 + \|Pu\|^2 \leqslant c_2^2 \, \|(Pf_1, Pf_2)\|^2 \leqslant c_2^2 \, \|\mathbf{f}\|^2. \tag{3.24}$$

Combining estimates (3.21) and (3.24) yields

$$\|\mathbf{w}\| \leqslant c \, \|\mathbf{f}\|$$

with $c = c(\alpha_1, \alpha_2, \beta_1, \beta_2, k_1, k_2) = \max\{c_1, c_2\}$.

The combination of estimate (3.18) and Theorem 3.1 applied to equation (3.17) shows that equation (3.17) is uniquely solvable. □

4. Uniqueness results for coupled string-beam equations

In this section we deal with string-beam equation $(\mathscr{S}\mathscr{B})$, i.e., the system

$$\mathbf{L}\mathbf{w} + \mathbf{N}(\mathbf{w}) = \mathbf{h}, \tag{4.1}$$

where $\mathbf{h} \in \mathbf{H}$ and denoting $\alpha = (\alpha_1, \alpha_2)^t$, $\beta = (\beta_1, \beta_2)^t$, $\mathbf{k} = (k_1, k_2)^t$ the operators $\mathbf{L} = \mathbf{L}(\alpha, \beta)$ and $\mathbf{N} = \mathbf{N}(\mathbf{k})$ are given by

$$\begin{aligned} \mathbf{L} &= (\mathbb{W}(\alpha_1, \beta_1), \mathbb{B}(\alpha_2, \beta_2))^t = (W(\alpha_1) + \beta_1 T, B(\alpha_2) + \beta_2 T)^t \\ \mathbf{N}(\mathbf{w}) &= (-k_1(u - v)^+, k_2(u - v)^+)^t \qquad (\mathbf{w} = (v, u) \in \mathbf{H}). \end{aligned} \tag{4.2}$$

Applying Theorem 3.1 we already know that system (4.1) has at least one solution. However, there is no hope to show in general that this existing solution is unique, too. But it can be shown that if system (4.1) admits a (unique) strictly positive solution then a *small* perturbation of the right hand side $\mathbf{h}$ of (4.1) leads to a *uniquely* solvable system (Theorem 4.1). In addition, for certain values of α, $\mathbf{k}$ and $\mathbf{h} = (W_1, W_2)^t$ we are able to show that (4.1) has a strictly positive solution and thus any (time-dependent) *small* perturbation of $\mathbf{h} = (W_1, W_2)^t$ results in a system (4.1) which is uniquely solvable (Theorem 4.2).

Before going into detail we would like to recall the concept of strict positivity used here. We call a couple $\mathbf{w} = (v, u)^t \in \mathbf{H}$ belonging to $C^1(\overline{Q}) \times C^1(\overline{Q})$ strictly positive if and only if $u - v > 0$ in Q and $\partial_x(u - v)(0, t) > 0$ as well as $\partial_x(u - v)(\pi, t) < 0$ for each $t \in [0, 2\pi]$.

The following result is of crucial importance in the proof of our main result—Theorem 4.1—but it is interesting by itself, too.

Lemma 4.1. *Suppose $\mathbf{h}_0 \in \mathbf{H}$ and $\mathbf{Lw} + \mathbf{N}(\mathbf{w}) = \mathbf{h}_0$ admits a strictly positive solution $\mathbf{w}_0$. Then there is a positive $\varepsilon = \varepsilon(\mathbf{h}_0)$ such that for any $\mathbf{h} = (h_1, h_2)^t \in H^{2,1} \times H^{1,1}$ satisfying $\|h_1\|_{2,1} + \|h_2\|_{1,1} < \varepsilon$ the system $\mathbf{Lw} + \mathbf{N}(\mathbf{w}) = \mathbf{h}_0 + \mathbf{h}$ has at least one strictly positive solution $\mathbf{w}$.*

P r o o f. Let $\mathbf{Mw} = (-k_1(u - v), k_2(u - v))^t$ with $\mathbf{w} = (v, u)^t \in \mathbf{H}$. Since $\mathbf{w}_0$ is strictly positive, the system $\mathbf{Lw}_0 + \mathbf{N}(\mathbf{w}_0) = \mathbf{h}_0$ is equivalent to the linear system $\mathbf{Lw}_0 + \mathbf{Mw}_0 = \mathbf{h}_0$. By Proposition 3.1 there is a unique solution $\mathbf{w}_1$ of $\mathbf{Lw} + \mathbf{Mw} = \mathbf{h}$ satisfying (with some positive constant c_1) the estimate

$$\|\mathbf{w}_1\| \leqslant c_1 \|\mathbf{h}\|. \tag{4.3}$$

Using $\mathscr{B}(X, Y)$ to denote the Banach space of bounded linear operators from Banach space X to Banach space Y with the corresponding norm $\|\cdot\|_{\mathscr{B}(X,Y)}$ we estimate the solution $\mathbf{w}_1$ of $\mathbf{Lw} + \mathbf{Mw} = \mathbf{h}$ (see Lemma 2.3 (iii)):

$$\begin{aligned} \|\mathbf{w}_1\|_{H^{1,1}\times H^{2,1}} &\leqslant \|\mathbf{L}^{-1}\|_{\mathscr{B}(\mathbf{H}, H^{1,1}\times H^{2,1})} \|\mathbf{h} - \mathbf{Mw}_1\| \\ &\leqslant \|\mathbf{L}^{-1}\|_{\mathscr{B}(\mathbf{H}, H^{1,1}\times H^{2,1})} \big(\|\mathbf{h}\| + \|\mathbf{M}\|_{\mathscr{B}(\mathbf{H},\mathbf{H})} \|\mathbf{w}_1\|\big) \\ &\leqslant \|\mathbf{L}^{-1}\|_{\mathscr{B}(\mathbf{H}, H^{1,1}\times H^{2,1})} \big(1 + c_1 \|\mathbf{M}\|_{\mathscr{B}(\mathbf{H},\mathbf{H})}\big) \|\mathbf{h}\|, \end{aligned}$$

where in the last inequality we have used (4.3). Thus we get with a positive c_2 the estimate

$$\|\mathbf{w}_1\|_{H^{1,1}\times H^{2,1}} \leqslant c_2 \|\mathbf{h}\|. \tag{4.4}$$

In the second step we get using (4.4)

(4.5)
$$\begin{aligned} \|\mathbf{w}_1\|_{H^{2,2}\times H^{3,2}} &\leqslant \|\mathbf{L}^{-1}\|_{\mathscr{B}(\mathbf{H}^{1,1}, H^{2,2}\times H^{3,2})} \big(\|\mathbf{h}\|_{1,1} + \|\mathbf{Mw}_1\|_{1,1}\big) \\ &\leqslant \|\mathbf{L}^{-1}\|_{\mathscr{B}(\mathbf{H}^{1,1}, H^{2,2}\times H^{3,2})} \big(\|\mathbf{h}\|_{1,1} + \|\mathbf{M}\|_{\mathscr{B}(\mathbf{H}^{1,1},\mathbf{H}^{1,1})} \|\mathbf{w}_1\|_{1,1}\big) \\ &\leqslant c_3 \|\mathbf{h}\|_{1,1}. \end{aligned}$$

Finally, using (4.5) we obtain

(4.6)
$$\begin{aligned}\|\mathbf{w}_1\|_{3,2} &\leqslant \|\mathbf{L}^{-1}\|_{\mathscr{B}(H^{2,1}\times H^{1,1},\mathbf{H}^{3,2})}\left(\|\mathbf{h}\|_{H^{2,1}\times H^{1,1}} + \|\mathbf{M}\mathbf{w}_1\|_{H^{2,1}\times H^{1,1}}\right)\\ &\leqslant \|\mathbf{L}^{-1}\|_{\mathscr{B}(H^{2,1}\times H^{1,1},\mathbf{H}^{3,2})}\left(\|\mathbf{h}\|_{H^{2,1}\times H^{1,1}} + \|\mathbf{M}\|_{\mathscr{B}(\mathbf{H}^{2,2},H^{2,1}\times H^{1,1})}\|\mathbf{w}_1\|_{2,2}\right)\\ &\leqslant c_4\,\|\mathbf{h}\|_{H^{2,1}\times H^{1,1}}.\end{aligned}$$

By Lemma 2.2 (ii) $\widehat{H}^{3,2}$ is continuously embedded into $\widehat{C}^{1,1}$, hence with a positive constant c_5 we have

$$\|\widehat{\mathbf{w}_1}\|_{\widehat{\mathbf{C}}^{1,1}} \leqslant c_5\,\|\mathbf{h}\|_{H^{2,1}\times H^{1,1}}. \tag{4.7}$$

If we choose $\|\widehat{\mathbf{w}_1}\|_{\widehat{\mathbf{C}}^{1,1}}$ small enough by making $\|\mathbf{h}\|_{H^{2,1}\times H^{1,1}}$ small, we see that $\mathbf{w} := \mathbf{w}_0 + \mathbf{w}_1$ (as a small perturbation of a strictly positive couple $\mathbf{w}_0$ in $\widehat{\mathbf{C}}^{1,1}$-norm) remains strictly positive, too. It is now clear that $\mathbf{w}$ satisfies the equation

$$\mathbf{L}\mathbf{w} + \mathbf{N}(\mathbf{w}) = \mathbf{L}\mathbf{w} + \mathbf{M}\mathbf{w} = \mathbf{h}_0 + \mathbf{h}.$$

□

We are now in a position to state and prove our main result.

Theorem 4.1. *Suppose the string-beam equation*

$$\mathbf{L}\mathbf{w} + \mathbf{N}(\mathbf{w}) = \mathbf{W} \tag{4.8}$$

with a time-independent right hand side $\mathbf{W} = (W_1(x), W_2(x))^t \in \mathbf{H}$ *admits a strictly positive solution* $\mathbf{w}_0$.

Then there exists a positive constant $\varepsilon = \varepsilon(\alpha, \beta, \mathbf{k}, \mathbf{W})$ *such that for any* $\mathbf{f} \in \mathbf{H}^{2,2}$ *satisfying*

$$\|\mathbf{f}\|_{2,2} < \varepsilon \tag{4.9}$$

there is a unique solution $\mathbf{w}$ *of*

$$\mathbf{L}\mathbf{w} + \mathbf{N}(\mathbf{w}) = \mathbf{W} + \mathbf{f}. \tag{4.10}$$

Moreover, $\mathbf{w} \in H^{2,2} \times H^{4,2}$ *and* $\mathbf{w}$ *is strictly positive.*

In addition, for any two solutions $\mathbf{w}_1$, $\mathbf{w}_2$ *of* (4.8) *with the corresponding right hand sides* $\mathbf{W} + \mathbf{f}_1$, $\mathbf{W} + \mathbf{f}_2$ *the estimate*

$$\|\mathbf{w}_1 - \mathbf{w}_2\|_{3,3} \leqslant c\,\|\mathbf{f}_1 - \mathbf{f}_2\|_{2,2} \tag{4.11}$$

holds with a positive constant c*, provided* $\|\mathbf{f}_i\|_{2,2} < \varepsilon$ $(i = 1, 2)$.

R e m a r k 4.1. In Theorem 4.2 we will show that under suitable conditions on α, $\mathbf{k}$ (appearing in $\mathbf{L}$, $\mathbf{M}$) and $\mathbf{W}$ equation (4.8) always admits a strictly positive solution.

R e m a r k 4.2. By Lemma 2.2 (ii) $\widehat{H}^{3,3}$ is continuously embedded into $\widehat{C}^{1,1}$. Hence the norm $\|\mathbf{w}_1 - \mathbf{w}_2\|_{3,3}$ in (4.11) may be replaced by $\|\mathbf{w}_1 - \mathbf{w}_2\|_{\widehat{\mathbf{C}}^{1,1}}$.

P r o o f (of Theorem 4.1). The proof is carried out in two steps.

Step 1. *There is an* $\varepsilon_1 = \varepsilon_1(\alpha, \beta, \mathbf{k}, \mathbf{W}) > 0$ *such that for any* $\mathbf{f} \in \mathbf{H}^{2,2}$ *satisfying* $\|\mathbf{f}\|_{2,2} < \varepsilon_1$ *there exists a strictly positive solution* $\mathbf{w_f}$ *of* $\mathbf{Lw} + \mathbf{N}(\mathbf{w}) = \mathbf{W} + \mathbf{f}$.

This has been shown already in Lemma 4.1.

Step 2. *There exists exactly one solution* $\mathbf{w}$ *of* (4.10) *provided* ε_1 *is chosen small enough. Moreover,* $\mathbf{w} \in H^{2,2} \times H^{4,2}$ *and* $\mathbf{w}$ *is strictly positive.*

Assume the contrary, i.e., there are $\mathbf{f}_n \in \mathbf{H}^{2,2}$ with $\lim\limits_{n\to\infty} \|\mathbf{f}_n\|_{2,2} = 0$ and solutions $\mathbf{w}_n$ of (4.10) with right hand side $\mathbf{W} + \mathbf{f}_n$ such that

$$\mathbf{w}_n \neq \mathbf{w}_{\mathbf{f}_n}. \tag{4.12}$$

Case (i): $\|\mathbf{w}_n\|$ *is unbounded in* n. Then taking a suitable subsequence we may assume $\lim\limits_{n\to\infty} \|\mathbf{w}_n\| = \infty$. Denoting $\mathbf{x}_n = \mathbf{w}_n / \|\mathbf{w}_n\|$ (and taking again a subsequence of $\mathbf{x}_n$) we see that $\mathbf{x}_n = (v_n, u_n)^t \rightharpoonup \mathbf{x}$, $(u_n - v_n)^+ \rightharpoonup z$, $\mathbf{N}(\mathbf{x}_n) \rightharpoonup \mathbf{g}$ with suitable $\mathbf{x}$, $\mathbf{g} \in \mathbf{H}$, $z \in H$.

By definition of $\mathbf{w}_n$ and $\mathbf{x}_n$ we have

$$\mathbf{L}\mathbf{x}_n + \mathbf{N}(\mathbf{x}_n) = \frac{1}{\|\mathbf{w}_n\|}(\mathbf{W} + \mathbf{f}_n),$$

and due to the compactness of $\mathbf{L}^{-1}$ (see Lemma 2.4 (iii))

$$\mathbf{x}_n = \mathbf{L}^{-1}\left(\frac{1}{\|\mathbf{w}_n\|}(\mathbf{W} + \mathbf{f}_n) - \mathbf{N}(\mathbf{x}_n)\right) \longrightarrow -\mathbf{L}^{-1}(\mathbf{g}) = \mathbf{x}.$$

Hence $\mathbf{N}(\mathbf{x}_n) \to \mathbf{N}(\mathbf{x})$ and $\mathbf{L}\mathbf{x} + \mathbf{N}(\mathbf{x}) = \mathbf{0}$. We apply Lemma 3.3 (ii) to conclude $\mathbf{x} = \mathbf{0}$, which contradicts $\|\mathbf{x}\| = 1$.

Case (ii): $\|\mathbf{w}_n\|$ *is bounded in* n. In this case we may assume (after passing to suitable subsequences) that $\mathbf{w}_n \rightharpoonup \mathbf{w}$, $\mathbf{N}(\mathbf{w}_n) \rightharpoonup \mathbf{h}$ with suitable $\mathbf{w}, \mathbf{h} \in \mathbf{H}$. From the equation

$$\mathbf{L}\mathbf{w}_n + \mathbf{N}(\mathbf{w}_n) = \mathbf{W} + \mathbf{f}_n \tag{4.13}$$

we conclude (using again the compactness of $\mathbf{L}^{-1}$)

$$\mathbf{w}_n = \mathbf{L}^{-1}(\mathbf{W} + \mathbf{f}_n - \mathbf{N}(\mathbf{w}_n)) \quad \longrightarrow \quad \mathbf{L}^{-1}(\mathbf{W} - \mathbf{h}) = \mathbf{w}$$

and hence $\mathbf{N}(\mathbf{w}_n) \to \mathbf{N}(\mathbf{w})$, which implies

$$\mathbf{L}\mathbf{w} + \mathbf{N}(\mathbf{w}) = \mathbf{W}. \tag{4.14}$$

If we compare (4.14) with the equation

$$\mathbf{L}\mathbf{w}_0 + \mathbf{N}(\mathbf{w}_0) = \mathbf{W} \tag{4.15}$$

which is valid due to our assumptions we conclude from Lemma 3.3 (ii) that $\mathbf{w} = \mathbf{w}_0$ and $\mathbf{w} \in H^{2,2} \times H^{4,2}$ (see Lemma 2.3 (ii) and Remark 3.4).

We now estimate the difference $\mathbf{w}_n - \mathbf{w}$ in two steps. First, we get

$$\begin{aligned} \|\mathbf{w}_n - \mathbf{w}\|_{1,1} &\leqslant \|\mathbf{L}^{-1}\|_{\mathscr{B}(\mathbf{H},\mathbf{H}^{1,1})} \|\mathbf{f}_n + \mathbf{N}(\mathbf{w}) - \mathbf{N}(\mathbf{w}_n)\| \\ &\leqslant \|\mathbf{L}^{-1}\|_{\mathscr{B}(\mathbf{H},\mathbf{H}^{1,1})} \left(\|\mathbf{f}_n\| + \|\mathbf{N}(\mathbf{w}) - \mathbf{N}(\mathbf{w}_n)\|\right) \end{aligned} \tag{4.16}$$

and conclude $\|\mathbf{w}_n - \mathbf{w}\|_{1,1} \to 0$ as $n \to \infty$. In the second step we estimate

(4.17)
$$\begin{aligned} \|\mathbf{w}_n - \mathbf{w}\|_{H^{2,2}\times H^{3,2}} &\leqslant \|\mathbf{L}^{-1}\|_{\mathscr{B}(\mathbf{H}^{1,1},H^{2,2}\times H^{3,2})} \|\mathbf{f}_n + \mathbf{N}(\mathbf{w}) - \mathbf{N}(\mathbf{w}_n)\|_{1,1} \\ &\leqslant \|\mathbf{L}^{-1}\|_{\mathscr{B}(\mathbf{H}^{1,1},H^{2,2}\times H^{3,2})} \left(\|\mathbf{f}_n\|_{1,1} + \|\mathbf{N}(\mathbf{w}) - \mathbf{N}(\mathbf{w}_n)\|_{1,1}\right) \end{aligned}$$

and conclude again $\|\mathbf{w}_n - \mathbf{w}\|_{H^{2,2}\times H^{3,2}} \to 0$ as $n \to \infty$, since $\|\mathbf{f}_n\|_{2,2} \to 0$ as well as $\|\mathbf{w}_n - \mathbf{w}\|_{1,1} \to 0$ and because $\mathbf{N}$ operates continuously from $\mathbf{H}^{1,1}$ to $\mathbf{H}^{1,1}$ (see Lemma 2.4 of [BDL]). If we put $\mathbf{w} = (v, u)^t$, $\mathbf{w}_n = (v_n, u_n)^t$ and apply Lemma 2.2 (ii) we may conclude from $\|\mathbf{w}_n - \mathbf{w}\|_{H^{2,2}\times H^{3,2}} \to 0$ that with $Q = [0, \pi] \times [0, 2\pi]$

$$\lim_{n\to\infty} \|u_n - u\|_{C^1(Q)} = 0. \tag{4.18}$$

Unfortunately, a similar result for $\|v_n - v\|_{C^1(Q)}$ does not follow from the convergence $\|\mathbf{w}_n - \mathbf{w}\|_{H^{2,2}\times H^{3,2}} \to 0$, since $\widehat{H}^{2,2}$ is not embedded into $\widehat{C}^{1,1}$. Therefore concerning $v_n - v$ we have to argue in a totally different way. It is exactly this point where the results of Appendix A now enter crucially.

Subtracting the first equation (4.14) from that of (4.13) it is easy to see that $v_n - v$ satisfies the equation

$$\mathbb{W}(v_n - v) = f_{n,1} + k_1 \left[(u_n - v_n)^+ - (u - v)^+\right]. \tag{4.19}$$

We would like to show that the right hand side

$$g_{n,1} := f_{n,1} + k_1 \left[(u_n - v_n)^+ - (u - v)^+\right] \tag{4.20}$$

satisfies

$$\lim_{n\to\infty} \|g_{n,1}\|_{C(I,H^1_{2\pi}(\mathbb{R}))} = 0. \tag{4.21}$$

To see this we apply Lemma A.3 and obtain

$$\begin{aligned}
\|g_{n,1}\|_{C(I,H^1_{2\pi}(\mathbb{R}))} &\leqslant \|f_{n,1}\|_{C(I,H^1_{2\pi}(\mathbb{R}))} + k_1 \left\|(u_n - v_n)^+ - (u-v)^+\right\|_{C(I,H^1_{2\pi}(\mathbb{R}))} \\
&\leqslant c\,\|f_{n,1}\|_{2,2} + k_1 \|u_n - v_n - (u-v)\|_{C(I,H^1_{2\pi}(\mathbb{R}))} \\
&\leqslant c\,\|\mathbf{f}_n\|_{2,2} + ck_1 \|u_n - v_n - (u-v)\|_{2,2} \\
&\leqslant c\,\left(\|\mathbf{f}_n\|_{2,2} + \sqrt{2}k_1 \|\mathbf{w}_n - \mathbf{w}\|_{2,2}\right) \longrightarrow 0 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Notice that due to Remark 3.4 and to the uniqueness of the solution to (4.8) (cf. Lemma 3.3), $u - v$ is nonnegative and time-independent since $\mathbf{w} = \mathbf{w}_0$ is a strictly positive solution of (4.8) with a time-independent right hand side $\mathbf{W}$.

Now the remaining nontrivial results of Appendix A are used. We apply Lemma A.1 to $u := v_n - v$ and $g := g_{n,1}$ (see formula (4.19), (4.20)) to see that

$$|\!|\!| v_n - v |\!|\!| \leqslant c\,\|g_{n,1}\|_{C(I,H^1_{2\pi}(\mathbb{R}))}. \tag{4.22}$$

Finally we use Lemma A.2 with $u = v_n - v$ (and the notation there) to obtain

$$\|v_n - v\|_{C^1(Q)} \leqslant c'\, |\!|\!| v_n - v |\!|\!|_X. \tag{4.23}$$

Since $|\!|\!| \cdot |\!|\!|_X \leqslant |\!|\!| \cdot |\!|\!|$ we can combine (4.22) and (4.23) to get

$$\|v_n - v\|_{C^1(Q)} \leqslant c''\, \|g_{n,1}\|_{C(I,H^1_{2\pi}(\mathbb{R}))}. \tag{4.24}$$

From (4.24) and (4.21) we conclude $\lim\limits_{n\to\infty} \|v_n - v\|_{C^1(Q)} = 0$ and including (4.18),

$$\lim_{n\to\infty} \|\mathbf{w}_n - \mathbf{w}\|_{\mathbf{C}^1(Q)} = 0. \tag{4.25}$$

Since $\mathbf{w} = \mathbf{w}_0$ is strictly positive it is now clear by (4.25) that $\mathbf{w}_n$ is strictly positive for large values of n, too. But this means that $\mathbf{w}_n$ as well as $\mathbf{w}_{\mathbf{f}_n}$ (see Step 1 for the definition of $\mathbf{w}_{\mathbf{f}_n}$) satisfy the linear string-beam equations $\mathbf{L}\mathbf{w} + \mathbf{M}\mathbf{w} = \mathbf{W} + \mathbf{f}_n$. Applying Proposition 3.1 we conclude $\mathbf{w}_n = \mathbf{w}_{\mathbf{f}_n}$, which contradicts (4.12).

To prove the estimate (4.11) we put $\mathbf{w} = \mathbf{w}_1 - \mathbf{w}_2$, $\mathbf{f} = \mathbf{f}_1 - \mathbf{f}_2$ and remark that $\mathbf{w}$ satisfies the linear string-beam equation $\mathbf{L}\mathbf{w} + \mathbf{M}\mathbf{w} = \mathbf{f}$. Clearly the estimate (4.11) is established if we show $(\mathbf{L} + \mathbf{M})^{-1} \in \mathscr{B}(\mathbf{H}^{2,2}, \mathbf{H}^{3,3})$.

Since we know from the estimate (3.18) of Proposition 3.1 that $(\mathbf{L}+\mathbf{M})^{-1}$ is bounded from $\mathbf{H}$ to $\mathbf{H}$ it suffices to show (due to the closed graph theorem—see [W])

$$(\mathbf{L}+\mathbf{M})^{-1}(\mathbf{H}^{2,2}) \subset \mathbf{H}^{3,3}. \tag{4.26}$$

Indeed, the boundedness of $(\mathbf{L}+\mathbf{M})^{-1}$ from $\mathbf{H}$ to $\mathbf{H}$ implies the closedness of $(\mathbf{L}+\mathbf{M})^{-1}$ from $\mathbf{H}$ to $\mathbf{H}$. Now, (4.26) implies the closedness of $(\mathbf{L}+\mathbf{M})^{-1}$ from $\mathbf{H}^{2,2}$ to $\mathbf{H}^{3,3}$. An application of the closed graph theorem implies $(\mathbf{L}+\mathbf{M})^{-1} \in \mathscr{B}(\mathbf{H}^{2,2}, \mathbf{H}^{3,3})$.

To see (4.26) consider the equation $\mathbf{Lw}+\mathbf{Mw}=\mathbf{f}$ with $\mathbf{f}\in\mathbf{H}^{2,2}$. Since $\mathbf{w}\in\mathscr{D}(\mathbf{L})\subset\mathbf{H}^{1,1}$ we conclude $\mathbf{Lw}=\mathbf{f}-\mathbf{Mw}\in\mathbf{H}^{1,1}$ and thus by regularity (see Lemma 2.3 (iii)) $\mathbf{w}\in\mathbf{H}^{2,2}$. But then $\mathbf{Lw}=\mathbf{f}-\mathbf{Mw}\in\mathbf{H}^{2,2}$ and using again Lemma 2.3 (iii) we conclude $\mathbf{w}\in\mathbf{H}^{3,3}$ proving (4.26). □

Let us now apply Theorem 4.1 to suspension bridges.

Theorem 4.2. *Suppose we are given a suspension bridge modelled by* (SB) *where the weight of the main cable and the weight of the roadbed is assumed to be constant (i.e.,* $W_1(x)\equiv W_1>0$, $W_2(x)\equiv W_2>0$*).*

Then there exist positive constants c *and* ε *such that for* $\frac{W_1}{W_2}<c$ *and for any pair* $\mathbf{f}=(f_1,f_2)^t\in\mathbf{H}^{2,2}$ *of external forcing terms satisfying*

$$\|\mathbf{f}\|_{2,2}<\varepsilon \tag{4.27}$$

there is a unique solution $\mathbf{w}=(v,u)^t$ *of* (SB). *Moreover,* $\mathbf{w}\in H^{2,2}\times H^{4,2}$ *and* $\mathbf{w}$ *is strictly positive. In addition, for any two solutions* $\mathbf{w}_1$, $\mathbf{w}_2$ *of* (SB) *with the corresponding right hand sides* $\mathbf{W}+\mathbf{f}_1$, $\mathbf{W}+\mathbf{f}_2$ *the estimate*

$$\|\mathbf{w}_1-\mathbf{w}_2\|_{3,3}\leqslant c\|\mathbf{f}_1-\mathbf{f}_2\|_{2,2} \tag{4.28}$$

holds with a positive constant c*, provided* $\|\mathbf{f}_i\|_{2,2}<\varepsilon$ $(i=1,2)$*.*

Proof. Let $W_1(x)\equiv W_1>0$ and $W_2(x)\equiv W_2>0$. Theorem 3.1 implies that (SB) with $f_1\equiv 0$, $f_2\equiv 0$ admits at least one weak solution $\mathbf{w}_0\in H^{1,1}\times H^{2,1}$. This solution does not depend on time due to Remark 3.4. Take this solution $\mathbf{w}_0=(v_0,u_0)^t$ and put $z=u_0-v_0$. Clearly z satisfies $(\mathscr{B}_2)$ with γ_i $(i=1,2)$ given by (B.1). Applying Proposition B.1 (for $\gamma_2\geqslant 4\gamma_1^2$) and Proposition B.2 (for $\gamma_2<4\gamma_1^2$) of Appendix B we conclude that z is positive in $]0,\pi[$ and satisfies $z'(0)>0$, $z'(\pi)<0$ provided $c>0$ is small enough. This proves the strict positivity of $\mathbf{w}_0=(v_0,u_0)^t$. Now, for $\varepsilon>0$ small and $\mathbf{f}=(f_1,f_2)^t\in\mathbf{H}^{2,2}$ satisfying $\|\mathbf{f}\|_{2,2}<\varepsilon$ apply Theorem 4.1 to get a unique and strictly positive solution $\mathbf{w}\in H^{2,2}\times H^{4,2}$ of (SB). The estimate (4.28) is an immediate consequence of (4.11). □

R e m a r k 4.3. In the cases $\kappa > \frac{4T^2}{EI}$ and $\kappa = \frac{4T^2}{EI}$ explicit formulas can be given for the constant c appearing in Theorem 4.2, namely:

Case $\kappa > \frac{4T^2}{EI}$:

$$c = c(\kappa, E, I, T, L) = \frac{T}{\sqrt{\kappa EI}} \frac{a_+ \tanh a_- - a_- \tanh a_+}{a_+ \tanh a_+ - a_- \tanh a_-}$$

where $a_\pm = L\sqrt{\frac{\kappa}{8T}}\sqrt{1 \pm \sqrt{1 - \frac{4T^2}{\kappa EI}}}$.

Case $\kappa = \frac{4T^2}{EI}$:

$$c = c(\kappa, T, L) = \frac{1}{2} \frac{\sinh\sqrt{\frac{\kappa L^2}{2T}} - \sqrt{\frac{\kappa L^2}{2T}}}{\sinh\sqrt{\frac{\kappa L^2}{2T}} + \sqrt{\frac{\kappa L^2}{2T}}}.$$

A. Appendix

The purpose of this appendix is to provide very special results concerning solutions of the string equation $\mathbb{W}u = g$ that are crucially needed in the proof of our basic result—Theorem 4.1. To do so we first need some notation.

For any $\omega \in \mathbb{R}^+$, $p \in \mathbb{N} \cup \{0\}$ we denote by $H^p_\omega(\mathbb{R})$ the space of all ω-periodic $u \in \mathscr{D}'(\mathbb{R}) \cap H^p_{\mathrm{loc}}(\mathbb{R})$ endowed with the norm $\|u\|_{H^p_\omega(\mathbb{R})} = \Big(\sum\limits_{0\leqslant i\leqslant p} \int\limits_0^\omega |u^{(i)}|^2\Big)^{1/2}$. If I is a compact interval, Y a Banach space and $p \in \mathbb{N} \cup \{0\}$ then $C^p(I, Y)$ is the space of p-times continuously Fréchet differentiable functions $f\colon I \to Y$ with norm $\|f\|_{C^p(I,X)} = \max\limits_{0\leqslant i\leqslant p} \sup\limits_{x\in I} \|f^{(i)}(x)\|_Y$. For simplicity we will write $C(I, Y)$ for $C^0(I, Y)$.

Lemma A.1. *Let $I = [0, \pi]$, $\mathbb{W} = \mathbb{W}(\alpha, \beta)$ and suppose $\mathbb{W}u = g$ with $g \in C(I, H^1_{2\pi}(\mathbb{R}))$. Then there exists a positive constant c independent of u and g such that*

$$\|\!|u|\!\| \leqslant c\, \|g\|_{C(I, H^1_{2\pi}(\mathbb{R}))}, \tag{A.1}$$

where $\|\!|u|\!\| = \max\limits_{0\leqslant i\leqslant 2} \sup\limits_{x\in I} \|\partial^i_x u(x, \cdot)\|_{H^{2-i}_{2\pi}(\mathbb{R})}$.

R e m a r k A.1. Notice that the right hand side g of $\mathbb{W}u = g$ belongs to H and (A.1) implies $u \in C(I, H^2_{2\pi}(\mathbb{R})) \cap C^1(I, H^1_{2\pi}(\mathbb{R}) \cap C^2(I, H^0_{2\pi}(\mathbb{R}))$.

P r o o f (of Lemma A.1). Clearly, the equation $\mathbb{W}u = g$ is related to the boundary value problem

$$(\mathscr{T}) \qquad \begin{cases} v_{tt} - \alpha^2 v_{xx} + \beta v_t = g, \\ v(0, t) = v(\pi, t) = 0, \\ v(x, t + 2\pi) = v(x, t), \qquad x \in I, t \in \mathbb{R}. \end{cases}$$

The strategy of our proof will be to show that $(\mathscr{T})$ has a *strong solution* v satisfying (A.1) and then to prove $u = v$. Notice that by a *strong solution* of $(\mathscr{T})$ we mean a real valued function $v \in C(I, H^2_{2\pi}(\mathbb{R})) \cap C^1(I, H^1_{2\pi}(\mathbb{R})) \cap C^2(I, H^0_{2\pi}(\mathbb{R}))$ satisfying the first equation of $(\mathscr{T})$ for all $x \in \,]0, \pi[$ and the other two in the sense of the space $H^0_{2\pi}(\mathbb{R})$. Using $a = \beta/\alpha$ and the transformations

$$w(x,t) := v(x, t/\alpha); \quad f(x,t) := \frac{1}{\alpha^2} g(x, t/\alpha) \tag{A.2}$$

it is easy to see that v is a strong solution of $(\mathscr{T})$ satisfying (A.1) (with u replaced by v) if and only if w is a strong solution of

$$(\mathscr{P}_\omega) \qquad \begin{cases} w_{tt} - w_{xx} + a w_t = f, \\ w(0,t) = w(\pi, t) = 0, \\ w(x, t+\omega) = w(x,t), \qquad x \in I, t \in \mathbb{R} \end{cases}$$

with $\omega = 2\pi\alpha$.

Now we may apply the results obtained in [V], chapter IV, section 1.3 to show that problem $(\mathscr{P}_\omega)$ admits a strong solution w which satisfies

$$|\!|\!| w |\!|\!| \leqslant c \, \| f \|_{C(I, H^1_\omega(\mathbb{R}))}$$

with $\omega = 2\pi\alpha$ and $|\!|\!| w |\!|\!| = \max\limits_{0 \leqslant i \leqslant 2} \sup\limits_{x \in I} \| \partial^i_x w(x, \cdot) \|_{H^{2-i}_\omega(\mathbb{R})}$. Using the inverses of the transformations (A.2) applied to w it is easy to see that problem $(\mathscr{T})$ has a strong solution v satisfying (A.1) with u replaced by v.

In the last step we shall show that v is a weak solution of $(\mathscr{T})$ meaning that $v \in \mathscr{D}(\mathbb{W})$ and $\mathbb{W}v = g$ and thus $u = v$ since $\mathbb{W}$ is one-to-one $(0 \neq \sigma(\mathbb{W}))$. To do so, let $\varphi \in \mathscr{C}$ (see (2.23) for definition of $\mathscr{C}$) and denote $J = [0, 2\pi]$. Then for all $x \in \,]0, \pi[$

$$\begin{aligned} &\int_J v_{tt}(x,t)\varphi(x,t)\,\mathrm{d}t - \alpha^2 \int_J v_{xx}(x,t)\varphi(x,t)\,\mathrm{d}t + \beta \int_J v_t(x,t)\varphi(x,t)\,\mathrm{d}t \\ &= \int_J g(x,t)\varphi(x,t)\,\mathrm{d}t. \end{aligned}$$

If we integrate each term of this equation with respect to x over I we get, setting $Q = I \times J$,

$$\int_Q v_{tt}\varphi - \alpha^2 \int_Q v_{xx}\varphi + \beta \int_Q v_t \varphi = \int_Q g\varphi. \tag{A.3}$$

Let us consider each integral on the left hand side of (A.3) separately. Recalling that $v(x,\cdot) \in H^2_{2\pi}(\mathbb{R})$ for all $x \in I$ and integrating twice by parts we get

$$\begin{aligned}
\int_Q v_{tt}\varphi &= \int_I \left(\int_J v_{tt}(x,t)\varphi(x,t)\,\mathrm{d}t \right) \mathrm{d}x \\
&= \int_I \left(\underbrace{v_t(x,t)\varphi(x,t)\Big|_{t=0}^{t=2\pi}}_{=0} - \int_J v_t(x,t)\varphi_t(x,t)\,\mathrm{d}t \right) \mathrm{d}x \\
&= \int_I \left(\underbrace{-v(x,t)\varphi_t(x,t)\Big|_{t=0}^{t=2\pi}}_{=0} + \int_J v(x,t)\varphi_{tt}(x,t)\,\mathrm{d}t \right) \mathrm{d}x = \int_Q v\varphi_{tt}
\end{aligned}$$

as well as

$$\int_Q v_t\varphi = \int_I \left(v(x,t)\varphi(x,t)|_{t=0}^{t=2\pi} - \int_J v(x,t)\varphi_t(x,t)\,\mathrm{d}t \right) \mathrm{d}x = -\int_Q v\varphi_t.$$

To attack $\int_Q v_{xx}\varphi$ we have to argue differently:

$$\int_Q v_{xx}\varphi = \int_I \left(\int_J v_{xx}(x,t)\varphi(x,t)\,\mathrm{d}t \right) \mathrm{d}x = \int_I \langle v_{xx}(x,\cdot), \varphi(x,\cdot)\rangle \,\mathrm{d}x$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(J)$. Integrating the identity

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\langle v_x(x,\cdot), \varphi(x,\cdot)\rangle - \langle v(x,\cdot), \varphi_x(x,\cdot)\rangle\right] = \langle v_{xx}(x,\cdot), \varphi(x,\cdot)\rangle - \langle v(x,\cdot), \varphi_{xx}(x,\cdot)\rangle$$

over I yields

$$\begin{aligned}
&\int_I \langle v_{xx}(x,\cdot), \varphi(x,\cdot)\rangle \,\mathrm{d}x - \int_I \langle v(x,\cdot), \varphi_{xx}(x,\cdot)\rangle \,\mathrm{d}x \\
&\qquad = \left[\langle v_x(x,\cdot), \varphi(x,\cdot)\rangle - \langle v(x,\cdot), \varphi_x(x,\cdot)\rangle\right]\Big|_{x=0}^{x=\pi} = 0
\end{aligned}$$

due to $\varphi \in \mathscr{C}$ and the boundary conditions ($\mathscr{T}$) for v. This shows $\int_Q v_{xx}\varphi = \int_Q v\varphi_{xx}$ and thus

$$\int_Q v[\varphi_{tt} - \alpha^2\varphi_{xx} - \beta\varphi_t] = \int_Q g\varphi.$$

$\square$

To show that the solution u of $\mathbb{W}u = g$ in Lemma A.1 is 'smooth' we need a specific embedding result. Set $I = [0, \pi]$, $J = [0, 2\pi]$, $Q = I \times J$ and $X = C(I, H^2_{2\pi}(\mathbb{R})) \cap C^1(I, H^1_{2\pi}(\mathbb{R}))$ with the norm $\|u\|_X = \max\limits_{0 \leqslant i \leqslant 1} \sup\limits_{x \in I} \|\partial^i_x u(x, \cdot)\|_{H^{2-i}_{2\pi}(\mathbb{R})}$.

Lemma A.2. *There is a positive constant c such that*

$$\|u\|_{C^1(Q)} \leqslant c\,\|u\|_X \qquad (u \in X). \tag{A.4}$$

P r o o f. Take $u \in X$, then $u \in C(I, H^2_{2\pi}(\mathbb{R}))$ which yields $u(x, \cdot) \in H^2_{2\pi}(\mathbb{R})$ and thus $u(x, \cdot) \in H^1_{2\pi}(\mathbb{R})$ as well as $\partial_t u(x, \cdot) \in H^1_{2\pi}(\mathbb{R})$ for all $x \in I$. Since $H^1_{2\pi}(\mathbb{R})$ is continuously embedded into the space $C_{2\pi}(\mathbb{R})$ of continuous 2π-periodic functions on $\mathbb{R}$ (see e.g., [V], Theorem 2.7.4) we conclude $u(x, \cdot) \in C_{2\pi}(\mathbb{R})$ for $x \in I$ and

$$|u(x,t) - u(x_0,t)| \leqslant c\,\|u(x,\cdot) - u(x_0,\cdot)\|_{H^1_{2\pi}(\mathbb{R})} \longrightarrow 0 \tag{A.5}$$

for x, $x_0 \in I$, $t \in J$, provided $x \to x_0$. Since $u(x_0, \cdot) \in C_{2\pi}(\mathbb{R})$ formula (A.5) yields $u \in C(Q)$. The same argument applied to $\partial_t u(x, \cdot)$ instead of $u(x, \cdot)$ establishes $\partial_t u \in C(Q)$. An application of $H^1_{2\pi}(\mathbb{R}) \hookrightarrow C_{2\pi}(\mathbb{R})$ to $u(x, \cdot)$, and $\partial_t u(x, \cdot)$, respectively, gives

$$\left.\begin{aligned} |u(x,t)| &\leqslant c\,\|u(x,\cdot)\|_{H^1_{2\pi}(\mathbb{R})} \leqslant c\,\|u\|_X \\ |\partial_t u(x,t)| &\leqslant c\,\|\partial_t u(x,\cdot)\|_{H^1_{2\pi}(\mathbb{R})} \leqslant c\|u\|_X \end{aligned}\right\} \quad (x \in I, t \in J)$$

and thus

$$\|u\|_{C(Q)} \leqslant c\,\|u\|_X; \qquad \|\partial_t u\|_{C(Q)} \leqslant c\,\|u\|_X. \tag{A.6}$$

To prove $\|\partial_x u\|_{C(Q)} \leqslant c\|u\|_X$ we argue as follows.

Since $u \in X$ implies $\frac{\mathrm{d}}{\mathrm{d}x} u \in C(I, H^1_{2\pi}(\mathbb{R}))$ we know by repeating the argument given in connection with (A.5) that $\frac{\mathrm{d}}{\mathrm{d}x} u \in C(Q)$. To see that u has a classical partial derivative with respect to x that is also continuous let us argue as follows.

Taking $t \in \mathbb{R}$, $x \in I$, $x + h \in I$ with $h \neq 0$ and using $H^1_{2\pi}(\mathbb{R}) \hookrightarrow C_{2\pi}(\mathbb{R})$ we conclude

$$\left|\frac{u(x+h,t) - u(x,t)}{h} - \frac{\mathrm{d}}{\mathrm{d}x} u(x,t)\right| \leqslant c\,\left\|\frac{u(x+h,\cdot) - u(x,\cdot)}{h} - \frac{\mathrm{d}}{\mathrm{d}x} u(x,\cdot)\right\|_{H^1_{2\pi}} \longrightarrow 0$$

as $h \to 0$. Hence for arbitrary $t \in \mathbb{R}$, $x \in I$ the function u has a classical partial derivative with respect to x in (x, t), namely $\partial_x u(x,t) = \frac{\mathrm{d}}{\mathrm{d}x} u(x,t)$. Since $\frac{\mathrm{d}}{\mathrm{d}x} u \in C(Q)$ we conclude that $\partial_x u \in C(Q)$ and for any $t \in J$, $x \in I$ the following estimate holds:

$$|\partial_x u(x,t)| = \left|\frac{\mathrm{d}}{\mathrm{d}x} u(x,t)\right| \leqslant c\,\left\|\frac{\mathrm{d}}{\mathrm{d}x} u(x,\cdot)\right\|_{H^1_{2\pi}(\mathbb{R})} \leqslant c\,\|u\|_X.$$

Combining the last estimate with (A.6) we obtain (A.4). $\square$

Lemma A.3. *Let $I = [0, \pi]$ and $g \in H^{2,2}$. Then g, $g^+ \in C(I, H^1_{2\pi}(\mathbb{R}))$ and there is a positive constant c independent of g such that*

$$\|g\|_{C(I,H^1_{2\pi}(\mathbb{R}))} + \|g^+\|_{C(I,H^1_{2\pi}(\mathbb{R}))} \leqslant c\,\|g\|_{2,2}. \tag{A.7}$$

If moreover $f \in H^{2,2}$ is nonnegative and time-independent then

$$\|g^+ - f^+\|_{C(I,H^1_{2\pi}(\mathbb{R}))} \leqslant \|g - f\|_{C(I,H^1_{2\pi}(\mathbb{R}))}. \tag{A.8}$$

Proof. The proof is carried out in four steps. To simplify notation we put $J = [0, 2\pi]$, $Q = I \times J$ and remark that the norm of $H^0_{2\pi}(\mathbb{R})$ is exactly that of $L^2(J)$. Moreover, it should be kept in mind that $g \in C(I, H^1_{2\pi}(\mathbb{R}))$ if and only if $g \in C(I, H^0_{2\pi}(\mathbb{R}))$ and $\partial_t g \in C(I, H^0_{2\pi}(\mathbb{R}))$.

Step 1. $g, g^+ \in C(I, H^0_{2\pi}(\mathbb{R}))$.

Since $\widehat{H}^{2,2} \hookrightarrow \widehat{C}(\mathbb{R}^2)$ there is a constant $c > 0$ such that

$$|g(x,t)| \leqslant c \qquad (x \in I, t \in \mathbb{R}). \tag{A.9}$$

If $x_n, x \in I$ and $x_n \to x$ then for all $t \in \mathbb{R}$

$$|g^+(x_n,t) - g^+(x,t)| \leqslant |g(x_n,t) - g(x,t)|,$$

hence

$$\|g^+(x_n,\cdot) - g^+(x,\cdot)\|_{L^2(J)} \leqslant \|g(x_n,\cdot) - g(x,\cdot)\|_{L^2(J)}, \tag{A.10}$$

where

$$\|g(x_n,\cdot) - g(x,\cdot)\|^2_{L^2(J)} = \int_J |g(x_n,t) - g(x,t)|^2 \,\mathrm{d}t \longrightarrow 0$$

as $n \to \infty$, since $g(x_n,t) \to g(x,t)$ for all $t \in J$. This shows $g, g^+ \in C(I, H^0_{2\pi}(\mathbb{R}))$.

Step 2. $\partial_t g, \partial_t g^+ \in C(I, H^0_{2\pi}(\mathbb{R}))$.

Since $\partial_t g \in H^{1,1}$ we know from Lemma 5.6.2 of [KJF] that $\partial_t g(\cdot,t) \in H^1_{\mathrm{loc}}(\mathbb{R})$ a.e. in $t \in \mathbb{R}$. Hence for all $x_1, x_2 \in I$ and a.e. in $t \in \mathbb{R}$

$$\partial_t g(x_1,t) - \partial_t g(x_2,t) = \int_{x_2}^{x_1} \partial_y \partial_t g(y,t)\,\mathrm{d}y, \tag{A.11}$$

which yields

$$|\partial_t g(x_1,t) - \partial_t g(x_2,t)|^2 = \left|\int_{x_2}^{x_1} \partial_y \partial_t g(y,t)\,\mathrm{d}y\right|^2. \tag{A.12}$$

If we integrate (A.12) with respect to t over J, we obtain

$$\begin{aligned}\|\partial_t g(x_1,\cdot)-\partial_t g(x_2,\cdot)\|^2_{L^2(J)} &\leqslant \int_J \left|\int_{x_2}^{x_1} \partial_y\partial_t g(y,t)\,\mathrm{d}y\right|^2 \mathrm{d}t \\ &\leqslant |x_1-x_2| \int_J \int_I |\partial_y\partial_t g(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t \\ &\leqslant |x_1-x_2|\cdot\|g\|^2_{2,2},\end{aligned}$$

which finally gives

$$\|\partial_t g(x_1,\cdot)-\partial_t g(x_2,\cdot)\|_{L^2(J)} \leqslant |x_1-x_2|^{1/2}\|g\|_{2,2}. \tag{A.13}$$

In particular, we conclude $\partial_t g \in C(I, H^0_{2\pi}(\mathbb{R}))$, hence $g \in C(I, H^1_{2\pi}(\mathbb{R}))$. To prove $\partial_t g^+ \in C(I, H^0_{2\pi}(\mathbb{R}))$ we multiply identity (A.11) by $\chi_{\{g>0\}}(x_1,t)$ and obtain

$$\chi_{\{g>0\}}(x_1,t)\partial_t g(x_1,t) - \chi_{\{g>0\}}(x_1,t)\partial_t g(x_2,t) = \chi_{\{g>0\}}(x_1,t)\int_{x_2}^{x_1} \partial_y\partial_t g(y,t)\,\mathrm{d}y.$$

Hence

(A.14)

$$\begin{aligned}&\chi_{\{g>0\}}(x_1,t)\partial_t g(x_1,t) - \chi_{\{g>0\}}(x_2,t)\partial_t g(x_2,t) \\ &= [\chi_{\{g>0\}}(x_1,t)-\chi_{\{g>0\}}(x_2,t)]\partial_t g(x_2,t) + \chi_{\{g>0\}}(x_1,t)\int_{x_2}^{x_1} \partial_y\partial_t g(y,t)\,\mathrm{d}y\end{aligned}$$

is valid for all $x_1, x_2 \in I$ and a.e. in $t \in \mathbb{R}$. Since $g(x,\cdot) \in H^1_{2\pi}(\mathbb{R})$ for all $x \in I$ we know that

$$\partial_t g^+(x,t) = \chi_{\{g(x,\cdot)>0\}}(t)\partial_t g(x,t) = \chi_{\{g>0\}}(x,t)\partial_t g(x,t)$$

for all $x \in I$ and a.e. in $t \in \mathbb{R}$. Thus (A.14) can be rewritten as

(A.15)

$$\begin{aligned}&\partial_t g^+(x_1,t) - \partial_t g^+(x_2,t) \\ &= [\chi_{\{g(x_1,\cdot)>0\}}(t)-\chi_{\{g(x_2,\cdot)>0\}}(t)]\partial_t g(x_2,t) + \chi_{\{g>0\}}(x_1,t)\int_{x_2}^{x_1} \partial_y\partial_t g(y,t)\,\mathrm{d}y.\end{aligned}$$

Using (A.15) we estimate

$$
\begin{aligned}
&\|\partial_t g^+(x_1,t) - \partial_t g^+(x_2,t)\|^2_{L^2(J)} \\
&\leqslant 2\int_J \left|(\chi_{\{g(x_1,\cdot)>0\}}(t) - \chi_{\{g(x_2,\cdot)>0\}}(t))\right|^2 |\partial_t g(x_2,t)|^2 \,\mathrm{d}t + 2\int_J \left|\int_{x_2}^{x_1} \partial_y\partial_t g(y,t)\,\mathrm{d}y\right|^2 \mathrm{d}t.
\end{aligned}
$$

Introducing $A(x_1,x_2)$ by setting

$$
A(x_1,x_2) = \int_J \left|(\chi_{\{g(x_1,\cdot)>0\}}(t) - \chi_{\{g(x_2,\cdot)>0\}}(t))\right|^2 |\partial_t g(x_2,t)|^2 \,\mathrm{d}t
$$

we end up with the inequality

$$
\|\partial_t g^+(x_1,\cdot) - \partial_t g^+(x_2,\cdot)\|^2_{L^2(J)} \leqslant 2A(x_1,x_2) + 2|x_1-x_2|\|g\|^2_{2,2} \tag{A.16}
$$

being valid for any $x_1,\ x_2 \in I$. Let us put $x_2 = x$ and $x_1 = x_n$ and assume $\lim\limits_{n\to\infty} x_n = x$. Then (A.16) reads

$$
\|\partial_t g^+(x_n,\cdot) - \partial_t g^+(x,\cdot)\|^2_{L^2(J)} \leqslant 2A(x_n,x) + 2|x_n-x|\|g\|^2_{2,2}. \tag{A.17}
$$

From inequality (A.17) we conclude $\partial_t g^+ \in C(I, H^0_{2\pi}(\mathbb{R}))$ and thus $g^+ \in C(I, H^1_{2\pi}(\mathbb{R}))$ provided $A(x_n,x) \to 0$ as $n\to\infty$. To see this recall

$$
\begin{aligned}
A(x_n,x) &= \int_J \left|\chi_{\{g(x_n,\cdot)>0\}}(t) - \chi_{\{g(x,\cdot)>0\}}(t)\right|^2 |\partial_t g(x,t)|^2 \,\mathrm{d}t \\
&= \int_{J\setminus\{g(x,\cdot)=0\}} \left|\chi_{\{g(x_n,\cdot)>0\}}(t) - \chi_{\{g(x,\cdot)>0\}}(t)\right|^2 |\partial_t g(x,t)|^2 \,\mathrm{d}t.
\end{aligned}
$$

The last equality holds true since $\partial_t g(x,\cdot)$ vanishes a.e. in t on the set $\{g(x,\cdot)=0\}$. In order to prove $\lim\limits_{n\to\infty} A(x_n,x) = 0$ let us introduce the functions

$$
f_n(t) = \left|\chi_{\{g(x_n,\cdot)>0\}}(t) - \chi_{\{g(x,\cdot)>0\}}(t)\right|^2 |\partial_t g(x,t)|^2.
$$

If $t \in J\setminus\{g(x,\cdot)=0\}$ then $g(x,t)>0$ or $g(x,t)<0$ and thus $g(x_n,t)>0$ or $g(x_n,t)<0$ for large n which means $f_n(t)=0$. Since $\partial_t g(x,\cdot)\in L^2(J)$ we conclude by the Lebesgue dominated convergence theorem that $\lim\limits_{n\to\infty} A(x_n,x)=0$.

Step 3. Proof of estimate (A.7).

To show inequality (A.7) we first note that if we put $x_n = x$ and $x=0$ in (A.10), noting that $g^+(0,\cdot) = g(0,\cdot) = 0$, we get

$$
\|g^+(x,\cdot)\|_{L^2(J)} \leqslant \|g(x,\cdot)\|_{L^2(J)}
$$

and thus

$$\sup_{x\in I}\|g^+(x,\cdot)\|_{L^2(J)} \leqslant \sup_{x\in I}\|g(x,\cdot)\|_{L^2(J)}. \tag{A.18}$$

Due to $\widehat{H}^{2,2} \hookrightarrow \widehat{C}(\mathbb{R}^2)$ there is a positive constant c such that

$$|g(x,t)| \leqslant c\|g\|_{2,2} \qquad (x \in I, t \in J),$$

which implies

$$\sup_{x\in I}\|g(x,\cdot)\|_{L^2(J)} \leqslant \sqrt{2\pi}c\|g\|_{2,2}. \tag{A.19}$$

Since we know that $g(x,\cdot) \in H^1_{2\pi}(\mathbb{R})$ for each $x \in I$ the formula

$$\partial_t g^+(x,t) = \chi_{\{g(x,\cdot)>0\}}(t)\partial_t g(x,t)$$

holds true a.e. in $t \in J$, which gives the estimate

$$\sup_{x\in I}\|\partial_t g^+(x,\cdot)\|_{L^2(J)} \leqslant \sup_{x\in I}\|\partial_t g(x,\cdot)\|_{L^2(J)}. \tag{A.20}$$

Finally, if we put $x_1 = x$, $x_2 = 0$ in (A.16) and notice that $\partial_t g^+(0,\cdot) = \partial_t g(0,\cdot) = 0$ and $A(x,0) = 0$ we obtain the estimate

$$\sup_{x\in I}\|\partial_t g^+(x,\cdot)\|_{L^2(J)} \leqslant \sqrt{2\pi}\|g\|_{2,2}. \tag{A.21}$$

The combination of (A.18), (A.19), (A.20) and (A.21) yields estimate (A.7).

Step 4. Proof of estimate (A.8).

First, we note that for any $x \in I$, $t \in J$

$$|g^+(x,t) - f^+(x,t)| \leqslant |g(x,t) - f(x,t)|,$$

which immediately gives

$$\|g^+ - f^+\|_{C(I,H^0_{2\pi}(\mathbb{R}))} \leqslant \|g - f\|_{C(I,H^0_{2\pi}(\mathbb{R}))}. \tag{A.22}$$

Second, we remark that $f^+ = f$ and $\partial_t f^+ = \partial_t f = 0$ since f is assumed to be nonnegative and time-independent. As a consequence,

$$\begin{aligned}&\|\partial_t g^+(x,\cdot) - \partial_t f^+(x,\cdot)\|_{L^2(J)} = \|\partial_t g^+(x,\cdot)\|_{L^2(J)}\\ &\quad = \|\chi_{\{g(x,\cdot)>0\}}\partial_t g(x,\cdot)\|_{L^2(J)} \leqslant \|\partial_t g(x,\cdot)\|_{L^2(J)} = \|\partial_t g(x,\cdot) - \partial_t f(x,\cdot)\|_{L^2(J)}\end{aligned}$$

for any $x \in I$ and thus

$$\|\partial_t g^+ - \partial_t f^+\|_{C(I,H^0_{2\pi}(\mathbb{R}))} \leqslant \|\partial_t g - \partial_t f\|_{C(I,H^0_{2\pi}(\mathbb{R}))}. \tag{A.23}$$

Now (A.22) and (A.23) yield estimate (A.8). □

B. Appendix

The aim of Appendix B is to show that under suitable conditions the stationary system $(\mathscr{S}\mathscr{B})$, i.e., $(\mathscr{S}\mathscr{B})$ with a time-independent right hand side $\mathbf{h} = (h_1(x), h_2(x))^t$, admits a strictly positive solution. Though this result is rather important for our paper (see Theorems 4.1 and 4.2) we nevertheless have put it into this appendix because its proof is technical and lengthy.

First of all, let us mention that due to Remark 3.4 any strictly positive solution $(u, v)^t$ of $(\mathscr{S}\mathscr{B})$ with a time-independent right hand side $\mathbf{h} = (h_1(x), h_2(x))^t$ is indeed a solution of the linear system

$$(\mathscr{B}_1) \qquad \begin{cases} -\gamma_1 v'' - (u - v) = h_1(x), \\ \gamma_2 u^{(4)} + (u - v) = h_2(x), \qquad (x \in \,]0, \pi[) \\ v(0) = v(\pi) = u(0) = u(\pi) = u''(0) = u''(\pi) = 0 \end{cases}$$

with

$$\gamma_i = \frac{\alpha_i^2}{k_i} \qquad (i = 1, 2) \tag{B.1}$$

where we have written (again) $h_i(x)$ for the rescaled functions $h_i(x)/k_i$. If we introduce a new function

$$z(x) = u(x) - v(x),$$

it can be easily seen that if $(v, u)^t$ is a solution of the boundary value problem $(\mathscr{B}_1)$ then z has to satisfy

$$(\mathscr{B}_2) \qquad \begin{cases} \gamma_2 z^{(4)} - \frac{\gamma_2}{\gamma_1} z'' + z = h_2(x) + \frac{\gamma_2}{\gamma_1} h_1''(x), \\ z(0) = z(\pi) = 0, \\ z''(0) = h_1(0)/\gamma_1; \quad z''(\pi) = h_1(\pi)/\gamma_1. \end{cases}$$

Under special assumptions on γ_1, γ_2; h_1, h_2 we can state the following assertions.

Proposition B.1. *Let $h_1(x) \equiv h_1$, $h_2(x) \equiv h_2$ be nonnegative real constants and $\gamma_2 \geqslant 4\gamma_1^2$. Moreover, let in case of $\gamma_2 > 4\gamma_1^2$ the condition*

$$\frac{h_1}{h_2} < ab\gamma_1 \frac{a \tanh b\frac{\pi}{2} - b \tanh a\frac{\pi}{2}}{a \tanh a\frac{\pi}{2} - b \tanh b\frac{\pi}{2}} \tag{B.2}$$

hold true, and in case of $\gamma_2 = 4\gamma_1^2$

$$\frac{h_1}{h_2} < a^2 \gamma_1 \frac{\sinh a\pi - a\pi}{\sinh a\pi + a\pi} \tag{B.3}$$

hold true, where

$$\text{(B.4)}\qquad a^2=\frac{1}{2\gamma_1}+\sqrt{\frac{1}{4\gamma_1^2}-\frac{1}{\gamma_2}};\qquad b^2=\frac{1}{2\gamma_1}-\sqrt{\frac{1}{4\gamma_1^2}-\frac{1}{\gamma_2}};\qquad a\geqslant b>0.$$

Then there exists a unique solution $z(x)$ of the boundary value problem $(\mathscr{B}_2)$ and it is symmetric with respect to $x=\frac{\pi}{2}$, positive for all $x=]0,\pi[$, and satisfies

$$z'(0)>0,\quad z'(\pi)<0.$$

P r o o f. The symmetry result is trivial since the differential equation as well as the boundary conditions of $(\mathscr{B}_2)$ are symmetric.

The proof will be carried out in two steps.

Step 1. It is not difficult to derive the explicit form of the solution of the boundary value problem $(\mathscr{B}_2)$. In case that $\gamma_2>4\gamma_1^2$ it has the form

$$z(x)=h_2+C_1[\sinh ax+\sinh a(\pi-x)]-C_2[\sinh bx+\sinh b(\pi-x)],$$

where a and b are given by (B.4) and

$$C_1=\left(\frac{h_1}{\gamma_1}+b^2h_2\right)\frac{1}{(a^2-b^2)\sinh a\pi},$$
$$C_2=\left(\frac{h_1}{\gamma_1}+a^2h_2\right)\frac{1}{(a^2-b^2)\sinh b\pi}.$$

In the latter case, i.e., $\gamma_2=4\gamma_1^2$, the solution has the form

$$\begin{aligned}z(x)=h_2&-\frac{h_2}{\sinh a\pi}[\sinh ax+\sinh a(\pi-x)]\\&+\frac{\frac{h_1}{\gamma_1}+a^2h_2}{2a\sinh a\pi}\left[x(\cosh ax-\cosh a(\pi-x))+\pi\frac{1-\cosh a\pi}{\sinh a\pi}\sinh ax\right],\end{aligned}$$

where $a^2=\frac{1}{2\gamma_1}$.

For the first derivative we obtain an expression

$$z'(0)=\frac{1}{a^2-b^2}\left[\left(\frac{h_1}{\gamma_1}+a^2h_2\right)b\tanh b\frac{\pi}{2}-\left(\frac{h_1}{\gamma_1}+b^2h_2\right)a\tanh a\frac{\pi}{2}\right]$$

in the first case and

$$z'(0)=\frac{1-\cosh a\pi}{\sinh a\pi}\left[\frac{ah_2}{2}\left(\frac{a\pi}{\sinh a\pi}-1\right)+\frac{h_1}{2\gamma_1a}\left(\frac{a\pi}{\sinh a\pi}+1\right)\right]$$

in the latter. So, it is not hard to see that conditions (B.2) and (B.3) ensure the positivity of the first derivative $z'(0)$ in both cases. Due to the symmetry with respect to $x = \frac{\pi}{2}$ we obtain $z'(\pi) < 0$.

Step 2. Now, we show that the positivity of the first derivative of the solution at zero (and due to the symmetry—the negativity of the first derivative at π) is a necessary as well as sufficient condition for the positivity of the solution in the whole interval $]0, \pi[$.

We will use the equivalent form of $(\mathscr{B}_2)$ with $h_1(x) \equiv h_1$ and $h_2(x) \equiv h_2$, i.e.,

$$\gamma_2 \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - a^2 \right) \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - b^2 \right) z = h_2, \tag{B.5}$$

where a^2, b^2 given by (B.4) are positive real constants due to the assumption $\gamma_2 \geqslant 4\gamma_1^2$.

If we denote $w(x) := z'' - b^2 z$, we can transform $(\mathscr{B}_2)$ into a system of two ordinary differential equations of the second order

$$z'' - b^2 z = w, \tag{B.6}$$

$$w'' - a^2 w = \frac{h_2}{\gamma_2} \tag{B.7}$$

with the boundary conditions

$$z(0) = z(\pi) = 0, \tag{B.8}$$

$$w(0) = w(\pi) = \frac{h_1}{\gamma_1}. \tag{B.9}$$

First, let us consider equation (B.7) with a nonnegative right hand side $\frac{h_2}{\gamma_2}$ and boundary conditions (B.9). Due to the maximum principle (see [PW]) $w(x)$ cannot assume a nonnegative maximum at an interior point of the interval $[0, \pi]$. Thus we have

$$w(x) < \frac{h_1}{\gamma_1} \qquad \text{for all } x \in \left]0, \pi\right[.$$

If $w(x)$ were nonnegative for all $x \in [0, \pi]$, it would mean that again due to the maximum principle the solution $z(x)$ of equation (B.6) with a nonnegative right hand side $w(x)$ and boundary conditions (B.8) would be negative, i.e.,

$$z(x) < 0 \qquad \text{for all } x \in \left]0, \pi\right[.$$

But this contradicts the fact that $z'(0) > 0$.

Hence there must be a point x_0 (and due to the symmetry $\pi - x_0$) such that

$$w(x_0) = w(\pi - x_0) = 0.$$

Due to the maximum principle there cannot be any other point in the interval $]0, \pi[$ where the function $w(x)$ would be equal to zero. (In case of $h_1 \equiv 0$ we have $x_0 \equiv 0$.)

First, let us consider the interval $]0, x_0[$ (and due to the symmetry also the interval $]\pi - x_0, \pi[$). We have $w(x) > 0$ for all $x \in]0, x_0[$ and if we use again the maximum principle for the equation (B.6), we obtain that $z(x)$ cannot have a nonnegative maximum in $]0, x_0[$ and thus—due to the assumption $z'(0) > 0$—cannot decrease below zero. So, we have

$$z(x) > 0 \qquad \text{for all } x \in]0, x_0] \cup [\pi - x_0, \pi[\,.$$

Second, let us consider the interval $]x_0, \pi - x_0[$ where we have $w(x) < 0$. Moreover, we know from the previous part that $z(x_0) = z(\pi - x_0) > 0$. Thus, now due to the dual minimum principle, $z(x)$ cannot assume a nonpositive minimum at an interior point of the interval $[x_0, \pi - x_0]$ and thus

$$z(x) > 0 \qquad \text{for all } x \in [x_0, \pi - x_0].$$

So, we can conclude that

$$z(x) > 0 \qquad \text{for all } x \in]0, \pi[\,.$$

□

In case of $\gamma_2 < 4\gamma_1^2$ the situation is more complicated. The coefficients a^2, b^2 of the decomposition (B.5) are the conjugate complex numbers

$$a^2 = \frac{1}{2\gamma_1} + \mathrm{i}\sqrt{\frac{1}{\gamma_2} - \frac{1}{4\gamma_1^2}}, \quad b^2 = \frac{1}{2\gamma_1} - \mathrm{i}\sqrt{\frac{1}{\gamma_2} - \frac{1}{4\gamma_1^2}}$$

and we cannot use the maximum principle. That is why *we will consider first only a special case when* $h_1(x) \equiv 0$ and we will argue in a completely different way.

Since we consider all coefficients as well as the right hand side of $(\mathscr{B}_2)$ to be constants, we can replace the space variable x by $x - \frac{\pi}{2}$ to map $[0, \pi]$ into the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, which enables us to work with rather simpler expressions. After this shift we have a boundary value problem

(B.10)
$$\gamma_2 z^{(4)} - \frac{\gamma_2}{\gamma_1} z'' + z = h_2, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
$$z\Big(-\frac{\pi}{2}\Big) = z\Big(\frac{\pi}{2}\Big) = z''\Big(-\frac{\pi}{2}\Big) = z''\Big(\frac{\pi}{2}\Big) = 0.$$

The main idea is as follows. First, we estimate $z(x)$ from below by a function which is nonnegative for all $x \in [-x_0, x_0] \subset [-\frac{\pi}{2}, \frac{\pi}{2}]$. Second, we prove that $z(x)$ is a concave function on intervals $[-\frac{\pi}{2}, -x_1] \cup [x_1, \frac{\pi}{2}]$, and finally we show that $x_1 < x_0$ which guarantees the positivity of $z(x)$ for all $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

Lemma B.1. *Let $\gamma_2 < 4\gamma_1^2$. Then there exists a unique solution $z(x)$ of the boundary value problem* (B.10) *which is positive for all $x \in \left]-x_0, x_0\right[$ where*

$$x_0 = \frac{1}{\alpha} \operatorname{arcosh}\left(\frac{\beta}{\alpha} \sinh \alpha \frac{\pi}{2}\right) \tag{B.11}$$

with

$$\alpha^2 = \frac{1}{4\gamma_1} + \frac{1}{2\sqrt{\gamma_2}} \qquad \text{and} \qquad \beta^2 = -\frac{1}{4\gamma_1} + \frac{1}{2\sqrt{\gamma_2}} \tag{B.12}$$

provided $\beta > 1$.

Proof. The solution of (B.10) has in case of $\gamma_2 < 4\gamma_1^2$ the explicit form

$$z(x) = h_2(1 + C_1 \sinh \alpha x \sin \beta x + C_2 \cosh \alpha x \cos \beta x), \tag{B.13}$$

where α and β are given by (B.12) and

(B.14)

$$C_1 = \frac{1}{2\alpha\beta(\sinh^2 \alpha\frac{\pi}{2} + \cos^2 \beta\frac{\pi}{2})} \left[(\alpha^2 - \beta^2) \cosh \alpha\frac{\pi}{2} \cos \beta\frac{\pi}{2} - 2\alpha\beta \sinh \alpha\frac{\pi}{2} \sin \beta\frac{\pi}{2}\right],$$

$$C_2 = \frac{-1}{2\alpha\beta(\sinh^2 \alpha\frac{\pi}{2} + \cos^2 \beta\frac{\pi}{2})} \left[(\alpha^2 - \beta^2) \sinh \alpha\frac{\pi}{2} \sin \beta\frac{\pi}{2} + 2\alpha\beta \cosh \alpha\frac{\pi}{2} \cos \beta\frac{\pi}{2}\right].$$

We denote

$$u = C_1 \sinh \alpha x \sin \beta x + C_2 \cosh \alpha x \cos \beta x.$$

If we find such an interval I that

$$|u(x)| < 1 \qquad \text{for all } x \in I,$$

then we may claim that $z(x) > 0$ on I. The absolute value of u can be estimated in the following way:

$$\begin{aligned} |u| &\leqslant |C_1|\,|\sinh \alpha x|\,|\sin \beta x| + |C_2| \cosh \alpha x|\cos \beta x| \\ &\leqslant \cosh \alpha x\,[|C_1|\,|\sin \beta x| + |C_2|\,|\cos \beta x|] \\ &\leqslant \sqrt{C_1^2 + C_2^2} \cosh \alpha x. \end{aligned}$$

Moreover, from (B.14) we have

$$\sqrt{C_1^2 + C_2^2} = \frac{\alpha^2+\beta^2}{2\alpha\beta} \frac{1}{\sqrt{\sinh^2 \alpha\frac{\pi}{2} + \cos^2 \beta\frac{\pi}{2}}}.$$

So,

$$\begin{aligned} |u| &\leqslant \frac{\alpha^2+\beta^2}{2\alpha\beta} \frac{\cosh \alpha x}{\sqrt{\sinh^2 \alpha\frac{\pi}{2} + \cos^2 \beta\frac{\pi}{2}}} \\ &\leqslant \frac{\alpha}{\beta} \frac{\cosh \alpha x}{\sinh \alpha\frac{\pi}{2}}, \end{aligned}$$

since $\alpha > \beta > 0$. Hence $|u| < 1$ if $\frac{\alpha}{\beta}\frac{\cosh \alpha x}{\sinh \alpha\frac{\pi}{2}} < 1$. Provided $\beta > 1$ this is true for all $x \in \left]-x_0, x_0\right[$, where

$$x_0 = \frac{1}{\alpha} \operatorname{arcosh}\left(\frac{\beta}{\alpha} \sinh \alpha \frac{\pi}{2}\right).$$

□

Remark B.1. 1. This estimate as well as the condition $\beta > 1$ are rather rough and restrictive but—as we will see later—they will suffice and agree with our further considerations.

2. Unfortunately, we can see that $x_0 < \frac{\pi}{2}$.

Lemma B.2. *Let $\gamma_2 < 4\gamma_1^2$. Then the solution $z(x)$ of the boundary value problem* (B.10) *given by* (B.13) *is a concave function for all $x \in [x_1, \frac{\pi}{2}]$ (and due to the symmetry for all $x \in [-\frac{\pi}{2}, -x_1]$), where*

(B.15)
$$\begin{aligned} x_1 &= \frac{\pi}{2} - \frac{\pi}{2\beta} \qquad &&\textit{for } \beta > 1, \\ x_1 &= 0 &&\textit{for } 0 < \beta \leqslant 1. \end{aligned}$$

Proof. It follows from the expression (B.13) that

$$\begin{aligned} z''(x) = {} & h_2 \left[C_1(\alpha^2 - \beta^2) - 2\alpha\beta C_2\right] \sinh \alpha x \sin \beta x \\ & + h_2 \left[C_2(\alpha^2 - \beta^2) + 2\alpha\beta C_1\right] \cosh \alpha x \cos \beta x. \end{aligned}$$

Moreover, from (B.14),

$$\begin{aligned} C_1(\alpha^2 - \beta^2) - 2\alpha\beta C_2 &= \frac{(\alpha^2+\beta^2)^2}{2\alpha\beta(\sinh^2 \alpha\frac{\pi}{2} + \cos^2 \beta\frac{\pi}{2})} \cosh \alpha\frac{\pi}{2} \cos \beta\frac{\pi}{2}, \\ C_2(\alpha^2 - \beta^2) + 2\alpha\beta C_1 &= -\frac{(\alpha^2+\beta^2)^2}{2\alpha\beta(\sinh^2 \alpha\frac{\pi}{2} + \cos^2 \beta\frac{\pi}{2})} \sinh \alpha\frac{\pi}{2} \sin \beta\frac{\pi}{2}. \end{aligned}$$

So, we obtain

$$z''(x) = h_2 \frac{(\alpha^2+\beta^2)^2}{2\alpha\beta(\sinh^2\alpha\frac{\pi}{2}+\cos^2\beta\frac{\pi}{2})}\Big[\cosh\alpha\frac{\pi}{2}\cos\beta\frac{\pi}{2}\sinh\alpha x\sin\beta x \\ -\sinh\alpha\frac{\pi}{2}\sin\beta\frac{\pi}{2}\cosh\alpha x\cos\beta x\Big].$$

We see that $z''(x) < 0$ (i.e., z is a strictly concave function) if and only if

$$\cosh\alpha\frac{\pi}{2}\cos\beta\frac{\pi}{2}\sinh\alpha x\sin\beta x < \sinh\alpha\frac{\pi}{2}\sin\beta\frac{\pi}{2}\cosh\alpha x\cos\beta x.$$

Dividing by $\cosh\alpha\frac{\pi}{2}\cosh\alpha x$ (which is positive) this is equivalent to

$$\text{(B.16)} \qquad \tanh\alpha x\cos\beta\frac{\pi}{2}\sin\beta x < \tanh\alpha\frac{\pi}{2}\sin\beta\frac{\pi}{2}\cos\beta x.$$

We claim that the inequality (B.16) holds true for all

$$x \in I = \Big]\frac{[\beta]}{\beta}\frac{\pi}{2} - \frac{\pi}{2\beta}; \frac{\pi}{2}\Big[\qquad \text{if } \beta > 1$$

and for all

$$x \in \Big]0; \frac{\pi}{2}\Big[\qquad \text{if } 0 < \beta \leqslant 1,$$

where $[\beta] = \sup\{n \in \mathbb{N}; n < \beta\}$. Thus we have $\frac{[\beta]}{\beta} < 1$.

First, we will consider the case $\beta > 1$.

We can divide the interval I into two parts:

$$I_1 = \Big]\frac{[\beta]}{\beta}\frac{\pi}{2}; \frac{\pi}{2}\Big[, \qquad I_2 = \Big]\frac{[\beta]}{\beta}\frac{\pi}{2} - \frac{\pi}{2\beta}; \frac{[\beta]}{\beta}\frac{\pi}{2}\Big[, \qquad I = I_1 \cup I_2 \cup \Big\{\frac{[\beta]}{\beta}\frac{\pi}{2}\Big\}.$$

We see that the point $x_p = \frac{[\beta]}{\beta}\frac{\pi}{2}$ is the nearest left neighbour of $\frac{\pi}{2}$ where either (i) $\sin\beta x_p = 0$ and $\cos\beta x_p = \pm 1$, or (ii) $\cos\beta x_p = 0$ and $\sin\beta x_p = \pm 1$. The length of the interval I_2 is a quarter of the period $\frac{2\pi}{\beta}$ of the functions $\sin\beta x$, $\cos\beta x$.

Now, let $\cos\beta\frac{\pi}{2} \neq 0$.

Case (i). We have $\sin\beta x_p = 0$ and thus

$$\cos\beta\frac{\pi}{2}\cos\beta x > 0 \qquad \text{for all } x \in I.$$

Case (ii). We have $\cos\beta x_p = 0$ and thus

$$\cos\beta\frac{\pi}{2}\cos\beta x > 0 \qquad \text{for all } x \in I_1.$$

However, if we have $\cos\beta\frac{\pi}{2}\cos\beta x > 0$, we can divide the inequality (B.16) by this expression obtaining

$$\tanh\alpha x\tan\beta x < \tanh\alpha\frac{\pi}{2}\tan\beta\frac{\pi}{2}$$

which is true since the function $\tanh\alpha x\tan\beta x$ is increasing for all positive x for which $\cos\beta\frac{\pi}{2}\cos\beta x > 0$ is guaranteed. So, the positivity is proved for all $x \in I$ in case (i), and for all $x \in I_1$ in case (ii).

Now, let us have a look at the interval I_2 in the latter case, i.e., $x \in I_2$ and $\cos x_p = 0$. Then either

$$\cos\beta\frac{\pi}{2} > 0 \quad\text{and}\quad \cos\beta x < 0 \quad\text{and}\quad \sin\beta\frac{\pi}{2} \leqslant 0 \quad\text{and}\quad \sin\beta x < 0,$$

or

$$\cos\beta\frac{\pi}{2} < 0 \quad\text{and}\quad \cos\beta x > 0 \quad\text{and}\quad \sin\beta\frac{\pi}{2} \geqslant 0 \quad\text{and}\quad \sin\beta x > 0.$$

In both cases the inequality (B.16) holds true.

The only case which is left is the point x_p and the case when $\cos\beta\frac{\pi}{2} = 0$. By a similar discussion of values and signs of the particular functions as in the previous paragraph, we can prove the correctness of (B.16) in these cases as well.

For $0 < \beta \leqslant 1$ we have

$$\beta\frac{\pi}{2} \leqslant \frac{\pi}{2} \qquad\text{and}\qquad \sin\beta\frac{\pi}{2} > 0, \qquad \cos\beta\frac{\pi}{2} \geqslant 0,$$

hence for all $x \in \left]0; \frac{\pi}{2}\right[$ we obtain $\cos\beta\frac{\pi}{2}\cos\beta x > 0$ or $\cos\beta\frac{\pi}{2} = 0$ and the inequality (B.16) holds true again. □

R e m a r k B.2. 1. We have proved in fact a stronger assertion than that formulated in Lemma B.2 since

$$\left[\frac{\pi}{2} - \frac{\pi}{2\beta}; \frac{\pi}{2}\right] \subset \left[\frac{[\beta]\pi}{2\beta} - \frac{\pi}{2\beta}; \frac{\pi}{2}\right].$$

The more precise estimate reads

$$z'' \leqslant 0 \quad \text{for all } x \in \left[\frac{\pi}{2\beta}([\beta]-1); \frac{\pi}{2}\right]$$

and we see that for $1 < \beta \leqslant 2$ we have $[\beta] = 1$. Thus

$$\frac{\pi}{2\beta}([\beta]-1) = 0.$$

It means that in such a case as well as in case $0 < \beta \leqslant 1$ the function z is concave for all $x \in [0, \frac{\pi}{2}]$ (and due to symmetry for all $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$).

2. For these reasons, in further consideration we will work with the condition $\beta > 1$ and with the weaker estimate (B.15) which is simpler and (as we will see later) strong enough.

Finally, we can return to the proof of positivity of the original solution $z(x)$ on the interval $]0,\pi[$.

Proposition B.2. *Let $h_1(x) \equiv h_1$, $h_2(x) \equiv h_2$ be nonnegative real constants and assume $\gamma_2 < 4\gamma_1^2$. Then there exists a positive constant c such that for $\frac{h_1}{h_2} < c$ the unique solution $z(x)$ of the boundary value problem $(\mathscr{B}_2)$ is positive in $]0,\pi[$ and satisfies*

$$z'(0) > 0, \quad z'(\pi) < 0.$$

P r o o f. We prove this assertion for the solution of the transformed problem

$$\text{(B.17)} \qquad \begin{aligned} &\gamma_2 z^{(4)} - \frac{\gamma_2}{\gamma_1} z'' + z = h_2, \quad x \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big], \\ &z\Big(-\frac{\pi}{2}\Big) = z\Big(\frac{\pi}{2}\Big) = 0, \quad z''\Big(-\frac{\pi}{2}\Big) = z''\Big(\frac{\pi}{2}\Big) = \frac{h_1}{\gamma_1}. \end{aligned}$$

We carry out the proof in two steps.

Step 1. *Proof for the special case $h_1 = 0$.*

From Lemma B.1 and Lemma B.2 we know that the solution $z(x)$ is positive for all $x \in\,]-x_0, x_0[$ and concave for all $x \in [-\frac{\pi}{2}, -x_1] \cup [x_1, \frac{\pi}{2}]$. We may ask whether

$$\text{(B.18)} \qquad x_1 < x_0$$

since it would mean that $z(x)$ is concave on $[x_0, \frac{\pi}{2}] \subset [x_1, \frac{\pi}{2}]$ and moreover $z(\frac{\pi}{2}) = 0$, $z(x_0) > 0$. So, if (B.18) holds true, the positivity of $z(x)$ is guaranteed for all $x \in \big]-\frac{\pi}{2}, \frac{\pi}{2}\big[$.

We can start with the inequality

$$\frac{\alpha}{\beta} < \frac{\alpha}{\beta}\frac{\pi}{2} < \sinh \frac{\alpha}{\beta}\frac{\pi}{2} = \frac{1}{2}(\mathrm{e}^{\frac{\alpha}{\beta}\frac{\pi}{2}} - \mathrm{e}^{-\frac{\alpha}{\beta}\frac{\pi}{2}})$$

and rewrite it as

$$\frac{\beta}{\alpha}\mathrm{e}^{-\frac{\alpha}{\beta}\frac{\pi}{2}} + 1 < \frac{\beta}{\alpha}\mathrm{e}^{\frac{\alpha}{\beta}\frac{\pi}{2}} - 1.$$

This is equivalent to

$$0 < \mathrm{e}^{-\frac{\alpha}{\beta}\frac{\pi}{2}} \left[\frac{\alpha}{\beta} + \mathrm{e}^{\frac{\alpha}{\beta}\frac{\pi}{2}}\right] < \mathrm{e}^{\frac{\alpha}{\beta}\frac{\pi}{2}} \left[\frac{\beta}{\alpha} - \mathrm{e}^{-\frac{\alpha}{\beta}\frac{\pi}{2}}\right].$$

But we have $\mathrm{e}^{-\frac{\alpha}{\beta}\frac{\pi}{2}} > \mathrm{e}^{-\alpha\frac{\pi}{2}}$ and $\mathrm{e}^{\frac{\alpha}{\beta}\frac{\pi}{2}} < \mathrm{e}^{\alpha\frac{\pi}{2}}$ for $\beta > 1$. Thus

$$\mathrm{e}^{-\alpha\frac{\pi}{2}} \left[\frac{\beta}{\alpha} + \mathrm{e}^{\frac{\alpha}{\beta}\frac{\pi}{2}}\right] < \mathrm{e}^{\alpha\frac{\pi}{2}} \left[\frac{\beta}{\alpha} - \mathrm{e}^{-\frac{\alpha}{\beta}\frac{\pi}{2}}\right],$$

which implies

$$e^{\alpha(\frac{\pi}{2}-\frac{\pi}{2\beta})} + e^{-\alpha(\frac{\pi}{2}-\frac{\pi}{2\beta})} < \frac{\beta}{\alpha}(e^{\alpha\frac{\pi}{2}} - e^{-\alpha\frac{\pi}{2}}).$$

So, we obtain

$$\cosh\alpha\left(\frac{\pi}{2} - \frac{\pi}{2\beta}\right) < \frac{\beta}{\alpha}\sinh\alpha\frac{\pi}{2}$$

or

$$\underbrace{\frac{\pi}{2} - \frac{\pi}{2\beta}}_{x_1} < \underbrace{\frac{1}{\alpha}\operatorname{arcosh}\left(\frac{\beta}{\alpha}\sinh\alpha\frac{\pi}{2}\right)}_{x_0}$$

and the condition (B.18) holds true. Thus we have proved that

$$z(x) > 0 \qquad \text{for all } x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

Moreover, since $z(x)$ is positive and concave in a right neighbourhood of $-\frac{\pi}{2}$ and in a left neighbourhood of $\frac{\pi}{2}$, we can conclude that

$$z'\left(-\frac{\pi}{2}\right) > 0 \qquad \text{and} \qquad z'\left(\frac{\pi}{2}\right) < 0.$$

Step 2. *Extension of the preceding step* ($h_1 = 0$) *to include positive* h_1*, provided* $\frac{h_1}{h_2}$ *is sufficiently small.*

It is convenient to introduce even functions

$$\varphi_1(x) = \sinh\alpha x \sin\beta x, \quad \varphi_2(x) = \cosh\alpha x \cos\beta x$$

and the abbreviations

$$\Phi_i = \varphi_i\left(\frac{\pi}{2}\right) \quad (i = 1, 2), \quad \delta = \frac{h_1}{h_2\gamma_1}.$$

For the definition of α and β see (B.12).

The solution of (B.17) has in case $\gamma_2 < 4\gamma_1^2$ the explicit form

(B.19) $$z(x) = h_2[1 + C_1\varphi_1(x) + C_2\varphi_2(x)].$$

In order to incorporate the boundary conditions of (B.17) we need $z''(x)$. An elementary calculation gives

(B.20) $$z''(x) = h_2[C_1(\alpha^2 - \beta^2) - 2\alpha\beta C_2]\varphi_1(x) + h_2[C_2(\alpha^2 - \beta^2) + 2\alpha\beta C_1]\varphi_2(x)$$

so that the boundary conditions

$$z\left(\frac{\pi}{2}\right) = 0, \quad z''\left(\frac{\pi}{2}\right) = \frac{h_1}{\gamma_1} = \delta h_2$$

(the boundary conditions for $x = -\frac{\pi}{2}$ are satisfied automatically by symmetry of $z(x)$) lead after dividing by h_2 to the linear system in C_1, C_2

$$\text{(B.21)} \quad \begin{pmatrix} \Phi_1 & \Phi_2 \\ (\alpha^2-\beta^2)\Phi_1 + 2\alpha\beta\Phi_2 & (\alpha^2-\beta^2)\Phi_2 - 2\alpha\beta\Phi_1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} -1 \\ \delta \end{pmatrix}.$$

The coefficient matrix, denoted henceforth by A, is invertible since

$$\det A = -2\alpha\beta(\Phi_1^2 + \Phi_2^2) \neq 0.$$

Hence equation (B.21) can be solved explicitly by

$$\text{(B.22)} \qquad \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = A^{-1}\begin{pmatrix} -1 \\ \delta \end{pmatrix} = A^{-1}\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \delta A^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If we denote

$$z_0(x) := 1 + A^{-1}\begin{pmatrix} -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}$$

then using (B.22) formula (B.19) can be rewritten as

$$\text{(B.23)} \qquad \begin{aligned} \frac{1}{h_2} z(x) &= 1 + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \cdot \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \\ &= 1 + A^{-1}\begin{pmatrix} -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} + \delta\, A^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \\ &= z_0(x) + \delta\, A^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}. \end{aligned}$$

Notice that $z_0(x)$ is the unique solution of the boundary value problem (B.18) but with $h_1 = 0$ and $h_2 = 1$. Hence by the preceding first step $z_0(x)$ is positive in the open interval $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and satisfies $z_0'(-\frac{\pi}{2}) > 0$ as well as $z_0'(\frac{\pi}{2}) < 0$. It is easy to see, using the last expression of formula (B.23), that there is a (usually small) constant $c > 0$ such that $\frac{1}{h_2} z(x)$ is positive in $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and $\frac{1}{h_2} z'(-\frac{\pi}{2}) > 0$ as well as $\frac{1}{h_2} z'(\frac{\pi}{2}) < 0$ provided $\delta < c$. Since h_2 is positive the same conclusions hold true for $z(x)$, too. $\square$

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