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Philippe Angot; Vít Dolejší; Miloslav Feistauer; Jiří Felcman

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ANALYSIS OF A COMBINED BARYCENTRIC FINITE VOLUME—NONCONFORMING FINITE ELEMENT METHOD FOR NONLINEAR CONVECTION-DIFFUSION PROBLEMS

Philippe Angot, Marseille, Vít Dolejší, Miloslav Feistauer, Jiří Felcman, Praha

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Abstract. We present the convergence analysis of an efficient numerical method for the solution of an initial-boundary value problem for a scalar nonlinear conservation law equation with a diffusion term. Nonlinear convective terms are approximated with the aid of a monotone finite volume scheme considered over the finite volume barycentric mesh, whereas the diffusion term is discretized by piecewise linear nonconforming triangular finite elements. Under the assumption that the triangulations are of weakly acute type, with the aid of the discrete maximum principle, a priori estimates and some compactness arguments based on the use of the Fourier transform with respect to time, the convergence of the approximate solutions to the exact solution is proved, provided the mesh size tends to zero.

Keywords: nonlinear convection-diffusion problem, barycentric finite volumes, Crouzeix-Raviart nonconforming piecewise linear finite elements, monotone finite volume scheme, discrete maximum principle, a priori estimates, convergence of the method

MSC 2000: 65M12, 65M50, 35k60, 76M10, 76M25

1. Introduction

Many processes in science and technology are described by convection-diffusion equations with convection dominating over diffusion. We can mention, e.g., processes of fluid dynamics, hydrology and environmental protection. There is an extensive literature on the numerical solution of convection-diffusion problems. Let us mention, e.g., the papers [1], [2], [22], [23], [27], [29], [32], [34], [35], the monographs [26], [28] and the references therein, devoted mainly to linear problems. The main difficulty which must be overcome is the accurate resolution of the so-called boundary layers. If the equation under consideration represents a nonlinear conservation law with a small dissipation, then beside boundary layers also shock waves appear (slightly smeared due to dissipation). This is particularly the case of the system describing the viscous gas flow.

In [6], [9], [10], [12] we developed numerical methods for the solution of high-speed viscous compressible flow in domains with complex geometry. These methods are based on the combination of a finite volume scheme for the discretization of inviscid convective terms and the finite element discretization of viscous terms. Numerical experiments proved the efficiency and robustness of these methods with respect to the precise resolution of boundary layers and shock capturing. (For the finite volume solution of an inviscid gas flow see, e.g., [3], [8], [16], [17], [18], [19], [20], [24], [33]). Since the complete viscous gas flow problem is rather complex, the theoretical analysis of the combined finite volume—finite element method has been carried out for the case of a simplified scalar nonlinear conservation law equation with a small dissipation which is the simplest prototype of the compressible Navier-Stokes equations. Papers [11], [13], [15] are concerned with the convergence and error estimates for the method using dual finite volumes over a triangular mesh combined with conforming piecewise linear triangular finite elements.

Another possibility is the combination of the so-called barycentric finite volumes constructed over a triangular grid with the well-known Crouzeix-Raviart nonconforming piecewise linear finite elements used for the numerical solution of incompressible viscous flow ([5], [8], [31]). The upwind version of the Crouzeix-Raviart finite element method was developed and analyzed in [27] for a linear stationary convection-diffusion equation. This was the inspiration for Schieweck and Tobiska who investigated in [29] upwind schemes for the steady incompressible Navier-Stokes equations.

In the present paper we are concerned with the convergence analysis of the combined barycentric finite volume—nonconforming piecewise linear finite element method for the numerical solution of the nonstationary initial-boundary value problem for a scalar nonlinear conservation law equation with a diffusion term. The main technique used in this paper is based on the discrete maximum principle, a priori estimates and discrete compactness results derived with the aid of the Fourier transform with respect to time.

2. Continuous Problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$. In the space-time cylinder $Q_T = \Omega \times (0, T)$ $(0 < T < \infty)$ we consider the following initial-boundary value problem:

Find $u\colon \overline{Q}_T \to \mathbb{R}$, $u = u(x,t)$, $x \in \Omega$, $t \in [0,T]$, such that

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s} - \nu \Delta u = g \quad \text{in } Q_T, \tag{2.1}$$

$$u|_{\partial\Omega\times(0,T)} = 0, \tag{2.2}$$

$$u(x,0) = u^0(x), \quad x \in \Omega, \tag{2.3}$$

where $\nu > 0$ is a given constant and $f_s\colon \mathbb{R} \to \mathbb{R}$, $s = 1,2$, $g\colon Q_T \to \mathbb{R}$, $u^0\colon \Omega \to \mathbb{R}$ are given functions.

We denote

$$V = H_0^1(\Omega) = W_0^{1,2}(\Omega). \tag{2.4}$$

In the space $H^1(\Omega)$ besides its norm we will often work with the seminorm

$$|u|_{H^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 \, \mathrm{d}x \right)^{1/2} \tag{2.5}$$

which is an equivalent norm on V: there exist constants $\bar{c}_1, \bar{c}_2 > 0$ such that

$$\bar{c}_1 \|v\|_{H^1(\Omega)} \leqslant |v|_{H^1(\Omega)} \leqslant \bar{c}_2 \|v\|_{H^1(\Omega)}. \tag{2.6}$$

We can write $|u|_{H^1(\Omega)} = (\!(u,u)\!)^{1/2}$, where

$$(\!(u,v)\!) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \quad u,v \in H^1(\Omega), \tag{2.7}$$

is a scalar product on V. Further we set

$$(u,v) = \int_\Omega uv \, \mathrm{d}x, \quad u,v \in L^2(\Omega). \tag{2.8}$$

We will assume that

$$f_s \in C^2(\mathbb{R}), \quad f_s(0) = 0, \quad s = 1,2, \tag{2.9}$$

$$g \in C([0,T]; W^{1,q}(\Omega)) \quad \text{for some } q > 2, \tag{2.10}$$

$$u^0 \in W^{1,p}(\Omega) \quad \text{for some } p > 2. \tag{2.11}$$

Now we derive the weak formulation of problem (2.1)–(2.3). Multiplying (2.1) by an arbitrary $v \in V$, integrating over Ω, using Green's theorem we obtain the identity

$$
\begin{aligned}
(2.12) \qquad & \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u(t) v \, \mathrm{d}x - \int_\Omega \sum_{s=1}^{2} f_s(u(t)) \frac{\partial v}{\partial x_s} \, \mathrm{d}x + \nu \int_\Omega \nabla u(t) \cdot \nabla v \, \mathrm{d}x \\
& = \int_\Omega g(t) v \, \mathrm{d}x, \quad \forall v \in V, \ \forall t \in [0, T].
\end{aligned}
$$

Here, for $t \in [0, T]$, $u(t)$ means the function "$x \in \Omega \to u(t)(x) = u(x, t)$." Let us set

$$
(2.13) \qquad b(\varphi, v) = - \int_\Omega \sum_{s=1}^{2} f_s(\varphi) \frac{\partial v}{\partial x_s} \, \mathrm{d}x \quad \text{for } \varphi \in L^\infty(\Omega), \quad v \in V.
$$

Definition 1. We say that a function u is a *weak solution* of problem (2.1)–(2.3), if it satisfies the conditions

$$
(2.14) \qquad u \in L^2(0, T; V) \cap L^\infty(Q_T),
$$

$$
(2.15) \qquad \frac{\mathrm{d}}{\mathrm{d}t} (u(t), v) + b(u(t), v) + \nu (\!(u(t), v)\!) = (g(t), v) \quad \forall v \in V,
$$

in the sense of distributions on $(0, T)$,

$$
(2.16) \qquad u(0) = u^0.
$$

The identity (2.15), which is (2.12) rewritten with the aid of the above notation, means that

$$
\begin{aligned}
(2.17) \quad & - \int_0^T (u(t), v) \, \psi'(t) \, \mathrm{d}t + \nu \int_0^T (\!(u(t), v)\!) \psi(t) \, \mathrm{d}t + \int_0^T b(u(t), v) \, \psi(t) \, \mathrm{d}t \\
& = \int_0^T (g(t), v) \, \psi(t) \, \mathrm{d}t \quad \forall v \in V, \ \forall \psi \in C_0^\infty((0, T)).
\end{aligned}
$$

It follows from [11] that problem (2.14)–(2.16) has a unique solution.

3. Discrete problem

Let Ω_h be a polygonal approximation of the domain Ω. By $\mathscr{T}_h$ we will denote a triangulation of Ω_h with standard properties (see e.g. [4]): $T \in \mathscr{T}_h$ are closed triangles and

$$\overline{\Omega}_h = \bigcup_{T \in \mathscr{T}_h} T, \tag{3.1}$$

$$\text{if } T_1, T_2 \in \mathscr{T}_h, \text{ then } T_1 \cap T_2 = \emptyset \text{ or} \tag{3.2}$$

$T_1 \cap T_2$ is a common side of T_1 and T_2 or $T_1 \cap T_2$ is a common vertex of T_1 and T_2,

$$P \in \overline{\Omega} \text{ for any vertex } P \text{ of each } T \in \mathscr{T}_h. \tag{3.3}$$

By $\mathscr{S}_h$ we denote the set of all sides of all triangles $T \in \mathscr{T}_h$. We introduce a numbering of triangles $T \in \mathscr{T}_h$ and their sides $S \in \mathscr{S}_h$ in such a way that

$$\begin{aligned} \mathscr{T}_h &= \{T_i;\ i \in I\}, \\ \mathscr{S}_h &= \{S_j;\ j \in J\}, \end{aligned}$$

where I and J are suitable index sets. By Q_j we denote the centre of a side $S_j \in \mathscr{S}_h$ and put $\mathscr{P}_h = \{Q_j;\ j \in J\}$. Moreover, we set

$$J^\circ = \{i \in J;\ Q_i \in \Omega_h\}. \tag{3.4}$$

Sometimes we will use the local notation S_{ij} and Q_{ij}, $j = 1, 2, 3$, for the sides of a triangle $T_i \in \mathscr{T}_h$ and their centres, respectively. Then

$$\begin{aligned} \{Q_j,\ j \in J\} &= \{Q_{ik},\ k = 1, 2, 3,\ i \in I\}, \\ \{S_j,\ j \in J\} &= \{S_{ik},\ k = 1, 2, 3,\ i \in I\}. \end{aligned} \tag{3.5}$$

By $h(T)$ and $\theta(T)$ we denote the length of the longest side and the magnitude of the smallest angle, respectively, of the triangle $T \in \mathscr{T}_h$ and put

$$h = \max_{T \in \mathscr{T}_h} h(T), \quad \theta_h = \min_{T \in \mathscr{T}_h} \theta(T). \tag{3.6}$$

Now let us construct the *barycentric mesh* $\mathscr{D}_h = \{D_i;\ i \in J\}$ over the basic mesh $\mathscr{T}_h$. The *barycentric finite volume* D_i is a closed polygon defined in the following way: We join the barycentre of every triangle $T \in \mathscr{T}_h$ with its vertices. Then around the side S_i, $i \in J^\circ$, we obtain a closed quadrilateral containing S_i. If $S_j \subset \partial\Omega_h$ is a side with vertices P_1, P_2 of a triangle $T \in \mathscr{T}_h$ adjacent to $\partial\Omega_h$, then we denote by D_j the triangle with the sides S_j and segments connecting the barycentre of T with P_1 and P_2. (See Figures 1, 2.)

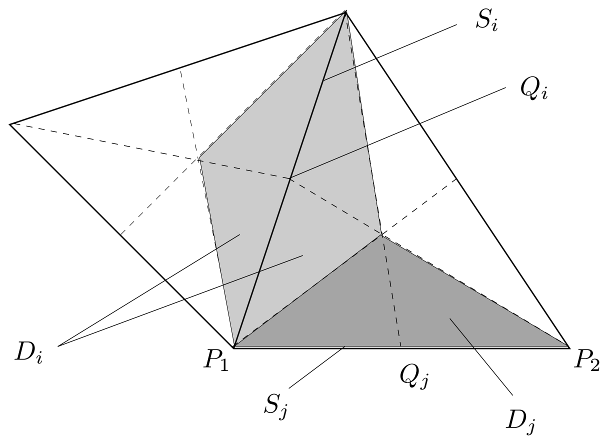

Fig. 1. Barycentric finite volumes, $D_i, D_j \in \mathscr{D}_h$, $Q_i, Q_j \in \mathscr{P}_h$, $S_i, S_j \in \mathscr{S}_h$, $S_j \subset \partial\Omega_h$.

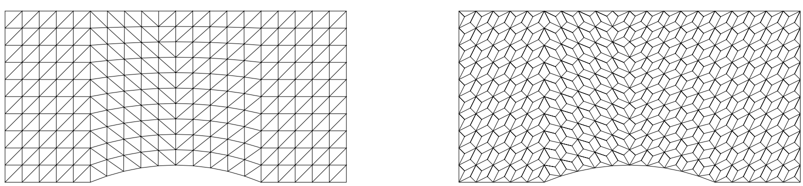

Fig. 2. Triangular mesh and associated barycentric finite volume mesh.

It is obvious that

$$\overline{\Omega}_h = \bigcup_{i \in J} D_i. \tag{3.7}$$

If $D_i \neq D_j$ and the set $\partial D_i \cap \partial D_j$ contains more than one point, we call D_i and D_j *neighbours* and set $\Gamma_{ij} = \partial D_i \cap \partial D_j$ (= a common side of D_i and D_j). Further, we define the set $s(i) = \{j \in J;\ D_j \text{ is a neighbour of } D_i\}$. If $Q_i \in \partial\Omega_h$, then we set $S(i) = s(i) \cup \{-1\}$ and $\Gamma_{i,-1} = S_i \subset \partial\Omega_h$, otherwise (for $i \in J^\circ$) we put $S(i) = s(i)$.

In the sequel we use the following notation: $|T|$ = area of $T \in \mathscr{T}_h$, $|D_i|$ = area of $D_i \in \mathscr{D}_h$ (i.e., $i \in J$), l_{ij} = length of the segment Γ_{ij}, $\mathbf{n}_{ij} = (n_{ij1}, n_{ij2})$ = unit outer normal to ∂D_i on Γ_{ij} (i.e., $\mathbf{n}_{ij}$ points from D_i to D_j). Moreover, let us consider a partition $0 = t_0 < t_1 < \ldots$ of the interval $(0, T)$ and set $\tau_k = t_{k+1} - t_k$ for $k = 0, 1, \ldots$. Obviously, we have

$$\partial D_i = \bigcup_{j \in S(i)} \Gamma_{ij}. \tag{3.8}$$

Let us define the following spaces over the grids $\mathscr{T}_h$ and $\mathscr{D}_h$:

$$\begin{aligned} X_h &= \left\{ v_h \in L^2(\Omega_h); v_h|_T \text{ is linear } \forall T \in \mathscr{T}_h, v_h \text{ is continuous at } Q_j \ \forall j \in J \right\}, \\ V_h &= \{ v_h \in X_h; v_h(Q_i) = 0 \ \forall i \in J - J^\circ \}, \\ Z_h &= \left\{ w_h \in L^2(\Omega_h); w_h|_{D_i} = \text{ const } \forall i \in J \right\}, \\ Y_h &= \{ w_h \in Z_h; w_h = 0 \text{ on } D_i \in \mathscr{D}_h \ \forall i \in J - J^\circ \}. \end{aligned} \tag{3.9}$$

Let us notice that $X_h \not\subset H^1(\Omega_h)$ and $V_h \not\subset V = H^1_0(\Omega_h)$. Therefore, we speak about *nonconforming, piecewise linear finite elements.* (By G. Strang, the use of nonconforming finite elements belongs to one of the basic *finite element variational crimes,* see [30].)

In the spaces from (3.9) we easily construct *simple bases*: The system $\{w_i;\ i \in J\}$ of functions $w_i \in X_h$ such that $w_i(Q_j) = \delta_{ij} =$ Kronecker delta, $i, j \in J$, forms a basis in X_h. The system $\{w_i, i \in J^\circ\}$ is a basis in V_h. Furthermore, denoting by $d_i = \chi_{D_i}$ the characteristic function of $D_i \in \mathscr{D}_h$, we have bases in Z_h and Y_h as the systems $\{d_i;\ i \in J\}$ and $\{d_i;\ i \in J^\circ\}$, respectively.

By I_h we denote the interpolation operator in the space of nonconforming finite elements (see [8], 8.9.79). If $v \in H^1(\Omega)$, then

$$I_h v \in X_h, \quad (I_h v)(Q_{ij}) = \frac{1}{|S_{ij}|} \int_{S_{ij}} v \, \mathrm{d}S, \quad j = 1, 2, 3,\ i \in I. \tag{3.10}$$

This integral exists due to the theorem on traces in the space $H^1(T)$:

$$\|\varphi\|_{L^2(\partial T)} \leqslant c \|\varphi\|_{H^1(T)}, \quad \varphi \in H^1(T),\ T \in \mathscr{T}_h \ \ (c = c(T)). \tag{3.11}$$

By $L_h\colon X_h \to Z_h$ we denote the so-called *lumping operator*: if $v\colon \mathscr{P}_h \to \mathbb{R}$, then we set

$$L_h v_h = \sum_{i \in J} v_h(Q_i)\, d_i \in Z_h. \tag{3.12}$$

Obviously, $L_h(V_h) = Y_h$.

In order to derive the discrete problem to (2.14)–(2.16) from Definition 1, we put

$$
\begin{aligned}
(3.13)\qquad (u,v)_h &= \int_{\Omega_h} (I_h u)(I_h v)\,\mathrm{d}x, \quad u,v \in H^1(\Omega_h),\\
(\!(u,v)\!)_h &= \sum_{i\in I} \int_{T_i} \nabla u\cdot\nabla v\,\mathrm{d}x, \quad u,v\in L^2(\Omega_h),\\
&\qquad u|_T, v|_T \in H^1(T)\ \forall T\in\mathscr{T}_h,\\
\tilde b_h(u,v) &= \sum_{i\in I}\int_{T_i}\sum_{s=1}^{2}\frac{\partial f_s(u)}{\partial x_s} v\,\mathrm{d}x, \quad u\in L^\infty(\Omega_h),\\
&\qquad v\in L^2(\Omega_h), u|_T\in H^1(T)\ \forall T\in\mathscr{T}_h.
\end{aligned}
$$

By $|\cdot|_h$ we denote the discrete L^2-norm induced by $(\cdot,\cdot)_h$. For $v_h \in X_h$ we set $I_h v_h = v_h$ and then

$$(3.14)\qquad (u_h,v_h)_h = (u_h,v_h)_{L^2(\Omega_h)}, |v_h|_h = \|v_h\|_{L^2(\Omega_h)}, \quad u_h,v_h\in X_h.$$

If $\Omega_h = \Omega$, then for "regular" functions we have

$$
\begin{aligned}
(3.15)\qquad (\!(u,v)\!)_h &= (\!(u,v)\!), && u,v\in H^1(\Omega),\\
\tilde b_h(u,v) &= b(u,v), && u\in H^1(\Omega)\cap L^\infty(\Omega),\ v\in L^2(\Omega).
\end{aligned}
$$

The form $(\!(\cdot,\cdot)\!)_h$ induces the seminorm

$$(3.16)\qquad \|u_h\|_{X_h} = \Big(\sum_{i\in I}\int_{T_i}|\nabla u_h|^2\,\mathrm{d}x\Big)^{1/2}, \quad u_h\in X_h.$$

Under the notation

$$(3.17)\qquad \|u_h\|_{X_h(T_i)} = \Big(\int_{T_i}|\nabla u_h|^2\,\mathrm{d}x\Big)^{1/2}, \quad i\in I,\ u_h\in X_h,$$

we have

$$(3.18)\qquad \|u_h\|^2_{X_h} = \sum_{i\in I}\|u_h\|^2_{X_h(T_i)}, \quad u_h\in X_h.$$

The following Cauchy inequality holds:

$$(3.19)\qquad (\!(u_h,v_h)\!)_h \leqslant \|u_h\|_{X_h}\|v_h\|_{X_h}, \quad u_h,v_h\in X_h.$$

In the case when the diffusion ν is small, it is suitable to modify the discrete "convection" form $\tilde{b}_h$ with the aid of the *finite volume approach*. Let $u \in H^1(\Omega_h), v_h \in V_h$. Then we write

$$\begin{aligned}
\int_{\Omega_h} \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s} v \,\mathrm{d}x &\approx \int_{\Omega_h} \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s} L_h v \,\mathrm{d}x \\
&= \sum_{i\in J} v(Q_i) \int_{D_i} \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s} \,\mathrm{d}x \\
&= \sum_{i\in J} v(Q_i) \int_{\partial D_i} \sum_{s=1}^{2} f_s(u) n_s \,\mathrm{d}S \\
&= \sum_{i\in J} v(Q_i) \sum_{j\in S(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{2} f_s(u) n_s \,\mathrm{d}S \\
&= \sum_{i\in J} v(Q_i) \sum_{j\in s(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{2} f_s(u) n_s \,\mathrm{d}S \\
&\approx \sum_{i\in J} v(Q_i) \sum_{j\in s(i)} H\left(u(Q_i), u(Q_j), \mathbf{n}_{ij}\right) l_{ij}.
\end{aligned}$$

The function H defined on $\mathbb{R}^2 \times \mathbf{S}$, where $\mathbf{S} = \{\mathbf{n} \in \mathbb{R}^2; |\mathbf{n}| = 1\}$, is called a *numerical flux*.

It is easy to see that the form

$$b_h(u,v) = \sum_{i\in J} v(Q_i) \sum_{j\in s(i)} H\left(u(Q_i), u(Q_j), \mathbf{n}_{ij}\right) l_{ij} \tag{3.20}$$

obtained above has sense for all $u, v \in X_h$. We will use it as an approximation of the form $\tilde{b}_h$.

Definition 2. We define the *approximate solution* of problem (2.1)–(2.3) as functions u_h^k, $t_k \in [0,T]$, given by the conditions

$$u_h^0 = I_h u^0 \ (\in V_h), \tag{3.21}$$

$$u_h^{k+1} \in V_h, \quad t_k \in [0,T), \tag{3.22}$$

$$\frac{1}{\tau}(u_h^{k+1} - u_h^k, v_h)_h + b_h(u_h^k, v_h) + \nu(\!(u_h^{k+1}, v_h)\!)_h = (g^{k+1}, v_h)_h, \qquad \forall v_h \in V_h,\ t_k \in [0,T), \tag{3.23}$$

where $g^k = g(\cdot, t_k)$. The function u_h^k is the approximate solution at time t_k.

Properties of the numerical flux. In what follows we use the following assumptions:

1. $H = H(y, z, \mathbf{n})$ is *locally Lipschitz-continuous* with respect to y, z: for any $M^* > 0$ there exists $c(M^*) > 0$ such that

$$|H(y,z,\mathbf{n}) - H(y^*,z^*,\mathbf{n})| \leqslant c(M^*)(|y-y^*| + |z-z^*|) \qquad \forall y, y^*, z, z^* \in [-M^*, M^*],\ \forall \mathbf{n} \in \mathscr{S}. \tag{3.24}$$

2. H is *consistent*:

$$H(u,u,\mathbf{n}) = \sum_{s=1}^{2} f_s(u) n_s, \quad \forall u \in \mathbb{R},\ \forall \mathbf{n} = (n_1, n_2) \in \mathscr{S}. \tag{3.25}$$

3. H is *conservative*:

$$H(y,z,\mathbf{n}) = -H(z,y,-\mathbf{n}) \quad \forall y, z \in \mathbb{R},\ \forall \mathbf{n} \in \mathscr{S}. \tag{3.26}$$

4. H is *monotone* in the following sense: For a given fixed number $M^* > 0$ the function $H(y,z,\mathbf{n})$ is nonincreasing with respect to the second variable z on the set

$$\mathscr{M}_{M^*} = \{(y,z,\mathbf{n}); y, z \in [-M^*, M^*], \mathbf{n} \in \mathscr{S}\}. \tag{3.27}$$

Lemma 1. *Problem* (3.21)–(3.23) *from Definition* 2 *has the following properties:*
1. *The bilinear forms* $(\cdot,\cdot)_h$ *and* $(\!(\cdot,\cdot)\!)_h$ *defined in* (3.13) *are scalar products on* V_h.
2. *For each* $u_h \in X_h$, $b_h(u_h, \cdot)$ *is a linear form on* V_h.
3. *If* $i \in J$ *and* $T \in \mathscr{T}_h$ *is a triangle for which* $Q_i \in T$, *then*

$$|T \cap D_i| = \frac{1}{3}|T|. \tag{3.28}$$

4. *The approximation* $(\cdot,\cdot)_h$ *of the* L^2*-scalar product can be defined with the aid of numerical integration using the centres* Q_{ij} *of sides* S_{ij} *of* $T_i \in \mathscr{T}_h$ *as integration points:*

$$(u,v)_h = \frac{1}{3}\sum_{i\in I}|T_i|\sum_{j=1}^{3} u(Q_{ij})v(Q_{ij}) = \int_\Omega (L_h u)(L_h v)\,\mathrm{d}x, \quad u, v \in X_h. \tag{3.29}$$

5. *We have*

$$(w_i, w_j)_h = \delta_{ij}|D_i|, \quad i, j \in J, \tag{3.30}$$

$$(u, w_i)_h = \frac{1}{3}\sum_{\{T\in\mathscr{T}_h; Q_i \in T\cap\mathscr{P}_h\}} |T| u(Q_i) = |D_i| u(Q_i), \quad i \in J,\ u \in X_h, \tag{3.31}$$

$$(g^k, w_i)_h = |D_i| g(Q_i, t_k), \quad i \in J,\ t_k \in [0, T]. \tag{3.32}$$

6. *Problem* (3.22)–(3.23) *has a unique solution* u_h^{k+1}.
7. *Function* $z \in X_h$ *and* $y \in V_h$ *can be expressed in the form*

$$z = \sum_{j\in J} z(Q_j)w_j \text{ and } y = \sum_{j\in J^\circ} y(Q_j)w_j, \tag{3.33}$$

respectively.
8. *Problem* (3.22)–(3.23) *is equivalent to the system of algebraic equations*

$$|D_i|u_h^{k+1}(Q_i) + \tau\nu \sum_{j\in J^\circ} ((w_i, w_j))_h u_h^{k+1}(Q_j) \tag{3.34}$$

$$= |D_i|u_h^k(Q_i) - \tau b_h(u_h^k, w_i) + \tau|D_i|g(Q_i, t_k), \quad i \in J^\circ, \tag{3.35}$$

for unknown values $u_h^{k+1}(Q_j)$, $j \in J^\circ$. *This system is uniquely solvable.*

P r o o f. Assertions 1, 2, 3 and 7 are obvious. By [4], Par. 4.1, the numerical quadrature

$$\int_{T_i} u \,\mathrm{d}x \approx \frac{|T_i|}{3} \sum_{j=1}^{3} u(Q_{ij}), \tag{3.36}$$

is exact for polynomials of degree $\leqslant 2$. This together with 3 implies assertion 4. Assertion 5 is a consequence of 3 and 4. Assertion 6 follows from the Lax-Milgram lemma, 8 is obtained from 5, 6 and 7. $\square$

4. Convergence

In what follows, for simplicity *we assume that the domain* Ω *is polygonal* and, hence, $\Omega_h = \Omega$. Let us consider a system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ $(h_0 > 0)$ of triangulations of the domain Ω, set $\tau = T/r$ for any integer $r > 1$ and define the partition of the interval $[0, T]$ formed by time instants $t_k = k\tau$, $k = 0, 1, \ldots, r$.

We define functions $u_{h\tau}, w_{h\tau}\colon (-\infty, \infty) \to V_h$ associated with an approximate solution $\{u_h^k\}_{k=0}^r$:

$$\begin{aligned} u_{h\tau}(t) &= u_h^0, \quad t \leqslant 0,\\ u_{h\tau}(t) &= u_h^k, \quad t \in (t_{k-1}, t_k], \quad k = 1, \ldots, r,\\ u_{h\tau}(t) &= u_h^r, \quad t \geqslant T; \end{aligned} \tag{4.1}$$

$$\begin{gathered} w_{h\tau} \text{ is a continuous, piecewise linear mapping of } [0, T] \text{ into } V_h,\\ w_{h\tau}(t_k) = u_h^k, \quad k = 0, \ldots, r,\\ w_{h\tau}(t) = 0 \text{ for } t < 0 \text{ or } t > T. \end{gathered} \tag{4.2}$$

Our goal is to prove that the functions $u_{h\tau}$, $w_{h\tau}$, constructed from the values of the approximate solution u_h^k, $t_k \in [0,T]$ with the aid of scheme (3.21)–(3.23), converge in some sense to the exact solution of problem (2.1)–(2.3) as $h,\tau \to 0$ in a suitable way. In what follows we shall work with a number of constants. By $c, c_1, c_2, \ldots, \hat{c}, \hat{c}_1, \ldots, \tilde{c}, \tilde{c}_1, \ldots$ we will denote constants independent of h, τ, ν, and $C, C_1, \ldots$ will denote constants that are independent of h, τ, but depend on ν. Moreover, c will be used as a generic constant attaining in general different values at different places.

Assumptions:

1. Let the system $\{\mathscr{T}_h\}_{h\in(0,h_0)}$ be *regular*, i.e. there exists $\vartheta_0 > 0$ such that

$$\theta_h \geqslant \vartheta_0 > 0 \quad \forall h \in (0,h_0). \tag{4.3}$$

2. Let the magnitude of all angles of all $T \in \mathscr{T}_h$, $h \in (0,h_0)$, is less than or equal to $\pi/2$, i.e.

$$\text{The triangulations } \mathscr{T}_h,\ h \in (0,h_0) \text{ are of } \textit{weakly acute type}. \tag{4.4}$$

3. The *inverse assumption* is satisfied: There exists $c_1 > 0$ such that

$$\frac{h}{h(T)} \leqslant c_1 \quad \forall T \in \mathscr{T}_h \ \forall h \in (0,h_0). \tag{4.5}$$

In view of [4], Remark 3.1.3, assumption (4.3) implies the existence of a constant $c_2 > 0$ such that

$$h^2 \leqslant c_2|T|, \quad T \in \mathscr{T}_h,\ h \in (0,h_0). \tag{4.6}$$

5. L^∞-stability

In virtue of (2.10) and (2.11), $u^0 \in C(\overline{\Omega})$ and $g \in C(\overline{Q}_T)$. Hence, there exist constants $\tilde{M}$ and $\tilde{K}$ such that

$$\|u_h^0\|_{L^\infty(\Omega)} \leqslant \tilde{M}, \quad \|g\|_{L^\infty(Q_T)} \leqslant \tilde{K}. \tag{5.1}$$

Let us put

$$M^* = \tilde{M} + T\tilde{K}. \tag{5.2}$$

If $u_h \in X_h$ and $|u_h(Q_i)| \leqslant M^*$ for all $i \in J$, then $\|u_h\|_{L^\infty(\Omega)} \leqslant M := 3M^*$. The main tool for proving the L^∞-stability is the *discrete maximum principle* represented by the following results.

Theorem 1. *For $i \in J^\circ$ and $j \in J$ let real numbers a_{ij}, b_{ij}, δ_i, φ_i, u_j, $\tilde{u}_j$, τ satisfy the following conditions:*

$$
\begin{aligned}
&\tau > 0,\\
&a_{ii} > 0 \quad \forall i \in J^\circ, \quad a_{ij} \leqslant 0 \;\; \forall i \in J^\circ,\; j \in J,\; i \neq j,\\
&b_{ij} \geqslant 0 \quad \forall i \in J^\circ,\; j \in J,\\
&\sum_{j \in J} a_{ij} = \sum_{j \in J} b_{ij} = \delta_i > 0 \;\forall i \in J^\circ,\\
&\sum_{j \in J} a_{ij}\tilde{u}_j = \sum_{j \in J} b_{ij}u_j + \tau\delta_i\varphi_i \;\forall i \in J^\circ,\\
&\tilde{u}_i = u_i = 0,\; \forall i \in J - J^\circ.
\end{aligned}
$$

Then

$$
\max_{j \in J} |\tilde{u}_j| \leqslant \max_{j \in J} |u_j| + \tau \max_{j \in J^\circ} |\varphi_i|. \tag{5.3}
$$

Proof follows from [22], Lemma 3.1.1, page 29. □

Lemma 2. *Let w_i, $i \in J$ be the basis functions of X_h defined above. Then under assumption* (4.4) *the following relations are valid:*

$$
((w_i, w_i))_h > 0,\; i \in J, \tag{5.4}
$$

$$
((w_i, w_j))_h \leqslant 0,\; i, j \in J,\; i \neq j, \tag{5.5}
$$

$$
\sum_{j \in J} ((w_i, w_j))_h = 0,\; i \in J. \tag{5.6}
$$

Proof. By the definition, we have

$$
((w_i, w_j))_h = \sum_{T \in \mathscr{T}_h} |T| \nabla w_i|_T \cdot \nabla w_j|_T. \tag{5.7}
$$

If $\nabla w_i|_T \cdot \nabla w_j|_T \neq 0$ then Q_i, Q_j must be the midpoints of the sides of the triangle T. So, let T be a triangle with nodes $Q_i = (x_{i1}, x_{i2})$, $Q_j = (x_{j1}, x_{j2})$, $Q_k = (x_{k1}, x_{k2})$. Taking into account that $w_i|_T$ is uniquely determined by its values at the vertices of the triangle $Q_iQ_jQ_k$ and using the standard results (Cf., e.q., [14], Section 4.4), we have

$$\begin{aligned}\nabla w_i|_T &= \frac{1}{D}(x_{j2} - x_{k2}, x_{k1} - x_{j1}),\\ \nabla w_j|_T &= \frac{1}{D}(x_{k2} - x_{i2}, x_{i1} - x_{k1}),\\ \nabla w_k|_T &= \frac{1}{D}(x_{i2} - x_{j2}, x_{j1} - x_{i1}),\end{aligned} \tag{5.8}$$

where

$$D = \begin{vmatrix} x_{i1}, & x_{i2}, & 1 \\ x_{j1}, & x_{j2}, & 1 \\ x_{k1}, & x_{k2}, & 1 \end{vmatrix}. \tag{5.9}$$

This implies that

$$|\nabla w_i|_T|^2 = \frac{1}{D^2}\,|Q_j - Q_k|^2 > 0. \tag{5.10}$$

Since $w_i + w_j + w_k = 1$ on T, we have

$$|\nabla w_i|^2 + \nabla w_i \cdot \nabla w_j + \nabla w_i \cdot \nabla w_k = 0 \text{ on } T. \tag{5.11}$$

Further, using (5.8), the well-known expression of the cosine of the angle between two vectors and denoting by α_i the angle in the triangle $Q_iQ_jQ_k$ at the vertex Q_i, we find that

$$\nabla w_i|_T \cdot \nabla w_j|_T = -\frac{1}{D^2}|Q_i - Q_j|\,|Q_i - Q_k| \cos\alpha_i \leqslant 0 \tag{5.12}$$

(similarly for $\nabla w_i \cdot \nabla w_k$ and $\nabla w_j \cdot \nabla w_k$). The last inequality is a consequence of the assumption (4.4) on the angles of $T \in T_h$, which implies that $\alpha_i \in (0, \pi/2]$. Now we multiply (5.10)–(5.12) by $|T|$, sum over all $T \in T_h$ and use (5.7). As a result we immediately obtain (5.4)–(5.6). □

Theorem 2. *If $\tau > 0$ and $h \in (0, h_0)$ satisfy the condition*

$$\tau c(M^*)|\partial D_i| \leqslant |D_i|, \quad i \in J, \tag{5.13}$$

where $c(M^*)$ is the constant from (3.24), and if (5.1) and (5.2) hold, then

$$\|u_h^k\|_{L^\infty(\Omega)} \leqslant M, \quad t_k \in [0, T], \tag{5.14}$$

$$\|u_{h\tau}\|_{L^\infty(Q_T)},\ \|w_{h\tau}\|_{L^\infty(Q_T)} \leqslant M. \tag{5.15}$$

P r o o f. In virtue of (3.34) and the fact that $u_h(Q_j) = 0$, $Q_j \in \partial\Omega$, identity (3.23) can be written in the form

$$\begin{aligned}(5.16)\qquad &|D_i|u_h^{k+1}(Q_i) + \tau\nu \sum_{j\in J} ((w_i, w_j))_h u_h^{k+1}(Q_j) \\ &= |D_i|u_h^k(Q_i) - \tau b_h(u_h^k, w_i) + \tau|D_i|g(Q_i, t_k), \quad i \in J^\circ,\ k \geqslant 0.\end{aligned}$$

By induction with respect to k we will prove that

$$(5.17)\qquad \|u_h^k(Q_i)\|_{L^\infty(\Omega)} \leqslant \tilde{M} + k\tau\tilde{K} \leqslant M^*, \quad t_k \in [0,T],\ Q_i \in \mathscr{P}_h.$$

Obviously, (5.17) holds for $k = 0$. Let us assume that (5.17) is valid for some $t_k \in [0, T)$.

Let us denote by L_i the left hand side of (5.16) and set $u_i = u_h^k(Q_i)$ and $\varphi_i = g(Q_i, t_k)$ (for simplicity we omit the superscript k). Then (5.16) reads

$$\begin{aligned}L_i &= |D_i|u_i - \tau b_h(u_h, w_i) + \tau|D_i|\varphi_i \\ &= |D_i|u_i - \tau \sum_{j\in s(i)} H(u_i, u_j, \mathbf{n}_{ij})l_{ij} + \tau|D_i|\varphi_i \\ &= |D_i|u_i + \tau \sum_{j\in s(i)} \Big[H(u_i, u_i, \mathbf{n}_{ij}) - H(u_i, u_j, \mathbf{n}_{ij}) - H(u_i, u_i, \mathbf{n}_{ij})\Big] l_{ij} \\ &\quad + \tau|D_i|\varphi_i, \quad i \in J^\circ.\end{aligned}$$

In view of the consistency of the numerical flux H (see (3.25)) and Green's theorem, we have

$$\sum_{j\in s(i)} H(u_i, u_i, \mathbf{n}_{ij})l_{ij} = \int_{\partial D_i} \sum_{s=1}^{2} f_s(u_i) n_s \,\mathrm{d}S = 0.$$

Hence, if we set

$$(5.18)\qquad \mathbb{H}_{ij} = \begin{cases} 0, & u_i = u_j \\ \dfrac{H(u_i, u_i, \mathbf{n}_{ij}) - H(u_i, u_j, \mathbf{n}_{ij})}{u_j - u_i}\, l_{ij}, & u_i \neq u_j, \end{cases}$$

we can write

$$(5.19)\qquad L_i = |D_i|u_i + \tau \sum_{j\in s(i)} \mathbb{H}_{ij}(u_j - u_i) + \tau|D_i|\varphi_i.$$

Due to the monotonicity of the numerical flux,

$$(5.20)\qquad \mathbb{H}_{ij} \geqslant 0, \quad i \in J^\circ,\ j \in s(i).$$

In virtue of the induction assumption, $|u_i| \leqslant \tilde{M} + k\tau\tilde{K} < M^*$ for $i \in J$. This and the local Lipschitz-continuity of H imply that

$$0 \leqslant \mathbb{H}_{ij} \leqslant c(M^*)l_{ij} = c(M^*)|\Gamma_{ij}|.$$

Using (3.8), we find that

$$0 \leqslant \sum_{j\in s(i)} \mathbb{H}_{ij} \leqslant c(M^*) \sum_{j\in s(i)} |\Gamma_{ij}| \leqslant c(M^*)|\partial D_i|$$

and hence, by (5.13),

$$|D_i| - \tau \sum_{j\in s(i)} \mathbb{H}_{ij} \geqslant 0, \quad i \in J^\circ. \tag{5.21}$$

From (5.19) it follows that (5.16) can be written in the form

$$\begin{aligned}|D_i|u_h^{k+1}(Q_i) &+ \tau\nu\sum_{j\in J}(\!(w_i, w_j)\!)_h u_h^{k+1}(Q_j) \\ &= \Big(|D_i| - \tau \sum_{j\in s(i)} \mathbb{H}_{ij}\Big)u_h^k(Q_i) + \tau \sum_{j\in s(i)} \mathbb{H}_{ij}\, u_h^k(Q_j) + \tau|D_i|\varphi_i^k, \quad i \in J^\circ.\end{aligned} \tag{5.22}$$

Taking into account (5.4)–(5.6), (5.20) and (5.21), we see that Theorem 1 can be applied if we set

$$\begin{aligned} a_{ij} &= |D_i|\delta_{ij} + \tau\nu(\!(w_i, w_j)\!)_h, \\ b_{ii} &= |D_i| - \tau \sum_{k\in s(i)} \mathbb{H}_{ik}, \\ b_{ij} &= \tau\mathbb{H}_{ij},\ i \neq j, \\ u_i &= u_h^k(Q_i), \\ \tilde{u}_i &= u_h^{k+1}(Q_i), \\ \delta_i &= |D_i|. \end{aligned} \tag{5.23}$$

Inequality (5.3) and the fact that $u_h^{k+1}(Q_j) = 0$ for $Q_j \in \partial\Omega_h$ imply that

$$\max_{i\in J} |u_h^{k+1}(Q_i)| \leqslant \max_{i\in J} |u_h^k(Q_i)| + \tau\|g(\cdot, t_k)\|_{L^\infty(\Omega)}.$$

In view of the induction assumption and (5.1), we find that

$$\max_{i\in J} |u_h^{k+1}(Q_i)| \leqslant \tilde{M} + (k+1)\tau\tilde{K} \leqslant M^*.$$

Hence, $\|u_h^{k+1}\|_{L^\infty(\Omega)} \leqslant M = 3M^*$, which we wanted to prove. $\square$

Lemma 3. *Assumption* (4.3) *and its consequence* (4.6) *imply that there exists a constant* $c_3 > 0$ *such that*

$$|D_i|/|\partial D_i| \geqslant c_3 h, \quad \forall i \in J, \ \forall h \in (0, h_0). \tag{5.24}$$

P r o o f. From (3.28) we deduce that

$$\forall i \in J \ \exists j_0 \in I \text{ such that } |D_i| \geqslant \frac{1}{3}|T_{j_0}|, \tag{5.25}$$

which together with (4.6) implies that

$$|D_i| \geqslant \frac{1}{3c_2}h^2 \qquad \forall i \in J. \tag{5.26}$$

Further, it is easy to see that $|\partial D_i| \leqslant \frac{8}{3}h$, which together with (5.26) gives assertion (5.24). □

As we see, we can consider the *stability condition*

$$0 \leqslant \tau \leqslant c_3 c(M)^{-1} h. \tag{5.27}$$

Obviously, (5.24) and (5.27) yield (5.13).

6. Consistency

Lemma 4. *(Discrete Friedrichs inequality) There exists a constant* $\hat{c}_1$ *independent of* h *such that*

$$\|u_h\|_{L^2(\Omega)} \leqslant \hat{c}_1 \|u_h\|_{X_h}, \quad u_h \in V_h, \ h \in (0, h_0). \tag{6.1}$$

P r o o f. In [31], Chap. I, Par. 4, Proposition 4.13 or [8], Lemma 8.9.92, this lemma is proved provided Ω is convex. For the case of a general polygonal domain see [7]. □

Definition 3. Let us define the space $L^2(0, T; V_h)$ as the set of all functions $v_h \colon (0, T) \to V_h$ such that

$$\begin{aligned} \|v_h\|_{L^2(0,T;V_h)} &\equiv \left(\int_0^T \|v_h(t)\|_{X_h}^2 \, \mathrm{d}t \right)^{1/2} \\ &= \left(\int_0^T \left(\sum_{i \in I} \int_{T_i} |\nabla v_h(t)|^2 \, \mathrm{d}x \right) \mathrm{d}t \right)^{1/2} < \infty. \end{aligned} \tag{6.2}$$

Lemma 5. *The interpolation operator I_h defined by* (3.10) *has the following properties:*

$$\text{If } \varphi \in V, \text{ then } I_h\varphi \in V_h. \tag{6.3}$$

Let $\varphi \in H^{k+1}(\Omega)$, where $k = 0$ or 1. Then for $h \in (0, h_0)$ we have

$$\|\varphi - I_h\varphi\|_{X_h} \leqslant ch^k \|\varphi\|_{H^{k+1}(\Omega)}, \tag{6.4}$$

$$\|\varphi - I_h\varphi\|_{L^2(\Omega)} \leqslant ch^{k+1} \|\varphi\|_{H^{k+1}(\Omega)}, \tag{6.5}$$

$$\|I_h\varphi\|_{X_h} \leqslant c\|\varphi\|_{H^1(\Omega)}, \tag{6.6}$$

$$\varphi \in H^1(\Omega) \Rightarrow \|\varphi - I_h\varphi\|_{X_h} \to 0 \text{ as } h \to 0 \tag{6.7}$$

with $c > 0$ independent of φ and h.

P r o o f. See [8], Lemma 8.9.81. □

Lemma 6. *There exists a constant $c > 0$ such that for any $h \in (0, h_0)$ we have*

$$\|v_h\|_{L^2(\Omega)} = \|L_h v_h\|_{L^2(\Omega)}, \quad v_h \in X_h, \tag{6.8}$$

$$\|v_h - L_h v_h\|_{L^2(\Omega)} \leqslant ch\|v_h\|_{X_h}, \quad v_h \in X_h, \tag{6.9}$$

$$(u_h, v_h) = (u_h, v_h)_h, \quad u_h, v_h \in X_h, \tag{6.10}$$

$$|(g^k, v_h) - (g^k, v_h)_h| \leqslant ch\|g^k\|_{W^{1,q}(\Omega)}\|v_h\|_{X_h}, \quad v_h \in V_h. \tag{6.11}$$

If $M > 0$ and $\kappa \in (0, 1)$, then there exists a constant $\tilde{c} = \tilde{c}(M, \kappa)$ such that

$$\begin{aligned} &|\tilde{b}_h(u_h, v_h) - b_h(u_h, v_h)| \leqslant \tilde{c}h^{1-\kappa} \left(\|u_h\|_{X_h}^2 + \|u_h\|_{X_h}\right) \|v_h\|_{X_h}, \\ &u_h \in V_h \cap L^\infty(\Omega), \ \|u_h\|_{L^\infty(\Omega)} \leqslant M, \ v_h \in V_h, \ h \in (0, h_0), \end{aligned} \tag{6.12}$$

where the forms $\tilde{b}_h$ and b_h are defined by (3.13) *and* (3.20), *respectively.*

P r o o f. 1. Let $v_h \in X_h$. We can write

$$\|v_h\|_{L^2(\Omega)}^2 = \int_\Omega |v_h|^2 \,\mathrm{d}x = \sum_{i \in I} \int_{T_i} |v_h|^2 \,\mathrm{d}x. \tag{6.13}$$

By the definition of X_h, $v_h|_{T_i}$ is a linear function. Since the quadrature formula

$$\int_{T_i} \varphi \,\mathrm{d}x \approx \frac{1}{3}|T_i| \sum_{j=1}^{3} \varphi(Q_{ij}) \tag{6.14}$$

is precise for quadratic functions on T_i, we immediately find that

$$\|v_h\|^2_{L^2(\Omega)} = \sum_{i\in I} \frac{1}{3}|T_i| \sum_{j=1}^{3} v_h(Q_{ij})^2. \tag{6.15}$$

On the other hand, from the definition of L_h and (3.28) it follows that

$$\begin{aligned} \|L_h v_h\|^2_{L^2(\Omega)} &= \int_\Omega |L_h v|^2 \,\mathrm{d}x = \sum_{j\in J} \int_{D_j} |L_h v|^2 \,\mathrm{d}x \\ &= \sum_{j\in J} |D_j| v_h(Q_j)^2 = \sum_{i\in I} \frac{1}{3}|T_i| \sum_{j=1}^{3} v_h(Q_{ij})^2. \end{aligned} \tag{6.16}$$

Now (6.15) and (6.16) yield (6.8).

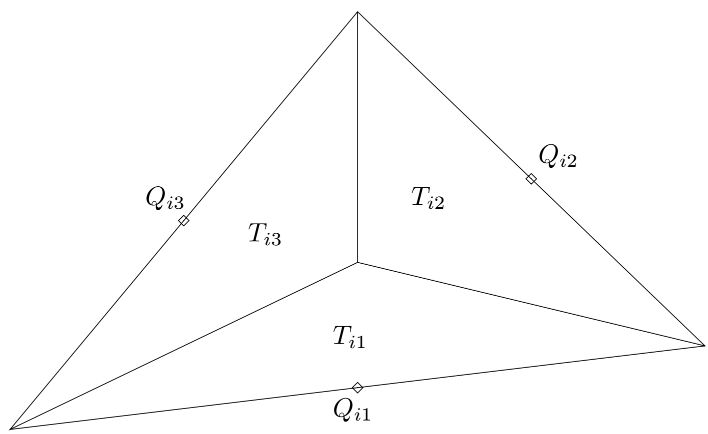

Fig. 3. Partition of a triangle T_i into subtriangles T_{i1}, T_{i2}, T_{i3}

2. Each $v_h \in X_h$ is linear on $T_i \in \mathscr{T}_h$ and can be expressed in the form

$$v_h(x_1, x_2) = v_h(Q_{ij}) + \left.\frac{\partial v_h}{\partial x_1}\right|_{T_i} (x_1 - x_1(Q_{ij})) + \left.\frac{\partial v_h}{\partial x_2}\right|_{T_i} (x_2 - x_2(Q_{ij})), \quad j = 1, 2, 3, \tag{6.17}$$

where $(x_1(Q_{ij}), x_2(Q_{ij}))$ are the coordinates of Q_{ij}.

Next we have $|\,(x_1 - x_1(Q_{ij}))\,| \leqslant h$, $|\,(x_2 - x_2(Q_{ij}))\,| \leqslant h$ for $(x_1, x_2) \in T_i$. Every triangle $T_i \in \mathscr{T}_h$ can be divided into three subtriangles T_{i1}, T_{i2}, T_{i3} (see Fig. 3).

Then we have

$$\|v_h - L_h v_h\|^2_{L^2(\Omega)} = \int_\Omega |v_h - L_h v_h|^2 \,\mathrm{d}x = \sum_{i\in I}\sum_{j=1}^{3}\int_{T_{ij}} |v_h - L_h v_h|^2 \,\mathrm{d}x$$
$$= \sum_{i\in I}\sum_{j=1}^{3}\int_{T_{ij}} \left|\frac{\partial v_h}{\partial x_1}\Big|_{T_i}\Big(x_1 - x_1(Q_{ij})\Big) + \frac{\partial v_h}{\partial x_2}\Big|_{T_i}\Big(x_2 - x_2(Q_{ij})\Big)\right|^2 \mathrm{d}x$$
$$\leqslant h^2 \sum_{i\in I}\sum_{j=1}^{3}\int_{T_{ij}} \left(\left|\frac{\partial v_h}{\partial x_1}\right| + \left|\frac{\partial v_h}{\partial x_2}\right|\right)^2 \mathrm{d}x \leqslant 2h^2 \sum_{i\in I}\int_{T_i}\left(\left|\frac{\partial v_h}{\partial x_1}\right|^2 + \left|\frac{\partial v_h}{\partial x_2}\right|^2\right)\mathrm{d}x$$
$$= 2h^2 \|v_h\|^2_{X_h},$$

which proves (6.9).

3. Assertion (6.10) immediately follows from (3.13) and the fact that $u_h = I_h u_h$ for $u_h \in X_h$.

4. Assertion (6.11) follows from relation (3.29), the fact that the quadrature formula (3.36) is exact for polynomials of degree $\leqslant 2$ and [4], Theorem 4.1.5.

5. Let us define the form

$$b_h^*(u_h, v_h) = \sum_{i\in J} v_h(Q_i) \sum_{j\in s(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{2} f_s(u_h) n_s \,\mathrm{d}S,\ \ u_h, v_h \in V_h. \tag{6.18}$$

We write

$$\tilde{b}_h(u_h, v_h) - b_h(u_h, v_h) = \Big[\tilde{b}_h(u_h, v_h) - \tilde{b}_h(u_h, L_h v_h)\Big] + \Big[\tilde{b}_h(u_h, L_h v_h) - b_h^*(u_h, v_h)\Big] + \Big[b_h^*(u_h, v_h) - b_h(u_h, v_h)\Big] \tag{6.19}$$

and estimate the expressions in square brackets separately. Obviously, due to (6.9) and the bound $\|u_h\|_{L^\infty(\Omega)} \leqslant M$,

$$\Big|\tilde{b}_h(u_h, v_h) - \tilde{b}_h(u_h, L_h v_h)\Big| = \left|\sum_{i\in I}\int_{T_i}\sum_{s=1}^{2} f_s'(u_h)\frac{\partial u_h}{\partial x_s}\,(v_h - L_h v_h)\,\mathrm{d}x\right| \tag{6.20}$$
$$\leqslant \max_{|\xi|\leqslant M}\max_{s=1,2} |f_s'(\xi)|\, \|\nabla u_h\|_{L^2(\Omega)} \|v_h - L_h v_h\|_{L^2(\Omega)}$$
$$\leqslant \tilde{c}h\|u_h\|_{X_h}\|v_h\|_{X_h}, \quad \tilde{c} = \tilde{c}(M).$$

Using notation from Fig. 3, we have

$$
\begin{aligned}
\text{(6.21)} \qquad \tilde{b}_h(u_h, L_h v_h) &= \sum_{i\in I} \int_{T_i} \sum_{s=1}^{2} \frac{\partial f_s(u_h)}{\partial x_s} L_h v_h \, \mathrm{d}x \\
&= \sum_{i\in I} \sum_{j=1}^{3} v_h(Q_{ij}) \int_{T_{ij}} \sum_{s=1}^{2} \frac{\partial f_s(u_h)}{\partial x_s} \, \mathrm{d}x \\
&= \sum_{i\in I} \sum_{j=1}^{3} v_h(Q_{ij}) \int_{\partial T_{ij}} \sum_{s=1}^{2} f_s(u_h) n_s \, \mathrm{d}S.
\end{aligned}
$$

It is evident (see Fig. 4) that for each $k \in J$ there exist i, $i^* \in I$ and j, $j^* \in \{1,2,3\}$ such that

$$
\text{(6.22)} \qquad D_k = T_{ij} \cup T_{i^*j^*}, \quad S_k = T_{ij} \cap T_{i^*j^*}, \quad Q_{ij} = Q_{i^*j^*} = Q_k.
$$

If D_k is a boundary finite volume then $i = i^*$, $j = j^*$ and $S_k = T_{ij} \cap \partial\Omega_h$.

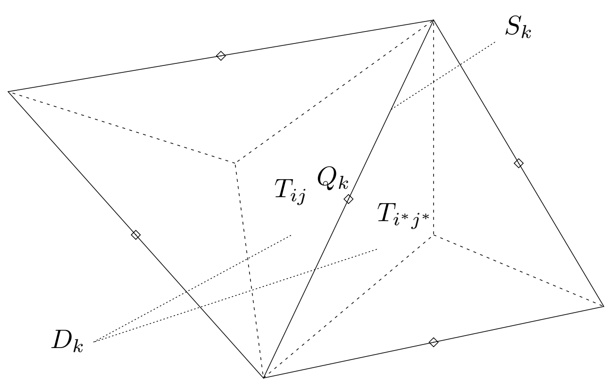

Fig. 4. Line of discontinuity S_k of finite volume D_k

The function $u_h \in X_h$ is in general discontinuous on $S_k - \{Q_k\}$. We denote $u_h|_{T_i} = (u_h)_k^p$ and $u_h|_{T_i^*} = (u_h)_k^n$. We denote the outer unit normals to T_i, T_{i^*} on S_k by $\mathbf{n}_k^p$, $\mathbf{n}_k^n$ and their components by n_{ks}^p, n_{ks}^n, $s = 1, 2$, respectively. (Obviously

$\mathbf{n}_k^p = -\mathbf{n}_k^n$.) We have

$$
\begin{aligned}
(6.23)\qquad & \int_{\partial T_{ij}} \sum_{s=1}^{2} f_s(u_h) n_s \,\mathrm{d}S + \int_{\partial T_{i^* j^*}} \sum_{s=1}^{2} f_s(u_h) n_s \,\mathrm{d}S \\
&= \int_{\partial D_k} \sum_{s=1}^{2} f_s(u_h) n_s \,\mathrm{d}S + \int_{S_k} \sum_{s=1}^{2} f_s\left((u_h)_k^p\right) n_{ks}^p \,\mathrm{d}S \\
&\quad + \int_{S_k} \sum_{s=1}^{2} f_s\left((u_h)_k^n\right) n_{ks}^n \,\mathrm{d}S \\
&= \int_{\partial D_k} \sum_{s=1}^{2} f_s(u_h) n_s \,\mathrm{d}S + \int_{S_k} \sum_{s=1}^{2} \left[f_s\left((u_h)_k^p\right) - f_s\left((u_h)_k^n\right)\right] n_{ks}^p \,\mathrm{d}S.
\end{aligned}
$$

Now from (6.21), (6.23) and the definition of the forms b_h^*, $\tilde{b}_h$ we have
(6.24)

$$
\tilde{b}_h(u_h, L_h v_h) - b_h^*(u_h, v_h) = \sum_{k \in J} v_h(Q_k) \int_{S_k} \sum_{s=1}^{2} \left[f_s\left((u_h)_k^p\right) - f_s\left((u_h)_k^n\right)\right] n_{ks}^p \,\mathrm{d}S.
$$

In the following we omit for simplicity the subscript k and write $u_h^p = (u_h)_k^p, u_h^n = (u_h)_k^n, n_s = n_{ks}^p$. By assumption (2.9) and the Taylor formula we can write

$$
\begin{aligned}
(6.25)\quad f_s(u_h^p) &= f_s(u_K) + f_s'(u_K)(u_h^p - u_K) + \frac{1}{2} f_s''(\eta_{sp})(u_h^p - u_K)^2,\ s = 1, 2, \\
f_s(u_h^n) &= f_s(u_K) + f_s'(u_K)(u_h^n - u_K) + \frac{1}{2} f_s''(\eta_{sn})(u_h^n - u_K)^2,\ s = 1, 2,
\end{aligned}
$$

where $u_K = u_h^p(Q_k) = u_h^n(Q_k)$, η_{sp} lies between u_h^p and u_K, η_{sn} lies between u_h^n and u_K. Using the above notation for $s = 1, 2$, we have
(6.26)

$$
f_s(u_h^p) - f_s(u_h^n) = f_s'(u_K)(u_h^p - u_h^n) + \frac{1}{2}\left[f_s''(\eta_{s1})(u_h^p - u_K)^2 - f_s''(\eta_{s2})(u_h^n - u_K)^2\right].
$$

Since u_h^p, u_h^n are linear functions, then ∇u_h^p and ∇u_h^n are constant and

$$
\begin{aligned}
(6.27)\qquad u_h^p(x) - u_h^n(x) &= (\nabla u_h^p - \nabla u_h^n) \cdot (x - Q_k), \quad x \in S_k, \\
u_h^p(x) - u_K &= \nabla u_h^p \cdot (x - Q_k), \quad x \in S_k, \\
u_h^n(x) - u_K &= \nabla u_h^n \cdot (x - Q_k), \quad x \in S_k, \\
|x - Q_k| &\leqslant \frac{h}{2}, \quad x \in S_k.
\end{aligned}
$$

Now from the assumptions on u_h, (6.26) and (6.27) we have

$$
\begin{aligned}
(6.28)\qquad & \left|\int_{S_k} \sum_{s=1}^{2} (f_s(u_h^p) - f_s(u_h^n))\, n_s \,\mathrm{d}S\right| \\
&\leqslant \left|\int\limits_{S_k} \sum_{s=1}^{2} f_s'(u_K)\,(u_h^p - u_h^n)\ \mathrm{d}S\right| \\
&\quad + \max_{|\xi|\leqslant M}\max_{s=1,2} |f_s''(\xi)|\, \frac{h^2}{4} \int\limits_{S_k} \big||\nabla u_h^p|^2 - |\nabla u_h^n|^2\big|\ \mathrm{d}S \\
&\leqslant \max_{|\xi|\leqslant M}\max_{s=1,2} |f_s''(\xi)|\, \frac{h^3}{4} \left(|\nabla u_h^p|^2 + |\nabla u_h^n|^2\right),
\end{aligned}
$$

since $\int_{S_k} (u_h^p - u_h^n)\,\mathrm{d}S = 0$, as one can easily show. Putting (6.28) into (6.24) and taking into account that $u(Q_k) = 0$ for $Q_k \in \partial\Omega_h$, we obtain

$$
\begin{aligned}
(6.29)\qquad & \left|\tilde{b}_h(u_h, L_h v_h) - b_h^*(u_h, v_h)\right| \\
&\leqslant \max_{|\xi|\leqslant M}\max_{s=1,2} |f_s''(\xi)|\, \frac{h^3}{4} \sum_{k\in J^o} |v_h(Q_k)| \left(|\nabla (u_h)_k^p|^2 + |\nabla (u_h)_k^n|^2\right) \\
&\leqslant \max_{|\xi|\leqslant M}\max_{s=1,2} |f_s''(\xi)|\, \frac{h^3}{2} \sum_{i\in I}\sum_{j=1}^{3} \left|\nabla u_h\big|_{T_i}\right|^2 \left|L_h v_h\big|_{T_{ij}}\right| \\
&\leqslant \hat{c}\max_{|\xi|\leqslant M}\max_{s=1,2} |f_s''(\xi)|\, \frac{h}{2} \sum_{i\in I}\sum_{j=1}^{3} |T_i| \left|\nabla u_h\big|_{T_i}\right|^2 \left|L_h v_h\big|_{T_{ij}}\right| \\
&\leqslant ch\|v_h\|_{L^\infty(\Omega)}\|u_h\|_{X_h}^2.
\end{aligned}
$$

Let us put $p = 2/\kappa$ ($\in (2,\infty)$). Similarly as in [29] we have

$$
(6.30)\qquad \|v_h\|_{L^p(\Omega)} \leqslant c(p)\|v_h\|_{X_h}, \quad v_h \in V_h,\ h \in (0, h_0).
$$

(Cf. also [31], Chap. II, Par. 23.) Then, using the inverse assumption (4.5), with the aid of the inverse inequality ([4], Theorem 3.2.6), we obtain

$$
(6.31)\qquad \|v_h\|_{L^\infty(\Omega)} \leqslant \tilde{c}(p)h^{-\frac{2}{p}}\|v_h\|_{L^p(\Omega)}, \quad v_h \in V_h,\ h \in (0, h_0).
$$

Now (6.29), (6.30) and (6.31) imply that

$$
(6.32)\qquad \left|\tilde{b}_h(u_h, L_h v_h) - b_h^*(u_h, v_h)\right| \leqslant c(M,\kappa)h^{1-\kappa}\|u_h\|_{X_h}^2\|v_h\|_{X_h}, \quad v_h \in V_h.
$$

Using (3.20), (6.18), the conservativity of the numerical flux H and the relations $\Gamma_{ij} = \Gamma_{ji}, l_{ij} = l_{ji}, \mathbf{n}_{ij} = -\mathbf{n}_{ji}$, we arrive at

(6.33)

$$
\begin{aligned}
&b_h^*(u_h, v_h) - b_h(u_h, v_h) \\
&= \sum_{i\in J} v_h(Q_i) \sum_{j\in s(i)} \left\{ \int_{\Gamma_{ij}} \sum_{s=1}^{2} f_s(u_h) n_s \, \mathrm{d}S - H\left(u_h(Q_i), u_h(Q_j), \mathbf{n}_{ij}\right) l_{ij} \right\} \\
&= \frac{1}{2} \sum_{i\in J} \sum_{j\in s(i)} \left[\int_{\Gamma_{ij}} \sum_{s=1}^{2} f_s(u_h) n_s \, \mathrm{d}S - H\left(u_h(Q_i), u_h(Q_j), \mathbf{n}_{ij}\right) l_{ij} \right] \left(v_h(Q_i) - v_h(Q_j)\right).
\end{aligned}
$$

If $i \in J$ and $j \in s(i)$ then we denote by T^{ij} the triangle from $\mathscr{T}_h$ such that $\Gamma_{ij} \subset T^{ij}$. It is easy to see that

$$
\begin{aligned}
&|Q_i - Q_j| \leqslant \frac{h}{2}, \quad |x - Q_i| \leqslant \frac{h}{2} \text{ for } x \in \Gamma_{ij},\ l_{ij} \leqslant \frac{2}{3}h, \\
&|u_h(Q_i) - u_h(Q_j)| \leqslant \frac{h}{2} |\nabla u_h|_{T^{ij}}|, \\
&|u_h(x) - u_h(Q_i)| \leqslant \frac{h}{2} |\nabla u_h|_{T^{ij}}| \quad \text{for } x \in \Gamma_{ij}, \\
&|v_h(Q_i) - v_h(Q_j)| \leqslant \frac{h}{2} |\nabla v_h|_{T^{ij}}|.
\end{aligned} \tag{6.34}
$$

In virtue of the consistency and local Lipschitz-continuity of H, the bound $\|u_h\|_{L^\infty(\Omega)} \leqslant M$ and (6.34), we conclude that

$$
\begin{aligned}
&\left| \int_{\Gamma_{ij}} \sum_{s=1}^{2} f_s(u_h) n_s \, \mathrm{d}S - H\left(u_h(Q_i), u_h(Q_j), \mathbf{n}_{ij}\right) l_{ij} \right| \\
&\leqslant \left| \int_{\Gamma_{ij}} \sum_{s=1}^{2} \left(f_s(u_h) - f_s(u_h(Q_i))\right) n_s \, \mathrm{d}S \right| \\
&\quad + \left| \sum_{s=1}^{2} f_s(u_h(Q_i)) - H\left(u_h(Q_i), u_h(Q_j), \mathbf{n}_{ij}\right) \right| l_{ij} \\
&= \left| \int_{\Gamma_{ij}} \left[H\left(u_h(x), u_h(x), \mathbf{n}_{ij}\right) - H\left(u_h(Q_i), u_h(Q_i), \mathbf{n}_{ij}\right) \right] \mathrm{d}S \right| \\
&\quad + \left| H\left(u_h(Q_i), u_h(Q_i), \mathbf{n}_{ij}\right) - H\left(u_h(Q_i), u_h(Q_j), \mathbf{n}_{ij}\right) \right| l_{ij} \\
&\leqslant 2c(M) \max_{x\in\Gamma_{ij}} |u_h(x) - u_h(Q_i)| \, l_{ij} + c(M) \, |u_h(Q_i) - u_h(Q_j)| \, l_{ij} \\
&\leqslant c(M) h^2 \left|\nabla u_h|_{T^{ij}}\right|.
\end{aligned} \tag{6.35}
$$

This, (6.33) and (6.34) immediately yield the estimate

$$|b_h^*(u_h, v_h) - b_h(u_h, v_h)| \leqslant \frac{1}{2} \sum_{i\in J} \sum_{j\in s(i)} c(M) h^3 \, |\nabla u_h|_{T^{ij}}| \, |\nabla v_h|_{T^{ij}}| \, . \tag{6.36}$$

Taking into account that each triangle $T \in \mathscr{T}_h$ appears in the sum in (6.36) as some T^{ij} at most six times and using (4.6), we find that

$$\begin{aligned} |b_h^*(u_h, v_h) - b_h(u_h, v_h)| \; &\leqslant 3c_2 c(M) h \sum_{i\in I} |T_i| \, |\nabla u_h|_{T_i}| \, |\nabla v_h|_{T_i}| \\ &= ch \sum_{i\in I} \int_{T_i} |\nabla u_h| \, |\nabla v_h| \, \mathrm{d}x \leqslant ch \|u_h\|_{X_h} \|v_h\|_{X_h}. \end{aligned}$$

This, (6.19), (6.20) and (6.32) finally yield (6.12). □

Lemma 7. *If $M > 0$, then there exists a constant $c^* = c^*(M)$ such that*

$$\begin{aligned} &|b_h(u_h, v_h)| \leqslant c^* \|u_h\|_{L^\infty(\Omega)} \|v_h\|_{X_h}, \\ &u_h \in V_h \cap L^\infty(\Omega), \ \|u_h\|_{L^\infty(\Omega)} \leqslant M, \ v_h \in V_h, \ h \in (0, h_0). \end{aligned} \tag{6.37}$$

Proof. Let $u_h, v_h \in V_h$ and $\|u_h\|_{L^\infty(\Omega)} \leqslant M$. Using (3.20), the conservativity of the numerical flux and the relations $\Gamma_{ij} = \Gamma_{ji}$, $l_{ij} = l_{ji}$, $\mathbf{n}_{ij} = -\mathbf{n}_{ji}$, we find that

$$\begin{aligned} b_h(u_h, v_h) &= \sum_{i\in J} v_h(Q_i) \sum_{j\in s(i)} H\left(u(Q_i), u(Q_j), \mathbf{n}_{ij}\right) l_{ij} \\ &= \frac{1}{2} \sum_{i\in J} \sum_{j\in s(i)} H\left(u(Q_i), u(Q_j), \mathbf{n}_{ij}\right) \left(v_h(Q_i) - v_h(Q_j)\right) l_{ij}. \end{aligned} \tag{6.38}$$

Let us use the symbol T^{ij} in the same way as in the proof of Lemma 6. Then (6.38), (2.9), (6.34), the consistency and local Lipschitz-continuity of H imply that

$$|b_h(u_h, v_h)| \leqslant \frac{1}{2} c(M) \max_{\overline{\Omega}} |u_h| \sum_{i\in J} \sum_{j\in s(i)} h^2 \left|\nabla v_h\big|_{T^{ij}}\right| . \tag{6.39}$$

From (6.39), (4.6), the fact that each $T \in \mathscr{T}_h$ appears in the above sum as some T^{ij} at most six times and from the Cauchy inequality we conclude that

$$\begin{aligned} |b_h(u_h, v_h)| \; &\leqslant 3c_2 c(M) \|u_h\|_{L^\infty(\Omega)} \sum_{i\in I} |T_i| \left|\nabla v_h\big|_{T_i}\right| \\ &= 3c_2 c(M) \|u_h\|_{L^\infty(\Omega)} \int_\Omega |\nabla v_h| \, \mathrm{d}x \\ &\leqslant 3c_2 c(M) \left(\mathrm{meas}(\Omega)\right)^{1/2} \|u_h\|_{L^\infty(\Omega)} \|v_h\|_{X_h}, \end{aligned} \tag{6.40}$$

which is (6.37) with $c^* := 3c_2 c(M) (\mathrm{meas}(\Omega))^{1/2}$. □

The above results imply the following

Theorem 3. *Let* (5.1), (5.2) and (5.13) *hold. Then the solution* u_h^{k+1} *of the discrete problem* (3.21)–(3.23) *satisfies the relation*

$$(6.41)\quad (u_h^{k+1} - u_h^k, v_h) + \tau\tilde{b}_h(u_h^k, v_h) + \tau\nu(\!(u_h^{k+1}, v_h)\!)_h = \tau(g^{k+1}, v_h) + l_h^k(v_h),$$
$$v_h \in V_h,\ t_k \in [0,T),\ h \in (0,h_0),$$

where

$$(6.42)\qquad l_h^k(v_h) = l_{1h}^k(v_h) + l_{2h}^k(v_h),$$

$$(6.43)\qquad l_{1h}^k(v_h) = \tau\left(\tilde{b}_h(u_h^k, v_h) - b_h(u_h^k, v_h)\right),$$

$$(6.44)\qquad l_{2h}^k(v_h) = -\tau\left((g^k, v_h) - (g^k, v_h)_h\right).$$

Moreover, for any $\kappa \in (0,1)$ *there exists a constant* $c > 0$ *independent of* v_h, k, τ *and* h *(but dependent on* κ *and* M*) such that*

$$(6.45)\qquad \left|l_{1h}^k(v_h)\right| \leqslant c\tau h^{1-\kappa}\left(\|u_h^k\|_{X_h}^2 + \|u_h^k\|_{X_h}\right)\|v_h\|_{X_h}.$$

There exists a constant $\hat{c}$ *independent of* v_h, k, τ *and* h *such that*

$$(6.46)\qquad \left|l_{2h}^k(v_h)\right| \leqslant \hat{c}\tau h\|g^k\|_{W^{1,q}(\Omega)}\|v_h\|_{X_h}.$$

P r o o f is an immediate consequence of Theorem 2 and Lemma 6. □

7. A priori estimates

Theorem 4. *Let* (5.1) *and* (5.2) *hold. Then there exists a constant* $\hat{C} > 0$ *independent of* h, τ *and* m *(but dependent on* ν*) such that*

$$(7.1)\qquad \max_{t_k\in[0,T]} \|u_h^k\|_{L^2(\Omega)} \leqslant \hat{C},$$

$$(7.2)\qquad \sum_{k=1}^{m} \|u_h^k - u_h^{k-1}\|_{L^2(\Omega)}^2 \leqslant \hat{C},\quad t_m \in (0,T],$$

$$(7.3)\qquad \nu\tau\sum_{k=0}^{m} \|u_h^k\|_{X_h}^2 \leqslant \hat{C},\quad t_m \in [0,T],$$

for all $\tau, h > 0$ *satisfying the conditions* $h \in (0,h_0)$ *and* (5.13).

P r o o f. In view of Lemma 3 and Theorem 2, conditions (5.1), (5.2) and (5.27) imply (5.14). If we set $v_h := u_k^{k+1}$ in (3.23) and use the relation

$$(y-z,y)=\frac{1}{2}\left(\|y\|^2_{L^2(\Omega)}-\|z\|^2_{L^2(\Omega)}+\|y-z\|^2_{L^2(\Omega)}\right) \tag{7.4}$$

valid for $y,z\in L^2(\Omega)$, we get

$$\begin{aligned}&\|u_h^{k+1}\|^2_{L^2(\Omega)}-\|u_h^k\|^2_{L^2(\Omega)}+\|u_h^{k+1}-u_h^k\|^2_{L^2(\Omega)}+2\tau\nu\|u_h^{k+1}\|^2_{X_h}\\&\quad=2\tau(g^k,u_h^{k+1})_h-2\tau b_h(u_h^k,u_h^{k+1}).\end{aligned} \tag{7.5}$$

In virtue of Theorem 2, Lemma 7 and Young's inequality for $\varepsilon>0$, we have

$$2\left|b_h(u_h^k,u_h^{k+1})\right|\leqslant (c^*M)^2/\varepsilon+\varepsilon\|u_h^{k+1}\|^2_{X_h}. \tag{7.6}$$

By (3.29) and the Cauchy inequality,

$$\left|(g^k,u_h^{k+1})_h\right|=\left|(L_hg^k,L_hu_h^{k+1})\right|\leqslant\|L_hg^k\|_{L^2(\Omega)}\|L_hu_h^{k+1}\|_{L^2(\Omega)}. \tag{7.7}$$

With the aid of the definition of the operator L_h and the continuous imbedding $W^{1,q}(\Omega)\hookrightarrow C(\overline{\Omega})$ it is easy to find that

$$\|L_hg^k\|_{L^2(\Omega)}\leqslant c\|g\|_{C(0,T;W^{1,q}(\Omega))} \tag{7.8}$$

with c independent of h and k. Further, from (3.14) and (3.29) we have

$$\|v_h\|_{L^2(\Omega)}=\|v_h\|_h=\|L_hv_h\|_{L^2(\Omega)},\quad \forall v_h\in X_h. \tag{7.9}$$

This, (7.7), (7.8), Lemma 4 and Young's inequality yield the estimate

$$\begin{aligned}2\left|(g^k,u_h^{k+1})_h\right|&\leqslant 2\hat{c}_1c\|g\|_{C(0,T;W^{1,q}(\Omega))}\|u_h\|_{X_h}\\&\leqslant \hat{c}_1^2c^2\|g\|^2_{C(0,T;W^{1,q}(\Omega))}/\varepsilon+\varepsilon\|u_h\|^2_{X_h},\quad \varepsilon>0.\end{aligned} \tag{7.10}$$

Now choosing $\varepsilon=\nu/2$, from (7.5), (7.6) and (7.10) we get

$$\begin{aligned}&\|u_h^{k+1}\|^2_{L^2(\Omega)}-\|u_h^k\|^2_{L^2(\Omega)}+\|u_h^{k+1}-u_h^k\|^2_{L^2(\Omega)}+\nu\tau\|u_h^{k+1}\|^2_{X_h}\leqslant\overline{C}\tau,\\&\overline{C}=2(\hat{c}_1^2c^2\|g\|^2_{C(0,T;W^{1,q}(\Omega))}+(c^*M)^2)/\nu.\end{aligned} \tag{7.11}$$

Summation over $k=0,\dots,m-1$ ($t_m\in(0,T]$) and the use of (7.9), (3.21), (6.6) and Lemma 4 yield

$$\begin{aligned}&\|u_h^m\|^2_{L^2(\Omega)}+\sum_{k=1}^m\|u_h^k-u_h^{k-1}\|^2_{L^2(\Omega)}+\tau\nu\sum_{k=1}^m\|u_h^k\|^2_{X_h}\\&\quad\leqslant\overline{C}T+\|u_h^0\|^2_{L^2(\Omega)}\leqslant\overline{C}T+\hat{c}_1^2\|u_h^0\|^2_{X_h}\\&\quad\leqslant\overline{C}T+\hat{c}_1^2c\|u^0\|^2_{H^1(\Omega)}\leqslant\hat{C},\quad t_m\in(0,T).\end{aligned} \tag{7.12}$$

Now, estimates (7.12) immediately imply (7.1)–(7.3). □

Theorem 5. *Let* (5.1) *and* (5.2) *hold. Then there exists a constant* $C > 0$ *such that functions* $u_{h\tau}$ *and* $w_{h\tau}$ *defined by* (4.1) *and* (4.2) *satisfy the estimates*

$$\|u_{h\tau}\|_{L^2(-1,T;L^2(\Omega))} \leqslant C, \tag{7.13}$$

$$\|w_{h\tau}\|_{L^2(0,T;L^2(\Omega))} \leqslant C, \tag{7.14}$$

$$\|u_{h\tau}\|_{L^2(-1,T;V_h)} \leqslant C, \tag{7.15}$$

$$\|w_{h\tau}\|_{L^2(0,T;V_h)} \leqslant C \tag{7.16}$$

for all $h \in (0, h_0)$ *and* $\tau > 0$ *satisfying condition* (5.13). *Moreover, there exists a constant* $\tilde{C} > 0$ *such that*

$$\|u_{h\tau} - w_{h\tau}\|_{L^2(Q_T)} \leqslant \tilde{C}\sqrt{\tau} \tag{7.17}$$

for all h *and* τ *with the above properties.*

Proof. Assertions (7.13) and (7.15) immediately follow from (7.1) and (7.3), respectively.

Now let us prove (7.17). We have

$$\begin{aligned}\|u_{h\tau} - w_{h\tau}\|^2_{L^2(Q_T)} &= \sum_{k=1}^{r} \int_{t_{k-1}}^{t_k} \left\| \frac{t-t_k}{\tau} \left(u_h^k - u_h^{k-1}\right)\right\|^2_{L^2(\Omega)} \,\mathrm{d}t \\ &\leqslant \sum_{k=1}^{r} \|u_h^k - u_h^{k-1}\|^2_{L^2(\Omega)} \int_{t_{k-1}}^{t_k} \left(\frac{t-t_k}{\tau}\right)^2 \mathrm{d}t \leqslant \hat{C}\frac{\tau}{3},\end{aligned}$$

as follows from (7.2).

Assertion (7.14) is a consequence of (7.13) and (7.17). Finally,

$$\|w_{h\tau}\|^2_{L^2(0,T;V_h)} = \sum_{k=1}^{r} \int_{t_{k-1}}^{t_k} \|w_{h\tau}(t)\|^2_{X_h} \,\mathrm{d}t \tag{7.18}$$

and for $t \in (t_{k-1}, t_k)$, using the convexity of the function "$u \to \|u\|^2_{X_h}$," we get

$$\|w_{h\tau}(t)\|^2_{X_h} = \left\| u_h^{k-1} + \frac{t-t_k}{\tau}\left(u_h^k - u_h^{k-1}\right)\right\|^2_{X_h} \leqslant \|u_h^{k-1}\|^2_{X_h} + \|u_h^k\|^2_{X_h}. \tag{7.19}$$

This and (7.3) already yield (7.16). □

We rewrite scheme (3.23) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(w_{h\tau}(t), v_h)_h + \nu(\!(u_{h\tau}, v_h)\!)_h + b_h(u_{h\tau}(t-\tau), v_h) = (g_{h\tau}(t), v_h)_h \quad \text{for a. e. } t \in (0,T),\ v_h \in V_h, \tag{8.1}$$

where

$$g_{h\tau}(t) = g^{k+1} \quad \text{for} \quad t \in (t_k, t_{k+1}). \tag{8.2}$$

The weak solution of the continuous problem (2.1)–(2.3) satisfies the condition $u(\cdot,t) \in V$ for a.e. $t \in (0,T)$ and the approximate solution $u_h^k \in V_h$ for $t_k \in [0,T]$. Since we use nonconforming FEM and thus $V_h \not\subset V$, the convergence analysis is more complex than in the conforming case investigated in [11]. Our further considerations will be based on results from [31] and [8], Section 8.9.

If $v_h \in V_h$, then the distributional derivatives are not elements of $L^2(\Omega)$. Therefore, we will define the discrete analogue $d_{ih}v_h$ of the derivatives $\frac{\partial v_h}{\partial x_i}$, $i = 1,2$:

$$(d_{ih}v_h)(x) = \left(\frac{\partial v_h}{\partial x_i}\right)(x), \quad x \in T,\ T \in \mathscr{T}_h. \tag{8.3}$$

Obviously, $d_{ih}v \in L^2(\Omega)$.

We introduce the space $F = \left[L^2(\Omega)\right]^3$ and the mapping $\omega\colon\ V \to F$ defined by

$$u \in V \longmapsto \omega u = \left(u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right) \in F. \tag{8.4}$$

The space F is equipped with the norm

$$\|\varphi\|_F = \left(\sum_{i=0}^{2} \|\varphi_i\|^2\right)^{1/2} \text{ for } \varphi = (\varphi_0, \varphi_1, \varphi_2) \in F. \tag{8.5}$$

We define a scalar product in F by

$$(\varphi, \psi)_F = \sum_{i=0}^{2} (\varphi_i, \psi_i)_{L^2(\Omega)}, \quad \varphi = (\varphi_0, \varphi_1, \varphi_2),\ \psi = (\psi_0, \psi_1, \psi_2) \in F. \tag{8.6}$$

Further, we define the imbedding operator $J_h\colon\ V_h \to F$ by

$$v_h \in V_h \longmapsto J_h v_h = (v_h, d_{1h}v_h, d_{2h}v_h) \in F. \tag{8.7}$$

From (8.5) and the discrete Friedrichs inequality (6.1) we have

$$\|J_h v_h\|_F^2 = \|v_h\|_{L^2(\Omega)}^2 + \|v_h\|_{X_h}^2 \leqslant (c_1^2+1)\|v_h\|_{X_h}^2 \text{ for } v_h \in V_h, \tag{8.8}$$

which leads to

$$\|J_h v_h\|_F \leqslant c\|v_h\|_{X_h} \text{ for } v_h \in V_h. \tag{8.9}$$

This implies that the operators J_h, $h \in (0, h_0)$, are *uniformly bounded*:

$$\|J_h\| = \sup_{0 \neq v_h \in V_h} \frac{\|J_h v_h\|_F}{\|v_h\|_{X_h}} \leqslant c, \quad h \in (0, h_0). \tag{8.10}$$

We will also work with the operator $I_h\colon V \to V_h$ defined by (3.10). Let us prove several auxiliary results.

Lemma 8. *1. For each $v \in V$,*

$$\lim_{h\to 0} J_h(I_h v) = \omega v \text{ strongly in } F. \tag{8.11}$$

2. If for a sequence $h_n \in (0, h_0)$, $n = 1, 2, \dots$ we have $h = h_n \to 0$ as $n \to \infty$, $v_h \in V_h$ and

$$\lim_{h\to 0} J_h v_h = \varphi \text{ weakly in } F, \tag{8.12}$$

then there exists $v \in V$ such that $\varphi = \omega v$.

P r o o f. 1. Let $v \in V$. In view of (8.4) and (8.7) we have

$$\|J_h(I_h v) - \omega v\|_F^2 = \|I_h v - v\|_{X_h}^2 + \|I_h v - v\|_{L^2(\Omega)}^2 \to 0 \text{ for } h \to 0 \tag{8.13}$$

as follows from Lemma 5.

2. To establish assertion 2, see the 2nd and 3rd part of the proof of 8.9.118 from [8] or Chap. 1, Sec. 5 from [21]. □

R e m a r k 1. The family of triplets $\{V_h, J_h, I_h\}_{h\in(0,h_0)}$ together with $\{V, F, \omega\}$ is called *the external approximation* of the space V. If (8.10) holds, the external approximation of V is called *stable.* If the operators I_h, J_h have properties (8.11) and (8.12), then the external approximation of V is called *convergent* (cf. [21], Chap. I, Sec. 5. or [31], Chap. I, Par. 3).

Lemma 9. *There exists a constant $c > 0$ such that*

$$\left|\sum_{T\in\mathscr{T}_h}\int_{\partial T} v_h\varphi n_i \,\mathrm{d}S\right| \leqslant ch\|\varphi\|_{H^1(\Omega)}\|v_h\|_{X_h}, \quad h\in(0,h_0), \tag{8.14}$$

for any $\varphi\in H^1(\Omega)$ and $v_h\in V_h$. Here n_i denotes the i-th component of the unit outer normal to ∂T.

Proof. See [8], Lemma 8.9.85 and Lemma 4. □

Now let us return to the systems of functions $u_{h\tau}$ and $w_{h\tau}$ defined in (4.1) and (4.2), respectively, for $h\in(0,h_0)$ and $\tau>0$ satisfying the stability condition (5.13). Then $u_{h\tau}$ and $w_{h\tau}$ satisfy estimates (7.13), (7.15) and (7.14), (7.16), respectively.

Lemma 10. *There exist sequences $h=h_n$, $\tau=\tau_h\to 0$ as $n\to\infty$ satisfying (5.13) and functions $u\in L^2(-1,T;V)$, $\varphi\in L_2(-1,T;F)$ such that*

$$u_{h\tau}\to u \text{ weakly in } L^2(-1,T;L^2(\Omega)), \tag{8.15}$$

$$J_h u_{h\tau}\to\varphi \text{ weakly in } L^2(-1,T;F), \tag{8.16}$$

and $\varphi=\omega u$.

Proof. In view of (7.15) and (8.9) we have

$$\|J_h u_{h\tau}\|_{L^2(-1,T;F)}\leqslant c\|u_{h\tau}\|_{L^2(-1,T;V_h)}\leqslant C, \quad h\in(0,h_0),\ \tau>0, \tag{8.17}$$

where $C>0$ is a constant independent of h and τ. Since the spaces $L^2(-1,T;L^2(\Omega))$ and $L^2(-1,T;F)$ are reflexive, we obtain sequences $h=h_n$, $\tau=\tau_n\to 0$ and functions u, φ such that (8.15) and (8.16) hold.

Further, we prove that $\varphi=\omega u$. If $\varphi=(\varphi_0,\varphi_1,\varphi_2)$, then obviously $u=\varphi_0$. We want to show that $\frac{\partial u}{\partial x_s}=\varphi_s$, $s=1,2$ in the sense of distributions on $\tilde{Q}_T=\Omega\times(-1,T)$. We can proceed similarly as in the proof of 8.9.81 from [8]. Let $\varphi\in C^\infty(\overline{\tilde{Q}}_T)$. Then (8.15) and (8.16) imply that

$$\int_{\tilde{Q}_T} u_{h\tau}\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x\,\mathrm{d}t\to\int_{\tilde{Q}_T} u\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x\,\mathrm{d}t, \tag{8.18}$$

$$\int_{\tilde{Q}_T} d_{sh}u_{h\tau}\varphi\,\mathrm{d}x\,\mathrm{d}t\to\int_{\tilde{Q}_T}\varphi_s\varphi\,\mathrm{d}x\,\mathrm{d}t. \tag{8.19}$$

Using Fubini's and Green's theorems, we get

$$\begin{aligned}\int_{\tilde{Q}_T} u_{h\tau}\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x\,\mathrm{d}t &= \int_{-1}^{T}\Big(\int_\Omega u_{h\tau}\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x\Big)\,\mathrm{d}t\\ &= -\int_{-1}^{T}\Big(\sum_{i\in I}\int_{T_i}\frac{\partial u_{h\tau}}{\partial x_s}\varphi\,\mathrm{d}x\Big)\,\mathrm{d}t+\int_{-1}^{T}\Big(\sum_{i\in I}\int_{\partial T_i}u_{h\tau}\varphi n_s\,\mathrm{d}S\Big)\,\mathrm{d}t.\end{aligned} \tag{8.20}$$

Since $\varphi \in C^\infty(\overline{\tilde{Q}}_T)$ and $u_{h\tau}\colon (-1,T) \to V_h$, we conclude from Lemma 9 and the Cauchy inequality that

$$(8.21)\qquad \left|\int_{-1}^T \Big(\sum_{i\in I}\int_{\partial T_i} u_{h\tau}\varphi n_s \,\mathrm{d}S\Big)\,\mathrm{d}t\right| \leqslant ch\int_{-1}^T \|u_{h\tau}(t)\|_{X_h}\|\varphi(t)\|_{H^1(\Omega)}\,\mathrm{d}t$$
$$\leqslant ch\|u_{h\tau}\|_{L^2(-1,T;V_h)}\|\varphi\|_{L^2(-1,T;H^1(\Omega))} \leqslant \tilde{C}h \to 0 \text{ as } h,\tau\to 0.$$

This, (8.18)–(8.20) and the relation

$$(8.22)\qquad \int_{-1}^T\Big(\sum_{i\in I}\int_{T_i}\frac{\partial u_{h\tau}}{\partial x_s}\varphi\,\mathrm{d}x\Big)\,\mathrm{d}t = \int_{\tilde{Q}_T} d_{sh}u_{h\tau}\varphi\,\mathrm{d}x\,\mathrm{d}t$$

imply that

$$(8.23)\qquad \int_{\tilde{Q}_T} u\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x\,\mathrm{d}t = -\int_{\tilde{Q}_T}\varphi_s\varphi\,\mathrm{d}x\,\mathrm{d}t.$$

Taking here $\varphi\in C_0^\infty(\tilde{Q}_T)\subset C^\infty(\overline{\tilde{Q}}_T)$, we find that $\frac{\partial u}{\partial x_s} = \varphi_s \in L^2(-1,T;L^2(\Omega))$, $s=1,2$. Hence, $u\in L^2(-1,T;H^1(\Omega))$ and $\varphi = \omega u$. As we see, we have

$$(8.24)\qquad \int_{\tilde{Q}_T} u\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x\,\mathrm{d}t = -\int_{\tilde{Q}_T}\frac{\partial u}{\partial x_s}\varphi\,\mathrm{d}x\,\mathrm{d}t \qquad \forall\varphi\in C^\infty(\overline{\tilde{Q}}_T),\ s=1,2.$$

The application of Green's theorem yields the identity

$$(8.25)\qquad \int_{-1}^T\Big(\int_{\partial\Omega} u\varphi n_s\,\mathrm{d}S\Big)\,\mathrm{d}t = 0 \qquad \forall\varphi\in C^\infty(\overline{\tilde{Q}}_T),\ s=1,2,$$

which implies that $u(t)=0$ on $\partial\Omega$ for a.e. $t\in(-1,T)$. Thus, $u\in L^2(-1,T;V)$. □

Lemma 11. *If $h=h_n$ and $\tau=\tau_n$ are sequences from* Lemma 10, *then*

$$(8.26)\qquad w_{h\tau}\to u \text{ weakly in } L^2(0,T;L^2(\Omega))$$
$$(8.27)\qquad J_h w_{h\tau}\to \omega u \text{ weakly in } L^2(0,T;F).$$

P r o o f. We use Lemma 10 and (7.17). □

Lemma 12. *Let $h=h_n\to 0$ as $n\to\infty$, $v_h\in X_h$, $v\in V$, $J_hv_h\to\omega v$ weakly in F. Then $v_h\to v$ strongly in $L^2(\Omega)$.*

P r o o f. See part 5) of the proof of Theorem 8.9.118 from [8]. □

In the sequel we will use the compactness criterion based on the Fourier transform $\hat{w}_{h\tau}$ of the function $w_{h\tau}$ with respect to time:

$$\hat{w}_{h\tau}(s) = \int_{\mathbb{R}} w_{h\tau}(t)\mathrm{e}^{-2\pi \mathrm{i} st}\,\mathrm{d}t. \tag{8.28}$$

Lemma 13. *We have*

$$\int_{\mathbb{R}} |s|^{2\gamma} \|\hat{w}_{h\tau}(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \leqslant c \quad \text{for } 0 < \gamma < 1/4 \tag{8.29}$$

with a constant c independent of h, τ.

P r o o f. For a.e. $t \in (0,T)$ we define $r_{h\tau}(t) \in V_h$ by the identity

$$((r_{h\tau}(t), v_h))_h = (g_{h\tau}(t), v_h)_h - \nu((u_{h\tau}(t), v_h))_h - b_h(u_{h\tau}(t-\tau), v_h) \quad \forall v_h \in V_h. \tag{8.30}$$

Hence, by (8.1),

$$\frac{\mathrm{d}}{\mathrm{d}t}(w_{h\tau}(t), v_h)_h = ((r_{h\tau}(t), v_h))_h, \quad v_h \in V_h, \text{ a.e. } t \in (0,T). \tag{8.31}$$

Substituting $v_h := r_{h\tau}(t)$ in (8.30) and using (6.37), (6.1), (2.10) and (5.15), we obtain

$$\begin{aligned}\|r_{h\tau}(t)\|^2_{X_h} &\leqslant \|r_{h\tau}(t)\|_{L^2(\Omega)}\|g_{h\tau}(t)\|_{L^2(\Omega)} + \nu\|u_{h\tau}(t)\|_{X_h}\|r_{h\tau}(t)\|_{X_h} \\ &\quad + c^*\|u_{h\tau}(t)\|_{L^\infty(\Omega)}\|r_{h\tau}(t)\|_{X_h} \leqslant c\|r_{h\tau}(t)\|_{X_h}\,(1 + \|u_{h\tau}(t)\|_{X_h})\end{aligned}$$

and thus, in view of (4.2),

$$\|r_{h\tau}(t)\|_{X_h} \leqslant c\left(1 + \|u_h^k\|_{X_h}\right), \quad t_{k-1} \leqslant t \leqslant t_k. \tag{8.32}$$

This, (7.3) and the Cauchy inequality imply that

$$\begin{aligned}\int_0^T \|r_{h\tau}(t)\|_{X_h}\,\mathrm{d}t &\leqslant \sqrt{T}\left(\int_0^T \|r_{h\tau}(t)\|^2_{X_h}\,\mathrm{d}t\right)^{1/2} \\ &\leqslant c\sqrt{T}\left(T + \tau\sum_{k=1}^{r}\|u_h^k\|^2_{X_h}\right)^{1/2} \leqslant \text{const.}\end{aligned} \tag{8.33}$$

Now we put

$$\bar{r}_{h\tau}(t) = \begin{cases} r_{h\tau}(t), & t \in (0,T), \\ 0, & t \notin (0,T). \end{cases} \tag{8.34}$$

Then the Fourier transform $\hat{\overline{r}}_{h\tau}$ of $\overline{r}_{h\tau}$ satisfies the relations

$$\|\hat{\overline{r}}_{h\tau}(s)\|_{X_h} = \left\|\int_{-\infty}^{\infty} \mathrm{e}^{-2\pi ist}\overline{r}_{h\tau}(t)\,\mathrm{d}t\right\|_{X_h} \leqslant \int_0^T \|r_{h\tau}(t)\|_{X_h}\,\mathrm{d}t \leqslant c, \quad s \in \mathbb{R}. \tag{8.35}$$

The distribution derivative of the function $(w_{h\tau}(t), v_h)_h$ over $\mathbb{R}$ has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(w_{h\tau}(t), v_h)_h = (\!(\overline{r}_{h\tau}(t), v_h)\!)_h + (u_h^0, v_h)_h\delta_0 - (u_h^r, v_h)_h\delta_T, \quad v_h \in V_h, \tag{8.36}$$

where δ_0 and δ_T are the Dirac distributions concentrated at $t = 0$ and $t = T$, respectively. The Fourier transform yields

$$\begin{aligned} 2\pi is(\hat{w}_{h\tau}(s), v_h)_h = (\!(\hat{\overline{r}}_{h\tau}(s), v_h)\!)_h + (u_h^0, v_h)_h \\ - (u_h^r, v_h)_h \exp(-2\pi isT), \quad s \in \mathbb{R}. \end{aligned} \tag{8.37}$$

Putting here $v_h := \hat{w}_{h\tau}(s)$, we have

$$\begin{aligned} 2\pi\mathrm{i}s(\hat{w}_{h\tau}(s), \hat{w}_{h\tau}(s))_h = (\!(\hat{\overline{r}}_{h\tau}(s), \hat{w}_{h\tau}(s))\!)_h \\ + (u_h^0, \hat{w}_{h\tau}(s))_h - (u_h^r, \hat{w}_{h\tau}(s))_h \exp(-2\pi isT), \quad c \in \mathbb{R}. \end{aligned} \tag{8.38}$$

From (6.10), (8.38) and the Cauchy inequality we find that

$$\begin{aligned} 2\pi|s|\|\hat{w}_{h\tau}(s)\|^2_{L^2(\Omega)} \leqslant \|\hat{\overline{r}}_{h\tau}(s)\|_{X_h}\|\hat{w}_{h\tau}(s)\|_{X_h} \\ + \|u_h^0\|_{L^2(\Omega)}\|\hat{w}_{h\tau}(s)\|_{L^2(\Omega)} \\ + \|u_h^r\|_{L^2(\Omega)}\|\hat{w}_{h\tau}(s)\|_{L^2(\Omega)}. \end{aligned} \tag{8.39}$$

In view of (8.39), (8.35), (7.1) and (6.1), we obtain

$$|s|\|\hat{w}_{h\tau}(s)\|^2_{L^2(\Omega)} \leqslant c\|\hat{w}_{h\tau}(s)\|_{X_h}, \quad s \in \mathbb{R}. \tag{8.40}$$

Let $0 < \gamma < 1/4$. Obviously, there exists a constant $c(\gamma)$ such that

$$|s|^{2\gamma} \leqslant c(\gamma)\frac{1 + |s|}{1 + |s|^{1-2\gamma}}, \; s \in \mathbb{R}, \tag{8.41}$$

and

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}s}{(1 + |s|^{1-2\gamma})^2} < \infty. \tag{8.42}$$

Using (8.40)–(8.42), the Cauchy inequality and (6.1), we find that

$$
\begin{aligned}
(8.43)\qquad &\int_{-\infty}^{\infty} |s|^{2\gamma} \|\hat{w}_{h\tau}(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s \\
&\leqslant c\int_{-\infty}^{\infty} \|\hat{w}_{h\tau}(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s + c\int_{-\infty}^{\infty} \frac{\|\hat{w}_{h\tau}(s)\|_{X_h}}{1+|s|^{1-2\gamma}} \,\mathrm{d}s \\
&\leqslant c\int_{-\infty}^{\infty} \|\hat{w}_{h\tau}(s)\|^2_{X_h} \,\mathrm{d}s \left\{1 + \left(\int_{-\infty}^{\infty} \frac{\mathrm{d}s}{(1+|s|^{1-2\gamma})^2}\right)^{1/2}\right\} \\
&\leqslant c\int_{-\infty}^{\infty} \|\hat{w}_{h\tau}(s)\|^2_{X_h} \,\mathrm{d}s.
\end{aligned}
$$

With the aid of (3.16), Fubini's theorem, the differentiation of the integral with respect to a parameter, Parseval's equality and (7.16), we obtain

$$
\begin{aligned}
(8.44)\quad &\int_{-\infty}^{\infty} \|\hat{w}_{h\tau}(s)\|^2_{X_h} \,\mathrm{d}s \\
&= \int_{-\infty}^{\infty} \sum_{i\in I} \int_{T_i} \left|\nabla \int_{-\infty}^{\infty} w_{h\tau}(t)\mathrm{e}^{-2\pi i t s} \,\mathrm{d}t\right|^2 \mathrm{d}x \,\mathrm{d}s \\
&= \sum_{i\in I} \int_{T_i} \left(\int_{-\infty}^{\infty} \left|\int_{-\infty}^{\infty} \nabla w_{h\tau}(t)\mathrm{e}^{-2\pi i t s} \,\mathrm{d}t\right|^2 \mathrm{d}s\right) \mathrm{d}x \overset{\text{Parseval's equality}}{=} \\
&= \sum_{i\in I} \int_{T_i} \left(\int_{-\infty}^{\infty} |\nabla w_{h\tau}(t)|^2 \,\mathrm{d}t\right) \mathrm{d}x = \int_0^T \left(\sum_{i\in I} \int_{T_i} |\nabla w_{h\tau}(t)|^2 \,\mathrm{d}x\right) \mathrm{d}t \\
&= \|w_{h\tau}\|^2_{L^2(0,T;V_h)} \leqslant C.
\end{aligned}
$$

□

Now we prove the *fundamental compactness result.*

Lemma 14. *Let us consider the sequences $h = h_n$, $\tau = \tau_n \to 0$ and $w_{h\tau} = w_{h_n\tau_n}$ from Lemma 11 satisfying (7.16) and (8.26). Then (8.27) holds and*

$$(8.45)\qquad w_{h\tau} \to u \text{ strongly in } L^2(Q_T),$$

where u is the limit function from Lemma 11.

Proof. Let us set

$$
\begin{aligned}
w(t) &= u(t), \quad t \in (0,T), \\
w(t) &= 0, \qquad t < 0 \text{ or } t > T.
\end{aligned}
$$

Then, in virtue of (8.26) and (8.27),

$$J_h w_{h\tau} \to \omega w \text{ weakly in } L^2(\mathbb{R}, F), \tag{8.46}$$

$$w_{h\tau} \to w \text{ weakly in } L^2(\mathbb{R}, L^2(\Omega)). \tag{8.47}$$

Our goal is to prove that

$$\mathscr{F}_{h\tau} = \int_{-\infty}^{\infty} \|w_{h\tau}(t) - w(t)\|_{L^2(\Omega)}^2 \, \mathrm{d}t \to 0. \tag{8.48}$$

In virtue of (7.14), $\mathscr{F}_{h\tau}$ is uniformly bounded for $h \in (0, h_0)$ and $\tau > 0$ satisfying (5.13). By Parseval's equality,

$$\mathscr{F}_{h\tau} = \int_{-\infty}^{\infty} \|\hat{w}_{h\tau}(s) - \hat{w}(s)\|_{L^2(\Omega)}^2 \, \mathrm{d}s, \tag{8.49}$$

where $\hat{w}$ is the Fourier transform of w.

For $\gamma > 0$ we define the space

$$\mathscr{H}^\kappa = \left\{ v;\ \omega v \in L^2(-1, 1; F),\ \int_{\mathbb{R}} |s|^{2\gamma} \|\hat{v}(s)\|_{L^2(\Omega)}^2 \, \mathrm{d}s < \infty \right\} \tag{8.50}$$

equipped with the scalar product

$$(v, w)_{\mathscr{H}^\kappa} = \int_{-1}^{T} (\omega v(t), \omega w(t))_F \, \mathrm{d}t + \int_{\mathbb{R}} |s|^{2\gamma} \left(\hat{v}(s), \hat{w}(s)\right) \mathrm{d}s. \tag{8.51}$$

It can be proved that $\mathscr{H}^\gamma$ is a Hilbert space. In virtue of Theorem 5 and Lemma 13, the system $\{w_{h\tau}\}$ is uniformly bounded in $\mathscr{H}^\gamma$ for all $h \in (0, h_0)$ and $\tau > 0$ satisfying condition (5.13). Then, taking into account (8.47), we can write

$$w_{h\tau} \to w \text{ weakly in } \mathscr{H}^\gamma \tag{8.52}$$

and thus in view of the boundedness of $\mathscr{F}_{h\tau}$ and relation (8.49),

$$\int_{-\infty}^{\infty} (1 + |s|^{2\gamma}) \|\hat{w}_{h\tau}(s) - \hat{w}(s)\|_{L^2(\Omega)}^2 \, \mathrm{d}s \leqslant C < \infty. \tag{8.53}$$

Now we write

$$
(8.54) \qquad \mathscr{F}_{h\tau} = \int\limits_{|s|\leqslant M} \|\hat{w}_{h\tau}(s) - \hat{w}(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s
$$
$$
+ \int\limits_{|s|>M} \left(1+|s|^{2\gamma}\right)\|\hat{w}_{h\tau}(s) - \hat{w}(s)\|^2_{L^2(\Omega)} \frac{\mathrm{d}s}{1+|s|^{2\gamma}}
$$
$$
\leqslant \mathscr{I}_{h\tau} + \frac{1}{1+M^{2\gamma}} \int\limits_{-\infty}^{\infty} \left(1+|s|^{2\gamma}\right)\|\hat{w}_{h\tau}(s) - \hat{w}(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s
$$
$$
\leqslant \mathscr{I}_{h\tau} + \frac{C}{1+M^{2\gamma}},
$$

where

$$
(8.55) \qquad \mathscr{I}_{h\tau} = \int_{|s|\leqslant M} \|\hat{w}_{h\tau}(s) - \hat{w}(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s.
$$

For a given $\varepsilon > 0$ we choose $M > 0$ such that

$$
(8.56) \qquad \frac{C}{1+M^{2\gamma}} \leqslant \frac{\varepsilon}{2}.
$$

Hence,

$$
(8.57) \qquad \mathscr{F}_{h\tau} \leqslant \mathscr{I}_{h\tau} + \frac{\varepsilon}{2}.
$$

Now we want to prove that

$$
(8.58) \qquad \mathscr{I}_{h\tau} \to 0 \text{ as } h, \tau \to 0.
$$

As we will show, this is a consequence of the Lebesgue theorem. We have

$$
(8.59) \qquad \hat{w}_{h\tau}(s) = \int_{-\infty}^{\infty} w_{h\tau}(t)\mathrm{e}^{-2\pi i t s}\,\mathrm{d}t = \int_{-\infty}^{\infty} w_{h\tau}(t)\chi(t)\mathrm{e}^{-2\pi i t s}\,\mathrm{d}t, \quad \forall s \in \mathbb{R},
$$

where χ is the characteristic function of the interval $[0, T]$ (hence $w_{h\tau} = \chi w_{h\tau}$). Then, using (7.14), we have

$$
(8.60) \qquad \|\hat{w}_{h\tau}(s)\|_{L^2(\Omega)} = \left\| \int_{-\infty}^{\infty} w_{h\tau}(t)\chi(t)\mathrm{e}^{-2\pi i t s}\,\mathrm{d}t \right\|_{L^2(\Omega)}
$$
$$
\leqslant \|w_{h\tau}(t)\|_{L^2(\mathbb{R}, L^2(\Omega))} \|\chi(t)\mathrm{e}^{-2\pi i t s}\|_{L^2(\mathbb{R})} \leqslant C
$$

and

$$\|\hat{w}_{h\tau}(s)-\hat{w}(s)\|^2_{L^2(\Omega)} \leqslant 2\left(C^2+\|\hat{w}(s)\|^2_{L^2(\Omega)}\right) \ \forall s\in\mathbb{R}. \tag{8.61}$$

The function on the right-hand side of (8.61) is integrable over the interval $(-M,M)$. By definition, (8.26) is equivalent to

$$\int_{\mathbb{R}\times\Omega}(w_{h\tau}-w)\varphi\,\mathrm{d}x\,\mathrm{d}t\to 0 \quad \forall\varphi\in L^2(\mathbb{R},L^2(\Omega)). \tag{8.62}$$

For $\vartheta\in L^2(\Omega)$ we have $\varphi(x,t)=\vartheta(x)\chi(t)\mathrm{e}^{-2\pi\mathrm{i}ts}\in L^2(\mathbb{R},L^2(\Omega))$ for any fixed $s\in\mathbb{R}$. Then, by the definition of the Fourier transform, Fubini's theorem and (8.62),

$$\begin{aligned}\int_\Omega(\hat{w}_{h\tau}(s)-\hat{w}(s))\,\vartheta\,\mathrm{d}x &= \int_\Omega\left(\int_{\mathbb{R}}(w_{h\tau}(x,t)-w(x,t))\,\mathrm{e}^{-2\pi\mathrm{i}ts}\chi(t)\,\mathrm{d}t\right)\vartheta(x)\,\mathrm{d}x\\ &= \int_{\mathbb{R}\times\Omega}(w_{h\tau}(x,t)-w(x,t))\,\vartheta(x)\chi(t)\mathrm{e}^{-2\pi\mathrm{i}ts}\,\mathrm{d}t\,\mathrm{d}x\to 0 \text{ as } h,\tau\to 0,\end{aligned}$$

which means that

$$\hat{w}_{h\tau}(s)\to\hat{w}(s)=\int_{-\infty}^{\infty}w(t)\chi(t)\mathrm{e}^{-2\pi\mathrm{i}ts}\,\mathrm{d}t \qquad \text{weakly in } L^2(\Omega)\ \forall s\in\mathbb{R}. \tag{8.63}$$

Due to (8.59), the Cauchy inequality and (7.16), we have

$$\begin{aligned}\|\hat{w}_{h\tau}(s)\|_{X_h} &= \left\|\int_{-\infty}^{\infty}w_{h\tau}(t)\chi(t)\mathrm{e}^{-2\pi\mathrm{i}ts}\,\mathrm{d}t\right\|_{X_h}\\ &\leqslant \|w_{h\tau}(t)\|_{L^2(0,T;V_h)}\|\chi(t)\mathrm{e}^{-2\pi\mathrm{i}ts}\|_{L^2(\mathbb{R})}\leqslant C.\end{aligned} \tag{8.64}$$

Hence

$$\|J_h\hat{w}_{h\tau}(s)\|_F\leqslant C \text{ for all } s\in\mathbb{R}. \tag{8.65}$$

Now (8.63), reflexivity of the space F, (8.65) and assertion 2 of Lemma 8 imply that

$$J_h\hat{w}_{h\tau}(s)\to\omega\hat{w}(s) \text{ weakly in } F \text{ for each } s\in\mathbb{R}. \tag{8.66}$$

Since $\hat{w}_{h\tau}(s)\in X_h$ ($h=h_n$, $\tau=\tau_n\to 0$ as $n\to\infty$), the application of Lemma 12 implies that

$$\hat{w}_{h\tau}(s)\to\hat{w}(s) \text{ strongly in } L^2(\Omega) \text{ for all } s\in\mathbb{R}. \tag{8.67}$$

Hence,

$$\|\hat{w}_{h\tau}(s)-\hat{w}(s)\|^2_{L^2(\Omega)}\to 0\ \forall s\in\mathbb{R}. \tag{8.68}$$

From (8.68), the bound (8.61) and the Lebesgue theorem we obtain (8.58). This proves the lemma. □

From Lemmas 10, 11 and 14 and assertion (7.17) we can conclude that there exist sequences $h = h_n \to 0$, $\tau = \tau_n \to 0$ as $n \to \infty$ satisfying (5.13) and a function u such that

$$\begin{aligned} & J_h u_{h\tau} \to \omega u \quad \text{weakly in } L^2(-1, T; F), \\ & J_h w_{h\tau} \to \omega u \quad \text{weakly in } L^2(-1, T; F), \\ & u_{h\tau} \to u \quad \text{strongly in } L^2(\tilde{Q}_T), \\ & w_{h\tau} \to u \quad \text{strongly in } L^2(\tilde{Q}_T) \end{aligned} \tag{8.69}$$

as $n \to \infty$. Since $L^\infty(Q_T)$ is the dual to the separable Banach space $L^1(Q_T)$, the above results and (5.15) imply that

$$\begin{aligned} & u_{h\tau} \to u \text{ weak-* in } L^\infty(Q_T), \\ & w_{h\tau} \to u \text{ weak-* in } L^\infty(Q_T). \end{aligned} \tag{8.70}$$

9. Limit Process

Let us consider sequences $h = h_n$, $\tau = \tau_n \to 0$ satisfying (5.13) and assume that the corresponding approximate solutions $u_{h\tau}$, $w_{h\tau}$ satisfy conditions (5.15) and (8.69). Our goal is to show that the limit function u is a weak solution of problem (2.1)–(2.3), i.e. u satisfies (2.14)–(2.16).

Multiplying (8.1) by any $\psi \in C_0^\infty([0,T)) := \{\varphi \in C^\infty([0,T]);\ \varphi(T) = 0\}$, integrating over $(0,T)$, applying the integration by parts in the first term and using (4.2), which implies that $w_{h\tau}(0) = u_h^0$, we find that

$$\begin{aligned} & -\int_0^T (w_{h\tau}(t), \psi'(t) v_h)_h \, \mathrm{d}t + \nu \int_0^T ((u_{h\tau}(t), \psi(t) v_h))_h \, \mathrm{d}t \\ & + \int_0^T b_h(u_{h\tau}(t-\tau), \psi(t) v_h) \, \mathrm{d}t \\ & = \int_0^T (g_{h\tau}(t), \psi(t) v_h)_h \, \mathrm{d}t + (u_h^0, v_h)\psi(0), \quad v_h \in V_h,\ \psi \in C_0^\infty([0,T)). \end{aligned} \tag{9.1}$$

For $t \in [0,T]$, $v_h \in V_h$, $\psi \in C_0^\infty([0,T))$ we set

$$\vartheta_{h\tau}(t; \psi, v_h) = (g_{h\tau}(t), \psi(t) v_h)_h - (g_{h\tau}(t), \psi(t) v_h)\,. \tag{9.2}$$

Taking into account (6.10), we see that (9.1) is equivalent to

$$
\begin{aligned}
(9.3)\qquad & -\int_0^T (w_{h\tau}(t), \psi'(t)v_h)\ \mathrm{d}t + \nu \int_0^T ((u_{h\tau}(t), \psi(t)v_h))_h\, \mathrm{d}t \\
& + \int_0^T b_h(u_{h\tau}(t-\tau), \psi(t)v_h)\, \mathrm{d}t \\
& = \int_0^T (g_{h\tau}(t), \psi(t)v_h)\ \mathrm{d}t + (u_h^0, v_h)\psi(0) + \int_0^T \vartheta_{h\tau}(t;\psi, v_h)\, \mathrm{d}t.
\end{aligned}
$$

In virtue of (6.11) and (2.10), we obtain

$$
(9.4)\qquad \left| \int_0^T \vartheta_{h\tau}(t;\psi, v_h)\, \mathrm{d}t \right| \leqslant ch\|v_h\|_{X_h}.
$$

Let $v \in C_0^\infty(\Omega)$, $v_h = I_h v$. From (6.1), (6.4) and (6.6) we have

$$
\begin{aligned}
(9.5)\qquad & \|v_h - v\|_{L^2(\Omega)} \leqslant \hat{c}_1 \|v_h - v\|_{X_h} \leqslant ch\|v\|_{H^2(\Omega)} \\
& \|v_h\|_{X_h} \leqslant c, \quad h \in (0, h_0).
\end{aligned}
$$

This implies that

$$
(9.6)\qquad J_h v_h \to \omega v \text{ strongly in } F.
$$

Hence,

$$
(9.7)\qquad \psi' v_h \to \psi' v \quad \text{strongly in } L^2(Q_T)
$$

and

$$
(9.8)\qquad J_h \psi v_h \to \omega \psi v \quad \text{strongly in } L^2(0,T;F).
$$

The analysis of the limit process will be divided into several lemmas. In what follows we consider sequences $h = h_n \to 0$, $\tau = \tau_n \to 0$ as $n \to \infty$ satisfying (5.13), such that (8.69) and (8.70) hold.

Lemma 15. *Let $\psi(t) \in C_0^\infty([0,T))$ and let $w_{h\tau}, v_{h\tau}$ be two sequences satisfying*

$$
\begin{aligned}
(9.9)\qquad & w_{h\tau} \to u \qquad \textit{strongly in } L^2(Q_T), \\
(9.10)\qquad & v_{h\tau} \to v \qquad \textit{strongly in } L^2(Q_T).
\end{aligned}
$$

Then

$$
(9.11)\qquad \int_0^T (w_{h\tau}(t), \psi'(t)v_{h\tau})\ \mathrm{d}t \to \int_0^T (u(t), \psi'(t)v)\ \mathrm{d}t
$$

as $h = h_n \to 0$ and $\tau = \tau_n \to 0$.

Proof is evident.

Lemma 16. *Let $\psi(t) \in C_0^\infty([0,T))$ and let $u_{h\tau}, v_{h\tau}$ be two sequences satisfying*

$$J_h u_{h\tau} \to \omega u \qquad \text{weakly in } L^2(0,T;F), \tag{9.12}$$

$$J_h v_h \to \omega v \qquad \text{strongly in } F. \tag{9.13}$$

Then

$$\int_0^T ((u_{h\tau}(t), \psi(t) v_{h\tau}))_h \,\mathrm{d}t \to \int_0^T ((u(t), \psi(t) v))\,\mathrm{d}t \tag{9.14}$$

as $h = h_n \to 0$ and $\tau = \tau_n \to 0$.

P r o o f. From (9.13) it follows that

$$J_h \psi(t) v_h \to \psi(t) v \text{ strongly in } L^2(0,T;F). \tag{9.15}$$

We have

$$\int_0^T ((u_{h\tau}(t), \psi(t)v_h))_h \,\mathrm{d}t = \int_0^T (J_h u_{h\tau}(t), J_h \psi(t) v_h)_F \,\mathrm{d}t - \int_0^T (u_{h\tau}(t), \psi(t) v_h)\,\mathrm{d}t,$$

$$\int_0^T ((u(t), \psi(t)v))_h \,\mathrm{d}t = \int_0^T (\omega u(t), \psi(t) \omega v)_F \,\mathrm{d}t - \int_0^T (u(t), \psi(t) v)\,\mathrm{d}t.$$

By the definition, (9.12) means that

$$\int_0^T (J_h u_{h\tau}(t), \vartheta(t))\,\mathrm{d}t \to \int_0^T (\omega u(t), \vartheta(t))\,\mathrm{d}t \quad \forall \vartheta \in L^2(0,T;F). \tag{9.16}$$

We can write

$$\begin{aligned}
&\left| \int_0^T ((u_{h\tau}(t), \psi(t) v_{h\tau}))_h \,\mathrm{d}t - \int_0^T ((u(t), \psi(t) v))\,\mathrm{d}t \right| &\quad (9.17)\\
&\leqslant \left| \int_0^T (J_h u_{h\tau}(t), J_h \psi(t) v_h)_F \,\mathrm{d}t - \int_0^T (\omega u(t), \psi(t) \omega v)_F \,\mathrm{d}t \right| \\
&\quad + \left| \int_0^T (u_{h\tau}(t), \psi(t) v_h)\,\mathrm{d}t - \int_0^T (u(t), \psi(t) v)\,\mathrm{d}t \right| \\
&\leqslant \int_0^T \|J_h u_{h\tau}(t)\|_F \|\psi(t)(J_h v_h - \omega v)\|_F \,\mathrm{d}t &\quad (9.18)\\
&\quad + \left| \int_0^T (J_h u_{h\tau}(t), \psi(t) \omega v)_F \,\mathrm{d}t - \int_0^T (\omega u(t), \psi(t) \omega v)\,\mathrm{d}t \right| \\
&\quad + \left| \int_0^T (u_{h\tau}(t), \psi(t) v_h)\,\mathrm{d}t - \int_0^T (u_{h_\tau}(t), \psi(t) v)\,\mathrm{d}t \right| \\
&\quad + \left| \int_0^T (u_{h\tau}(t), \psi(t) v)\,\mathrm{d}t - \int_0^T (u(t), \psi(t) v)\,\mathrm{d}t \right| \to 0,
\end{aligned}$$

as follows from the boundedness of the sequence $\{J_h u_{h\tau}\}$ in $L^2(0,T;F)$, (9.12), (9.15) and (9.16), where we substitute $\vartheta(t) = \psi(t)v$. $\square$

Lemma 17. *Let $\psi(t) \in C_0^\infty([0,T))$ and let $u_{h\tau}, v_h$ be two sequences satisfying*

$$u_{h\tau} \to u \qquad \text{strongly in } L^2(\tilde{Q}_T), \quad \tilde{Q}_T = \Omega \times (-1,T), \tag{9.19}$$

$$J_h u_{h\tau} \to \omega u \qquad \text{weakly in } L^2(-1,T;F), \tag{9.20}$$

$$J_h v_h \to \omega v \qquad \text{strongly in } F, v \in C_0^\infty(\Omega). \tag{9.21}$$

Then

$$\int_0^T b_h(u_{h\tau}(t-\tau), v_h)\psi(t)\,\mathrm{d}t \to \int_0^T b(u(t),v)\psi(t)\,\mathrm{d}t \tag{9.22}$$

as $h = h_n \to 0,\ \tau = \tau_n \to 0$.

Proof. We write

$$\begin{aligned}
&b_h(u_{h\tau}(t-\tau), v_h) - b(u(t),v) \\
&= b_h(u_{h\tau}(t-\tau), v_h) - \tilde{b}_h(u_{h\tau}(t-\tau), v_h) && (=: \sigma(1)) \\
&+ \tilde{b}_h(u_{h\tau}(t-\tau), v_h) - \tilde{b}_h(u_{h\tau}(t-\tau), v) && (=: \sigma(2)) \\
&+ \tilde{b}_h(u_{h\tau}(t-\tau), v) - b(u(t-\tau), v) && (=: \sigma(3)) \\
&+ b(u(t-\tau),v) - b(u(t),v) && (=: \sigma(4))
\end{aligned}$$

($\tilde{b}_h$ is defined in (3.13)) and successively estimate the terms $\sigma(1)$–$\sigma(4)$:
In virtue of (6.12),

$$|\sigma(1)| \leqslant ch^{1-\kappa}(\|u_{h\tau}(t-\tau)\|^2_{X_h} + \|u_{h\tau}(t-\tau)\|_{X_h})\|v_h\|_{X_h}$$

with $\kappa \in (0,1)$. Hence,

$$\begin{aligned}
&\left|\int_0^T \Big[b_h(u_{h\tau}(t-\tau), v_h) - \tilde{b}_h(u_{h\tau}(t-\tau), v_h)\Big]\ \psi(t)\,\mathrm{d}t\right| \\
&\leqslant ch^{1-\kappa}\|v_h\|_{X_h}\left[\int_0^T \|u_{h\tau}(t-\tau)\|^2_{X_h}\,\mathrm{d}t + \left(\int_0^T \|u_{h\tau}(t-\tau)\|^2_{X_h}\,\mathrm{d}t\right)^{1/2}\right] \\
&\leqslant ch^{1-\kappa},
\end{aligned} \tag{9.23}$$

as follows from (7.15), (9.20) and (9.21). Similarly, using the Cauchy inequality and (9.5), we obtain

$$
(9.24)\qquad \left|\int_0^T \left[\tilde b_h(u_{h\tau}(t-\tau),v_h)-\tilde b_h(u_{h\tau}(t-\tau),v)\right]\psi(t)\,\mathrm{d}t\right|
$$

$$
\leqslant 2\max_{|\xi|\leqslant M}\max_{s=1,2}|f_s'(\xi)|\int_0^T \|u_{h\tau}(t-\tau)\|_{X_h}^2\,\mathrm{d}t\|v_h-v\|_{L^2(\Omega)}
$$

$$
\leqslant c\|J_h v_h-\omega v\|_F\to 0.
$$

Further,

$$
\begin{aligned}
\sigma(3)&=\sum_{i\in I}\int_{T_i}\sum_{s=1}^2\left(\frac{\partial f_s(u_{h\tau}(t-\tau))}{\partial x_s}-\frac{\partial f_s(u(t-\tau))}{\partial x_s}\right)v\,\mathrm{d}x\\
&=\sum_{i\in I}\int_{T_i}\sum_{s=1}^2\left(f_s'(u_{h\tau}(t-\tau))\frac{\partial u_{h\tau}(t-\tau)}{\partial x_s}-f_s'(u(t-\tau))\,\frac{\partial u(t-\tau)}{\partial x_s}\right)v\,\mathrm{d}x\\
&=\overbrace{\sum_{i\in I}\int_{T_i}\sum_{s=1}^2\left(f_s'(u_{h\tau}(t-\tau))-f_s'(u(t-\tau))\right)\frac{\partial u_{h\tau}(t-\tau)}{\partial x_s}\,v\,\mathrm{d}x}^{\sigma^*(3)}\\
&\quad+\sum_{i\in I}\int_{T_i}\sum_{s=1}^2 f_s'(u(t-\tau))\left(\frac{\partial u_{h\tau}(t-\tau)}{\partial x_s}-\frac{\partial u(t-\tau)}{\partial x_s}\right)v\,\mathrm{d}x.
\end{aligned}
$$

Using the mean value theorem in the integral form, we find that

$$
\begin{aligned}
|\sigma^*(3)|&\leqslant\max_{\xi\in[-M,M]}\max_{s=1,2}|f_s''(\xi)|\sum_{i\in I}\int_{T_i}|u_{h\tau}(t-\tau)-u(t-\tau)|\,|\nabla u_{h\tau}(t-\tau)|\;|v|\,\mathrm{d}x\\
&\leqslant c\left\|u_{h\tau}(t-\tau)-u(t-\tau)\right\|_{L^2(\Omega)}\left\|u_{h\tau}(t-\tau)\right\|_{X_h}.
\end{aligned}
$$

Since $\psi(t)=0$ for $t\geqslant T$, the substitution $t:=t-\tau$ yields

$$
(9.25)\qquad \left|\int_0^T\left[\tilde b_h(u_{h\tau}(t-\tau),v)-b(u(t-\tau),\,v)\right]\psi(t)\,\mathrm{d}t\right|
$$

$$
\begin{aligned}
&\leqslant c\left(\|u_{h\tau}-u\|_{L^2(Q_T)}^2+\tau\|u_h^0-u^0\|_{L^2(\Omega)}^2\right)^{1/2}\left(\|J_h\,u_{h\tau}\|_F^2+\tau\|u_h^0\|_{X_h}^2\right)^{1/2}\\
&\quad+\left|\int_0^T\psi(t+\tau)\sum_{i\in I}\int_{T_i}\sum_{s=1}^2 f_s'(u(t))\left(\frac{\partial u_{h\tau}(t)}{\partial x_s}-\frac{\partial u(t)}{\partial x_s}\right)v\,\mathrm{d}x\,\mathrm{d}t\right.\\
&\quad\left.+\tau\sum_{i\in I}\int_{T_i}\sum_{s=1}^2 f_s'(u^0)\left(\frac{\partial u_h^0}{\partial x_s}-\frac{\partial u^0}{\partial x_s}\right)v\,\mathrm{d}x\right|\longrightarrow 0\quad\text{as } h,\,\tau\to 0
\end{aligned}
$$

due to (9.19), (9.20) and (6.7) valid for $\varphi=u_0$.

Similar calculation yields the estimate

$$\begin{aligned}(9.26)\quad &\left|\int_0^T (b(u(t-\tau),v) - b(u(t),v))\,\psi(t)\,\mathrm{d}t\right| \\ &\leqslant c\int_0^T\int_\Omega |u(x,t-\tau)-u(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t\,\|v\|_{H^1(\Omega)} \to 0 \quad \text{as } h,\tau\to 0,\end{aligned}$$

which is a consequence of the continuity in the mean of $u \in L^2(\tilde{Q}_T)$. (Cf., e.g., [25], Theorem 2.4.2.)

Now (9.23)–(9.26) immediately imply (9.22). □

Lemma 18. *Let $\psi(t) \in C_0^\infty([0,T))$ and let $v_{h\tau}, g_{h\tau}$ be two sequences such that*

$$(9.27)\qquad v_h \to v \qquad \text{strongly in } L^2(\Omega),$$

$$(9.28)\qquad g_{h\tau}(t) = g^k = g(\cdot,t_k),\ \ \forall t\in[t_k,t_{k+1}),$$

where g satisfies (2.10). *Then*

$$(9.29)\qquad \int_0^T (g_{h\tau}(t),v_h)\,\psi(t)\,\mathrm{d}t \to \int_0^T (g(t),v)\,\psi(t)\,\mathrm{d}t \quad \text{as } h,\tau\to 0.$$

P r o o f. Obviously, by (2.10) and (9.27),

$$\begin{aligned}(9.30)\quad &\left|\int_0^T ((g_{h\tau}(t),v_h) - (g(t),v))\,\psi(t)\,\mathrm{d}t\right| \\ &\leqslant c\int_0^T \|g_{h\tau}(t)\|_{L^2(\Omega)}\|v_h - v\|_{L^2(\Omega)}\,\mathrm{d}t \\ &\quad + c\int_0^T \|g_{h\tau}(t)-g(t)\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\,\mathrm{d}t \\ &\leqslant c\|v_h - v\|_{L^2(\Omega)} + c\Big(\int_0^T \|g_{h\tau}(t)-g(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\Big)^{1/2}.\end{aligned}$$

In virtue of the uniform continuity of the mapping $g\colon [0,T]\to L^2(\Omega)$, we have

$$\int_0^T \|g_{h\tau}(t)-g(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t = \sum_{k=0}^{r-1}\int_{t_k}^{t_{k+1}} \|g(t_k)-g(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \to 0 \quad \text{as } \tau\to 0.$$

This, (9.27) and (9.30) yield (9.29). □

Finally, using (6.1), (6.7) and (9.5) we conclude that

$$(u_h^0, v_h) \to (u^0, v) \quad \text{as } h \to 0. \tag{9.31}$$

Now, summarizing (9.1), (9.2), (9.3), (9.4), (9.11), (9.14), (9.22), (9.29) and (9.31), we see that the limit function $u \in L^2(0,T;V) \cap L^\infty(Q_T)$ satisfies the identity

$$\begin{aligned} -\int_0^T (u(t),v)\,\psi'(t)\,\mathrm{d}t + \nu\int_0^T ((u(t),v))\psi(t)\,\mathrm{d}t + \int_0^T b\,(u(t),v)\,\psi(t)\,\mathrm{d}t \\ = \int_0^T (g(t),v)\,\psi(t)\,\mathrm{d}t + (u^0,v)\psi(0), \quad v \in C_0^\infty(\Omega),\ \psi \in C_0^\infty([0,T)). \end{aligned} \tag{9.32}$$

Since the space $C_0^\infty(\Omega)$ is dense in V, (9.32) holds for all $v \in V$. Moreover, $C_0^\infty((0,T)) \subset C_0^\infty([0,T))$ and identity (9.32) implies (2.17). Hence, u satisfies (2.14)–(2.15).

It is possible to show that $u' \in L^2(0,T;V^*)$ and

$$\langle u'(t),v\rangle + \nu((u(t),v)) + b(u(t),v) = (g(t),v), \quad v \in V, \quad \text{a.e. } t \in (0,T), \tag{9.33}$$

where $\langle\cdot,\cdot\rangle$ is the duality between V^* and V. (Cf., e.g., [31], Sec. 8.6.)

If we multiply (9.33) by any $\psi \in C_0^\infty([0,T))$, integrate over $(0,T)$ and transform the first term with the aid of integration by parts, we obtain the identity

$$\begin{aligned} -\int_0^T (u(t),v)\,\psi'(t)\,\mathrm{d}t + \nu\int_0^T ((u(t),v))\psi(t)\,\mathrm{d}t + \int_0^T b\,(u(t),v)\,\psi(t)\,\mathrm{d}t \\ = \int_0^T (g(t),v)\,\psi(t)\,\mathrm{d}t + (u(0),v)\psi(0), \quad v \in V,\ \psi \in C_0^\infty([0,T)). \end{aligned} \tag{9.34}$$

The comparison of (9.32) (with $v \in V$) and (9.34) immediately implies that $u(0) = u^0$. Hence, we have proved that u is a solution of problem (2.14)–(2.16).

On the basis of the above considerations we come to the following conclusion:

Let us consider approximate solutions of problem (2.14)–(2.16) obtained from (3.21)–(3.23) with $\tau, h > 0$ satisfying condition (5.13). Then the system of functions $u_{h\tau}$, $w_{h\tau}$ defined by (4.1) and (4.2) can be split into sequences converging in the sense of (8.69) and (8.70). Every limit function of such a sequence is a solution of problem (2.14)–(2.16). (As we see, we have proved the existence of a weak solution of (2.1)–(2.3).) Taking into account the uniqueness of the solution of (2.14)–(2.16) we obtain the convergence of the whole systems $\{u_{h\tau}\}$, $\{w_{h\tau}\}$ to the weak solution u of problem (2.1)–(2.3). Thus, we come to the *main result* of this paper:

Theorem 6. *Let us assume that the domain $\Omega \subset \mathbb{R}^2$ is bounded and polygonal and that conditions* (2.9)–(2.11), (3.1)–(3.3), (3.24)–(3.27), (4.3)–(4.5), (5.1) *and*

(5.2) *are satisfied. For* $h \in (0, h_0)$ *and* $\tau \in (0, T)$ *let us construct approximate solutions with the aid of the finite volume—finite element scheme* (3.21)–(3.23) *and define functions* $u_{h\tau}$ *and* $w_{h\tau}$ *by* (4.1) *and* (4.2). *Then the systems* $\{u_{h\tau}\}$, $\{w_{h\tau}\}$ *with* $h \in (0, h_0)$, $\tau \in (0, T)$ *satisfying the "stability condition"* (5.13) *fulfil estimates* (5.14) *and* (7.13)–(7.16). *Moreover,*

$$
\begin{aligned}
&J_h u_{h\tau},\ J_h w_{h\tau} \to \omega u \quad \text{weakly in } L^2(0, T; F),\\
&u_{h\tau},\ w_{h\tau} \to u \quad \text{weak-* in } L^\infty(Q_T),\\
&u_{h\tau},\ w_{h\tau} \to u \quad \text{strongly in } L^2(Q_T), \quad \text{as } h,\ \tau \to 0,\ h, \tau \text{ satisfy } (5.13),
\end{aligned}
$$

where u *is the unique weak solution of problem* (2.1)–(2.3) *(i.e.,* u *satisfies* (2.14)–(2.16)*).*

R e m a r k 2. There are several unsolved problems connected with the above results:

- error estimates and a posteriori error estimates,
- analysis of the problem in a nonpolygonal domain, i.e., the effect of the approximation of a curved boundary,
- analysis of the problem with nonhomogeneous Dirichlet boundary conditions and/or mixed Dirichlet-Neumann boundary conditions.

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Authors' addresses: *Philippe Angot*, IRPHE Chateau-Gombert, Technopole de Chateau-Gombert, 38, rue Frederic Joliot Curie, 13451 Marseille, France; *Vít Dolejší*, *Miloslav Feistauer*, *Jiří Felcman*, Institute of Numerical Mathematics, Faculty of Mathematics and Physics, Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic.