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A NUMERICAL SOLUTION OF THE DIRICHLET PROBLEM ON SOME SPECIAL DOUBLY CONNECTED REGIONS

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Abstract. The aim of this paper is to give a convergence proof of a numerical method for the Dirichlet problem on doubly connected plane regions using the method of reflection across the exterior boundary curve (which is analytic) combined with integral equations extended over the interior boundary curve (which may be irregular with infinitely many angular points).

Keywords: Dirichlet problem, integral equations, numerical method

MSC 2000: 31A25, 31A20, 45B05, 65R20, 65N99

1. Integral equations

We shall identify the real plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ in the usual sense. A mapping from a subset of $\mathbb{R}^2$ into $\mathbb{R}^2$ is then regarded as a complex function of the complex variable.

For the reduction of two integral equations for the Dirichlet problem on doubly connected regions to only one equation we use the so-called reflection function. Note that J. M. Sloss in [20] proved the existence of reflection functions for a sufficiently large family of smooth curves. In [21] he used this reflection function for treating the Dirichlet problem on multiply connected regions bounded by curves with continuously varying curvature. In [4], [5] a suitable integral representation for the solution of the Dirichlet and the Neumann problems on regions with nonsmooth boundaries was established. Let us only recall that if L is an analytic Jordan curve (in $\mathbb{R}^2$) then,

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under some additional conditions on L, there are

$$R_g\text{—an open connected neighbourhood of } L$$

and

$$g\text{—a function holomorphic and one-to-one on } R_g$$

such that

$$\overline{g}(R_g \cap \operatorname{Int} L) \subset \operatorname{Ext} L, \tag{1.1}$$

$$\overline{g}(R_g \cap \operatorname{Ext} L) \subset \operatorname{Int} L, \tag{1.2}$$

$$\overline{g}(z) = z \qquad \text{for } z \in L, \tag{1.3}$$

$$\overline{g}\big(\overline{g}(z)\big) = z \qquad \text{for } z \in R_g \tag{1.4}$$

(see [20] or [4], [5]); here $\overline{g}$ is the complex conjugate of g, $\operatorname{Int} L$ and $\operatorname{Ext} L$ denote the bounded and the unbounded complementary domain of L, respectively. The function $\overline{g}$ is then called the reflection function of the curve L. In the sequel we will always suppose that L is an analytic Jordan curve having a reflection function $\overline{g}$ defined on a neighbourhood R_g of L.

Let us recall the following elementary property of the reflection function (see [4], remark 1.3). Let h be a function harmonic on $R_g \cap \operatorname{Int} L$ the first partial derivatives of which are continuously extendible from $R_g \cap \operatorname{Int} L$ to $(R_g \cap \operatorname{Int} L) \cup L$; its composition with $\overline{g}$, to be denoted by $h * \overline{g}$, is defined on $R_g \cap \operatorname{Ext} L$ and is continuously extendible to $(R_g \cap \operatorname{Ext} L) \cup L$. If $\mathbf{n}_e$ denotes the exterior normal of $\operatorname{Int} L$ on L and $\mathbf{n}_i$ the corresponding interior normal then (on L)

$$\frac{\partial h}{\partial \mathbf{n}_e} = \frac{\partial (h * \overline{g})}{\partial \mathbf{n}_i} = -\frac{\partial (h * \overline{g})}{\partial \mathbf{n}_e}. \tag{1.5}$$

Let K be a rectifiable Jordan curve such that

$$K \subset R_g \cap \operatorname{Int} L \tag{1.6}$$

and suppose that

$$\operatorname{Int} L \cap \operatorname{Ext} K \subset R_g.$$

We shall consider the Dirichlet problem on the domain

$$S^+ = \operatorname{Int} L \cap \operatorname{Ext} K. \tag{1.7}$$

Further let us denote

$$S^- = \overline{g}(S^+), \qquad S = S^+ \cup L \cup S^-, \tag{1.8}$$

$$\widehat{K} = \overline{g}(K). \tag{1.9}$$

Then $\widehat{K}$ is a Jordan curve (also rectifiable),

$$K \cup L \subset \operatorname{Int} \widehat{K}$$

and further

$$\begin{gathered} S^- = \operatorname{Ext} L \cap \operatorname{Int} \widehat{K}, \qquad S = \operatorname{Ext} K \cap \operatorname{Int} \widehat{K}, \\ \partial S^+ = K \cup L, \qquad \partial S^- = L \cup \widehat{K}, \qquad \partial S = K \cup \widehat{K}. \end{gathered}$$

Suppose that K has a parameterization ψ defined on an interval $\langle a, b\rangle$ [that is, $K = \psi(\langle a, b\rangle)$ and $\psi(t_1) \neq \psi(t_2)$ for any $t_1, t_2 \in \langle a, b\rangle$ with $0 < |t_1 - t_2| < b - a$]. For $z \in \mathbb{R}^2$ let ϑ_z stand for a single-valued (continuous) branch of the argument of $[\psi - z]$ on the set $\langle a, b\rangle \setminus \psi^{-1}(z)$. For $0 < r \leqslant \infty$ let $\gamma_{z,r}$ be the family of all connected components of the set

$$\bigl\{\, t \in \langle a, b\rangle \bigm| 0 < |\psi(t) - z| < r \,\bigr\}$$

$(\gamma_z \equiv \gamma_{z,\infty})$.

The so-called cyclic variation of ψ at $z \in \mathbb{R}^2$ is denoted by $v_r^\psi(z)$; $v^\psi(z) \equiv v_\infty^\psi(z)$. Let us recall that

$$v_r^\psi(z) = \sum_{I \in \gamma_{z,r}} \operatorname{var}[\vartheta_z; I], \tag{1.10}$$

where we denote as usual by $\operatorname{var}[\vartheta; I]$ the total variation of ϑ on I. We will always suppose that

$$V_K \equiv \sup_{z \in K} v^\psi(z) < \infty. \tag{1.11}$$

Under this assumption v^ψ is bounded on $\mathbb{R}^2$ and the double layer potential $W_\psi(\cdot, f) = W_K(\cdot, f)$ for $f \in \mathscr{C}(K)$ is defined on $\mathbb{R}^2$ by

$$W_K(z, f) = \frac{1}{\pi} \sum_{I \in \gamma_z} \int_I f\bigl(\psi(t)\bigr)\, \mathrm{d}\vartheta_z(t) \tag{1.12}$$

($z \in \mathbb{R}^2$). One can similarly define $W_L(\cdot, f)$ for $f \in \mathscr{C}(L)$ [an analogue of (1.11) for L is guaranteed since L is analytic]. It is known (see [7], for example) that if (1.11) is fulfilled then finite limits

$$W_K^e(\zeta, f) \equiv \lim_{\substack{z\to\zeta\\ z\in \operatorname{Ext} K}} W_K(z, f), \qquad W_K^i(\zeta, f) \equiv \lim_{\substack{z\to\zeta\\ z\in \operatorname{Int} K}} W_K(z, f)$$

exist for each $\zeta \in K$. Recall how to evaluate these limits. Given $\zeta \in K$ let $t_1 \in \langle a, b)$, $t_2 \in (a, b\rangle$ be such that $\zeta = \psi(t_1) = \psi(t_2)$. Due to $v^\psi(\zeta) < \infty$ the limits

$$\tau_+(\zeta) \equiv \tau_+^K(\zeta) = \lim_{t\to t_1+} \frac{\psi(t)-\zeta}{|\psi(t)-\zeta|}, \tag{1.13}$$

$$\tau_-(\zeta) \equiv \tau_-^K(\zeta) = \lim_{t\to t_2-} \frac{\psi(t)-\zeta}{|\psi(t)-\zeta|} \tag{1.14}$$

exist. Let ι denote the value of the index of a point from $\operatorname{Int} K$ with respect to ψ (that is, $\iota = 1$ if ψ is positively oriented, $\iota = -1$ if ψ is oriented negatively). We say that a vector $\tau \in \mathbb{R}^2$ points into $E \subset \mathbb{R}^2$ at $\zeta \in \mathbb{R}^2$ provided there is a $\delta > 0$ such that $\{\zeta + r\tau \mid 0 < r < \delta\} \subset E$. To a point $\zeta \in K$ we assign two values $\alpha_+(\zeta), \alpha_-(\zeta)$ such that

$$\tau_+(\zeta) = \mathrm{e}^{i\alpha_+(\zeta)}, \qquad \tau_-(\zeta) = \mathrm{e}^{i\alpha_-(\zeta)} \tag{1.15}$$

and at the same time

(a) $\alpha_+(\zeta) < \alpha_-(\zeta) < \alpha_+(\zeta) + 2\pi$ if $\tau_+(\zeta) \neq \tau_-(\zeta)$,

(b) $\alpha_-(\zeta) = \alpha_+(\zeta) + (1-\iota)\pi$ if $\tau_+(\zeta) = \tau_-(\zeta)$ and the vector $\mathrm{e}^{i(\alpha_+(\zeta)+\pi)}$ points into $\operatorname{Ext} K$ at ζ,

(c) $\alpha_-(\zeta) = \alpha_+(\zeta) + (1+\iota)\pi$ if $\tau_+(\zeta) = \tau_-(\zeta)$ and the vector $\mathrm{e}^{i(\alpha_+(\zeta)+\pi)}$ points into $\operatorname{Int} K$ at ζ.

Put

$$\Delta(\zeta) = \pi - \big(\alpha_-(\zeta) - \alpha_+(\zeta)\big). \tag{1.16}$$

It is known that, under the condition (1.11),

$$W_K^e(\zeta, f) = W_K(\zeta, f) - \iota f(\zeta)\left(1 - \frac{1}{\pi}\iota\Delta(\zeta)\right), \tag{1.17}$$

$$W_K^i(\zeta, f) = W_K(\zeta, f) + \iota f(\zeta)\left(1 + \frac{1}{\pi}\iota\Delta(\zeta)\right) \tag{1.18}$$

for $\zeta \in K$, $f \in \mathscr{C}(K)$, see [8], theorem 2.11, for example. Note that [8] deals with non-tangential limits. For the general case of $\mathbb{R}^n$, $n \geqslant 2$, see [12], [10]. It is easy to show that the set $\{\zeta \in K \mid |\Delta(\zeta)| > 0\}$ formed by the angular points of K is at most countable (cf. 1.7 in [9]).

If we define for $\zeta \in K$, $f \in \mathscr{C}(K)$

$$\overline{W}_K f(\zeta) = W_K(\zeta, f) + \frac{1}{\pi}\Delta(\zeta) f(\zeta) \tag{1.19}$$

then we have

$$W_K^i(\zeta, f) = \overline{W}_K f(\zeta) + \iota f(\zeta), \tag{1.20}$$

$$W_K^e(\zeta, f) = \overline{W}_K f(\zeta) - \iota f(\zeta), \tag{1.21}$$

$$\overline{W}_K f(\zeta) = W_K^i(\zeta, f) - \iota f(\zeta) = W_K^e(\zeta, f) + \iota f(\zeta). \tag{1.22}$$

Now define

$$\mathcal{H}_K f(z) = W_K(z, f) - W_K\big(\overline{g}(z), f\big) \tag{1.23}$$

for $f \in \mathscr{C}(K)$, $z \in S^+$. Since $\overline{g}(\zeta) = \zeta$ for $\zeta \in L$ and $W_K(\cdot, f)$ is continuous outside K, we have for $\zeta \in L$

$$\lim_{\substack{z \to \zeta \\ z \in S^+}} \mathcal{H}_K = W_K(\zeta, f) - W_K(\zeta, f) = 0. \tag{1.24}$$

For $\zeta \in K$ we obtain

$$\begin{aligned} \mathcal{H}_K^e f(z) \equiv \lim_{\substack{z \to \zeta \\ z \in S^+}} \mathcal{H}_K f(z) &= W_K^e(\zeta, f) - W_K\big(\overline{g}(\zeta), f\big) \\ &= \overline{W}_K f(\zeta) - \iota f(\zeta) - W_K\big(\overline{g}(\zeta), f\big). \end{aligned} \tag{1.25}$$

Consider now the equation

$$\mathcal{H}_K^e f(\zeta) = u(\zeta), \qquad \zeta \in K, \tag{1.26}$$

where $u \in \mathscr{C}(K)$ is a given function, $f \in \mathscr{C}(K)$ unknown. If f is a solution of (1.26) (for a given u) then it follows from (1.25), (1.24) that the function $\mathcal{H}_K$ (defined on S^+) is the solution of the Dirichlet problem on S^+ with the boundary condition u on K and the zero boundary condition on L.

Denoting

$$\overline{\mathcal{H}} f(\zeta) \equiv \overline{W}_K f(\zeta) - W_K\big(\overline{g}(\zeta), f\big) \tag{1.27}$$

for $f \in \mathscr{C}(K)$, $\zeta \in K$, we can write (1.26) in the form

$$(I - \iota\overline{\mathcal{H}})f = -\iota u \tag{1.28}$$

[see (1.25), (1.22)]. The problem of solvability of (1.28) was investigated in [4]. It has been shown that if the condition

$$\frac{1}{\pi} \lim_{r\to 0+} \sup_{\zeta\in K} v_r^{\psi}(\zeta) < 1 \tag{1.29}$$

is fulfilled then the space of solutions [in $\mathscr{C}(K)$] of the homogeneous equation $(I - \iota\overline{\mathcal{H}})f = 0$ is equal to the space of all functions constant on K. In [4] a necessary and sufficient condition on $u \in \mathscr{C}(K)$ guaranteeing solvability of (1.28) [and equivalently of (1.26)] is given (the solution is then determined uniquely up to a constant—see [4], theorem 2.16).

Given $z \in \mathbb{R}^2$ put $h_z(z) = +\infty$,

$$h_z(\zeta) = \frac{1}{\pi} \ln \frac{1}{|\zeta - z|}, \qquad \zeta \in \mathbb{R}^2 \setminus \{z\}. \tag{1.30}$$

Fix $z_0 \in \operatorname{Int} K$ and put for $z \in R_g$

$$v(z) = h_{z_0}(z) - h_{z_0}\big(\overline{g}(z)\big). \tag{1.31}$$

As we have already noted one can define $W_L(\cdot, f)$ for $f \in \mathscr{C}(L)$ analogously to the definition of $W_K(\cdot, f)$ for $f \in \mathscr{C}(K)$. Analogously we also define

$$\mathcal{H}_L f(z) = W_L(z, f) - W_L\big(\overline{g}(z), f\big) \tag{1.32}$$

for $f \in \mathscr{C}(L)$, $z \in S^+$. We will always suppose that the curve L is positively oriented. Then

$$\lim_{\substack{z\to\zeta \\ z\in S^+}} \mathcal{H}_L f(z) = 2f(\zeta) \tag{1.33}$$

for $f \in \mathscr{C}(L)$, $\zeta \in L$ (see [4], section 2.15).

As shown in [4] the solution of the Dirichlet problem on S^+ with boundary conditions $u_K \in \mathscr{C}(K)$ on K and $u_L \in \mathscr{C}(L)$ on L can be found in the form

$$h(z) = \mathcal{H}_K f_K(z) + \mathcal{H}_L f_L(z) + av(z),$$

where $f_K \in \mathscr{C}(K)$, $f_L \in \mathscr{C}(L)$, $a \in \mathbb{R}$.

Since $v(\zeta) = 0$ for $\zeta \in L$ and since (1.24) is valid, it suffices to choose $f_L = \frac{1}{2} u_L$ due to (1.33). Then it suffices to choose f_K as a solution of the equation

$$(I - \iota\overline{\mathcal{H}}) f_K = -\iota\Big(u_K - \tfrac{1}{2}\mathcal{H}_L u_L\big|_K - a \cdot v\big|_K\Big). \tag{1.34}$$

Here $a \in \mathbb{R}$ has to be chosen in such a way that the equation (1.34) be solvable. It is shown in [4] that such $a \in \mathbb{R}$ exists and that it is determined uniquely. For our purposes we formulate the result in the following form.

Theorem 1.1. *Suppose that* (1.29) *is fulfilled. Then for each boundary conditions* $u_k \in \mathscr{C}(K)$, $u_L \in \mathscr{C}(L)$ *there are* $a \in \mathbb{R}$, $f_K \in \mathscr{C}(K)$ *such that the function* h *of the form*

$$h(z) = \mathcal{H}_K f_K(z) + \tfrac{1}{2}\mathcal{H}_L u_L(z) + a \cdot v(z) \tag{1.35}$$

solves on S^+ *the Dirichlet problem with the boundary conditions* u_K, u_L. *The constant* a *is determined uniquely, the function* f_K *is determined uniquely up to a constant;* f_K *is a solution of the equation* (1.34).

2. Abstract form of a convergence theorem

In this part we shall prove one assertion which can be viewed as a convergence theorem. Using this assertion we shall prove the convergence of a numerical method for solving the equation (1.34). The idea comes from W. L. Wendland [25] (in fact, the assertion is only a variant of Theorem 3.5 from [25]; see also [24]).

Lemma 2.1. *Given a normed linear space* L *let*

$$B_n \colon L \to L, \qquad B \colon L \to L \qquad (n = 1, 2, 3, \ldots)$$

be linear operators. Suppose that B_n^{-1}, B^{-1} *exist,* B_n^{-1} *are bounded and there is an* $M \in \mathbb{R}$ *such that* $\|B_n^{-1}\| \leqslant M$. *Let* L_0 *be a closed subspace of* L, $B(L_0) = L_0$. *Suppose that*

$$B_n x \to B x \tag{2.1}$$

for each $x \in L_0$. *If* $x_n \in L$, $x_n \to x \in L_0$, *then*

$$B_n^{-1} x_n \to B^{-1} x. \tag{2.2}$$

In particular: Let L be a Banach space, A_n, $A\colon L \to L$ bounded linear operators and suppose that there is a $\lambda \in \mathbb{R}$, $\lambda < 1$, such that

$$\|A_n\| \leqslant \lambda, \qquad \|A\| \leqslant \lambda. \tag{2.3}$$

Let L_0 be a closed subspace of L, $(I - A)(L_0) = L_0$. Suppose that

$$A_n x \to Ax \tag{2.4}$$

for each $x \in L_0$. Then

$$(I - A_n)^{-1} x_n \to (I - A)^{-1} x \tag{2.5}$$

whenever $x_n \in L$, $x_n \to x \in L_0$.

Proof. We have

$$B_n^{-1} - B^{-1} = B_n^{-1}(B - B_n)B^{-1}.$$

Let $x \in L_0$, $\varepsilon > 0$. Then $B^{-1}x \in L_0$ [as $B(L_0) = L_0$ by the assumption] and it follows from (2.1) that there is n_0 such that

$$\big\|(B - B_n)(B^{-1}x)\big\| < \frac{\varepsilon}{M}$$

for $n > n_0$. Hence for $n > n_0$ we get

$$\|B_n^{-1}x - B^{-1}\| = \big\|B_n^{-1}(B - B_n)(B^{-1}x)\big\| \leqslant \|B_n^{-1}\|\big\|(B - B_n)(B^{-1}x)\big\| < \varepsilon$$

and thus

$$B_n^{-1}x \to B^{-1}x.$$

If now $x_n \in L$, $x_n \to x \in L_0$, then

$$\begin{aligned}\|B_n^{-1}x_n - B^{-1}x\| &\leqslant \|B_n^{-1}x_n - B_n^{-1}x\| + \|B_n^{-1}x - B^{-1}x\| \\ &\leqslant M\|x_n - x\| + \|B_n^{-1}x - B^{-1}x\| \to 0\end{aligned}$$

and the first part of Lemma is proved.

The second part follows immediately from the first. $\square$

Note that the above assertion is the Lemma 3.1 from [2] slightly modified for our purposes. The following assertion is the Lemma 3.2 from [2].

Lemma 2.2. *Let X be a Banach space, $X_0 \subset X$, a closed subspace. Let Q, B be bounded linear operators,*

$$Q\colon\ X \to X, \qquad B\colon\ X \to X_0,$$

let $\|Q\| < 1$, and suppose that $Q\colon\ X_0 \to X_0$. Then

$$(I - Q - B)^{-1}(0) \subset X_0. \tag{2.6}$$

Suppose in addition that B is compact. If for each $f \in X_0$ the equation

$$(I - Q - B)g = f \tag{2.7}$$

(in unknown g) has a unique solution in X_0 then (2.7) is uniquely solvable in X for each $f \in X$.

Lemma 2.3. *Let X be a Banach space, $X_0 \subset X$ its closed subspace. Let $Q, B\colon X \to X$ be bounded linear operators and suppose that B is compact, $\|Q\| < 1$ and*

$$B\colon\ X \to X_0, \qquad Q\colon\ X_0 \to X_0.$$

Let $H_n \subset X$ $(n = 1, 2, 3, \ldots)$ be subspaces in X and let

$$P_n\colon\ X \to H_n$$

be projections, $\|P_n\| = 1$, and suppose that for each $f \in X_0$

$$\big\|P_n f - f\big\| \to 0 \qquad \text{for } n \to \infty. \tag{2.8}$$

Further let B_n be compact operators, $B_n\colon\ X \to X_0$, and suppose that B_n are collectively compact (which means that the set

$$\big\{B_n f \bigm| n \in \mathbb{N}, f \in X, \|f\| \leqslant 1\big\} \tag{2.9}$$

is relatively compact) and that for each $f \in X_0$

$$B_n f \to Bf \qquad \text{for } n \to \infty. \tag{2.10}$$

Consider the equations

$$(I - Q - B)u = f, \tag{2.11}$$

$$(I - QP_n - B_n)u_n = f, \tag{2.12}$$

where $f \in X$ is given and u, $u_n \in X$ are unknown. Suppose that for each $f \in X_0$ the equation (2.11) *has a unique solution in X_0. Then there is n_0 such that for each $n > n_0$ and each $f \in X$ the equation* (2.12) *is uniquely solvable in X. At the same time there are constants c_1, c_2 such that the corresponding solutions of* (2.11), (2.12) *satisfy the estimates*

$$\|u_n\| \leqslant c_1\|u\| \leqslant c_2\|f\|, \tag{2.13}$$

$$\|u - u_n\| \leqslant c_2\big(\|QP_nu - Qu\| + \|B_nu - Bu\|\big). \tag{2.14}$$

P r o o f. First note that it follows from Lemma 2.2 that under the given assumptions the equation (2.11) is uniquely solvable not only in X_0 but also in X (for $f \in X$).

Let us show that for all sufficiently large n also the equation (2.12) is uniquely solvable in X. Since B_n is compact and $\|QP_n\| < 1$ (as $\|Q\| < 1$, $\|P_n\| = 1$), by the Riesz-Schauder theory it suffices to show that the homogeneous equation

$$(I - QP_n - B_n)u_n = 0 \tag{2.15}$$

has (in X) only the trivial solution. Suppose that there are infinitely many n for which (2.15) has a non-trivial solution. One can suppose for simplicity that for each natural n the equation (2.15) possesses a non-trivial solution. Then there are

$$u_n \in X, \qquad \|u_n\| = 1$$

such that (2.15) is valid, which can be written in the form

$$(I - QP_n)u_n = B_nu_n,$$

that is

$$u_n = (I - QP_n)^{-1}B_nu_n \tag{2.16}$$

[as $\|QP_n\| < 1$ the inverse $(I - QP_n)^{-1}$ exists]. Since B_n are collectively compact and $\|u_n\| = 1$, there is a subsequence $\{u_{n_k}\}$ such that the sequence $\{B_{n_k}u_{n_k}\}$ is convergent. For simplicity let us denote this subsequence by the same symbol $\{u_n\}$. We can thus write

$$B_n u_n = v_n \to v \in X_0 \tag{2.17}$$

(as by assumption $B_n\colon X \to X_0$ and X_0 is closed). Using Lemma 2.1 we now obtain

$$(I - QP_n)^{-1} v_n \to (I - Q)^{-1} v \in X_0$$

($Q\colon X_0 \to X_0$ by assumption), whence [see (2.16)]

$$u_n \to (I - Q)^{-1} v = u \in X_0. \tag{2.18}$$

We have

$$\|B_n u_n - Bu\| \leqslant \|B_n u_n - B_n u\| + \|B_n u - Bu\|. \tag{2.19}$$

Since $u \in X_0$ it follows from the assumption [see (2.10)] that

$$\|B_n u - Bu\| \to 0. \tag{2.20}$$

Since B_n are collectively compact they are also uniformly bounded and thus

$$\big\|B_n(u_n - u)\big\| \to 0. \tag{2.21}$$

Now it follows from (2.21), (2.20) and (2.19) that

$$v_n = B_n u_n \to Bu$$

and thus (as $v_n \to v$)

$$Bu = v. \tag{2.22}$$

Since we have put $u = (I - Q)^{-1} v$ [see (2.18)] we thus obtain

$$u = (I - Q)^{-1} Bu,$$

so that

$$(I - Q - B)u = 0. \tag{2.23}$$

Since $\|u_n\| = 1$ (and $u_n \to u$), we have also $\|u\| = 1$, which contradicts the unique solvability of the equation (2.11). The first part of the assertion is proved.

Now we shall prove (2.13). Existence of c_2 in the second inequality in (2.13) is clear since under the given assumptions the operator $(I - Q - B)$ has a bounded inverse.

Suppose that the first inequality in (2.13) is not valid (that is there exists no c_1 such that ...). Then there are $f_n \in X$, $u_n \in X$, $g_n \in X$ (more precisely, there is a subsequence of indices n) such that

$$(I - Q - B)u_n = f_n, \tag{2.24}$$

$$(I - QP_n - B_n)g_n = f_n \tag{2.25}$$

and

$$\|u_n\| = 1, \qquad \|g_n\| \to +\infty.$$

The equalities (2.24), (2.25) yield

$$(I - Q - B)u_n = (I - QP_n - B_n)g_n. \tag{2.26}$$

Denoting

$$h_n = \frac{g_n}{\|g_n\|}$$

(2.26) can be written in the form

$$(I - QP_n)h_n = \frac{1}{\|g_n\|}(I - Q - B)u_n + B_n h_n,$$

or

$$h_n = (I - QP_n)^{-1}\left\{\frac{1}{\|g_n\|}(I - Q - B)u_n + B_n h_n\right\}.$$

Since

$$\|u_n\| = 1, \qquad \|g_n\| \to +\infty,$$

we have

$$\frac{1}{\|g_n\|}(I - Q - B)u_n \to 0.$$

The operators B_n are collectively compact by the assumption. As $\|h_n\| = 1$ there is a subsequence such that

$$B_n h_n \equiv v_n \to v \in X_0.$$

Now it follows from Lemma 2.1 that

$$h_n = (I - QP_n)^{-1}\left\{\frac{1}{\|g_n\|}(I - Q - B)u_n + B_n h_n\right\} \to (I - Q)^{-1}v = h \in X_0.$$

In the same way as in the first part of the proof one can find that

$$B_n h_n \to Bh$$

(which means that $v = Bh$) and obtain that

$$h = (I - Q)^{-1} Bh$$

and thus

$$(I - Q - B)h = 0.$$

But $\|h\| = 1$ (as $\|h_n\| = 1$), $h \in X_0$, which contradicts the assumption that (2.11) is uniquely solvable in X_0.

It suffices to prove (2.14). Consider $n > n_0$ [where n_0 is such that for each $n > n_0$ the equation (2.12) is uniquely solvable for any $f \in X$]. Given $f \in X$ let u be the solution of (2.11) [as we have noted, (2.11) is uniquely solvable in X due to Lemma 2.2] and let u_n be the solution of (2.12). Then

$$u - Qu - Bu = u_n - QP_n u_n - B_n u_n,$$

which can be written in the form

$$(u - u_n) - QP_n(u - u_n) - B_n(u - u_n) = Qu - QP_n u + Bu - B_n u,$$

that is $(u - u_n)$ is a solution of the equation (2.12) with the right hand side equal to

$$Qu - QP_n u + Bu - B_n u.$$

Now (2.14) follows immediately from the second inequality in (2.13). □

Proposition 2.4. *Let X be a Banach space, $X_0 \subset X$ its closed subspace, $q, s \in X_0$, $s \neq 0$. Let $Q, C\colon X \to X$ be two bounded linear operators, $C\colon X \to X_0$, C compact, $Q\colon X_0 \to X_0$, $\|Q\| < 1$. Let H_n be subspaces in X, $P_n\colon X \to H_n$ projections ($n = 1, 2, 3, \dots$) such that $\|P_n\| = 1$, $P_n s = s$ ($n \in \mathbb{N}$) and suppose that for each $f \in X_0$*

$$\|P_n f - f\| \to 0 \qquad \text{for } n \to \infty.$$

Consider the equations

$$(I - Q - C)u + aq = f, \tag{2.27}$$

$$(I - QP_n - CP_n)u_n + a_n q = f, \tag{2.28}$$

where $f \in X$ is given while $u, u_n \in X$, $a, a_n \in \mathbb{R}$ are unknown. Suppose that for each $f \in X_0$ the equation (2.27) has a solution (u, a) in $X_0 \times \mathbb{R}$ where $a \in \mathbb{R}$ is determined uniquely; further suppose that if (u_0, a) is a solution of (2.27) ($u_0 \in X_0$) then the set of all solutions of (2.27) is equal to

$$\{\,(u_0 + ts, a) \bigm| t \in \mathbb{R}\,\}. \tag{2.29}$$

Then there is n_0 such that for each $n > n_0$ and for each $f \in X$ the equation (2.28) has a solution in $X \times \mathbb{R}$. The constant a_n is then determined uniquely and the set of all solutions of (2.28) is of the form

$$\{\,(u_n^0 + ts, a_n) \bigm| t \in \mathbb{R}\,\}, \tag{2.30}$$

where (u_n^0, a_n) is a solution of (2.28).

Further let $r\colon X \to \mathbb{R}$ be a bounded linear functional such that $r(s) \neq 0$, $f \in X_0$, (u, a) is a solution of (2.27) such that $r(u) = 0$ and let (u_n, a_n) be a solution of (2.28) such that $r(P_n u_n) = 0$. Then

$$u_n \to u, \qquad a_n \to a. \tag{2.31}$$

P r o o f. Let h be an arbitrary bounded linear functional on X such that $h(s) \neq 0$. For $x \in X$ put

$$Bx = Cx - h(x)q, \qquad B_n x = CP_n x - h(x)q.$$

It is easy to see that $B\colon X \to X_0$, $B_n\colon X \to X_0$, B is compact, B_n are collectively compact and $B_n f \to Bf$ for each $f \in X_0$.

For $f \in X_0$ consider the equation

$$(I - Q - B)u = f. \tag{2.32}$$

Then $u \in X$ is a solution of (2.32) if and only if $(u, h(u))$ solves (2.27). Since (2.27) is solvable in $X_0 \times \mathbb{R}$ and the set of all solutions of (2.27) is of the form (2.29), we see that (2.32) is in X_0 uniquely solvable. Now it follows from Lemma 2.3 that for all sufficiently large n the equation

$$(I - QP_n - B_n)u_n = f \tag{2.33}$$

is uniquely solvable in X (for each $f \in X$). If u_n is a solution of (2.33) then $(u_n, h(u_n))$ is a solution of (2.28). The fact that the set of all solutions of (2.27) is of the form (2.29) means that $\{ts \mid t \in \mathbb{R}\}$ is the null space of the operator $(I - Q - C)$. Since $P_n s = s$ we see that s is contained also in the null space of $(I - QP_n - CP_n)$.

Suppose that there is an $s_1 \in X$ such that s, s_1 are linearly independent and s_1 is contained in the null space of $(I - QP_n - CP_n)$. Suppose that h was chosen such that, in addition, $h(s_1) = 0$. Let u_n be a solution of (2.33). Then also $u_n + s_1$ solves (2.33) since

$$\begin{aligned}(I - QP_n - B_n)(u_n + s_1) &= (I - QP_n - CP_n)(u_n + s_1) + h(u_n + s_1)q \\ &= (I - QP_n - CP_n)u_n + h(u_n)q = f;\end{aligned}$$

this contradicts the fact that (2.33) has a unique solution. Now we see that the operator $(I - QP_n - CP_n)$ has the same null space as the operator $(I - Q - P)$ which implies that the set of all solutions of (2.28) is of the form (2.30), indeed.

Given $f \in X_0$ let $\widetilde{u}$ be a solution of (2.32), $\widetilde{u}_n$ a solution of (2.33). From the inequality (2.14) in Lemma 2.3 we obtain that for some $c_2 \in \mathbb{R}$

$$\|\widetilde{u} - \widetilde{u}_n\| \leqslant c_2\Big(\big\|Q(P_n\widetilde{u} - \widetilde{u})\big\| + \big\|B_n\widetilde{u} - B\widetilde{u}\big\|\Big).$$

Since $\widetilde{u} \in X_0$ (due to the assumption $f \in X_0$) we have

$$P_n\widetilde{u} \to \widetilde{u}, \qquad B_n\widetilde{u} \to B\widetilde{u}$$

and thus

$$\widetilde{u}_n \to \widetilde{u}.$$

If (u, a) is a solution of (2.27) and (u_n, a) is a solution of (2.28) then we have

$$a_n = h(\widetilde{u}_n) \to h(\widetilde{u}) = a.$$

Let u, u_n be, in addition, such that $r(u) = 0$, $r(P_n u_n) = 0$. By the above argument u, u_n are of the form

$$u = \widetilde{u} + t \cdot s, \qquad u_n = \widetilde{u}_n + t_n \cdot s,$$

where

$$t = -\frac{r(\widetilde{u})}{r(s)}, \qquad t_n = -\frac{r(P_n\widetilde{u})}{r(s)}.$$

We have

$$\|P_n\widetilde{u}_n - \widetilde{u}\| \leqslant \|P_n\widetilde{u}_n - P_n\widetilde{u}\| + \|P_n\widetilde{u} - \widetilde{u}\| \leqslant \|P_n\|\,\|\widetilde{u}_n - \widetilde{u}\| + \|P_n\widetilde{u} - \widetilde{u}\|,$$

whence $P_n\widetilde{u}_n \to \widetilde{u}$ (as $\|P_n\| = 1$, $\widetilde{u}_n \to \widetilde{u}$, $P_n\widetilde{u} \to \widetilde{u}$). Now we see that $t_n \to t$ and we conclude that $u_n \to u$. The assertion is proved. $\square$

Now consider again the Dirichlet problem on S^+ as described in the first section. We shall look for the solution in the form (1.35). As we have noted the problem can be reduced to the integral equation (1.34).

For simplicity we will suppose throughout this part that K is positively oriented. Then the equation (1.34) can be written in the form

$$(I - \overline{\mathcal{H}})f_K + av = g_0, \tag{3.1}$$

where

$$g_0 = -\Big(u_K - \tfrac{1}{2}H_L u_L\big|_K\Big) \tag{3.2}$$

(instead of $v|_K$ we write simply v). We know that the solution of (3.1) is not unique—it is true that $a \in \mathbb{R}$ is determined uniquely, but f_K is determined up to a constant. If we add a condition that the value of f_K at a fixed point in K vanishes (for example) then (3.1) will have a unique solution.

One of the ways how to solve (3.1) numerically is to look for a piecewise constant approximation of f_K. Dividing K into n arcs on which the approximate solution is constant we get a system of n equations with $(n+1)$ unknowns (n values of f_K and the constant a). If we choose f_K vanishing at a fixed point then we can consider a piecewise constant approximation which is equal to zero on a given arc. Then one column in the system of equations can be removed and we obtain a system of n equations with n unknowns (one of which is a). Solving this system we get an approximation of f_K and at the same time the constant a (an approximation of a).

In the following we describe this method in more detail, and using Proposition 2.4 we will show that for all sufficiently large n the matrix of the mentioned system of (linear) equations is regular and the method converges [under the assumption (1.29)]. The question concerning the rate of convergence is more complicated and is not investigated here.

In the first section $\overline{\mathcal{H}}f$, $W_K(\cdot, f)$, $\overline{W}_K f$ etc. were defined for $f \in \mathscr{C}(K)$. Considering the integral on the right hand side of (1.12) to be the Lebesgue-Stieltjes integral (identifying ϑ_z with the corresponding Lebesgue-Stieltjes measure) $W_K(\cdot, f)$ is defined whenever f is a bounded Baire function on K (for example). Let $\mathscr{B}(K)$ stand for the space of all bounded Baire functions on K equipped with the supremum norm. If $W_K(\cdot, f)$ is defined for $f \in \mathscr{B}(K)$ then also $\overline{W}_K f, \overline{\mathcal{H}}f$ are defined for $f \in \mathscr{B}(K)$ [by equalities (1.19), (1.27)], that is $\overline{W}_K, \overline{\mathcal{H}}$ can be regarded as operators on $\mathscr{B}(K)$. It is easily seen that

$$\overline{W}_K, \overline{\mathcal{H}}\colon\ \mathscr{B}(K) \to \mathscr{B}(K).$$

Denote further

$$\widehat{W}_K f(\zeta) = W_K\big(\overline{g}(\zeta), f\big)$$

for $f \in \mathscr{B}(K)$, $\zeta \in K$, so that $\overline{\mathcal{H}} = \overline{W}_K - \widehat{W}_K$.

Lemma 3.1. *Let* $\operatorname{var}[\psi; \langle a, b\rangle] < \infty$. *Then the operator* $\widehat{W}_K\colon f \mapsto \widehat{W}_K f$ *is compact as an operator on* $\mathscr{B}(K)$ *and*

$$\widehat{W}_K\colon \mathscr{B}(K) \to \mathscr{C}(K).$$

P r o o f. This follows from the reasoning described in the proof of Lemma 2.2 in [4]. It only suffices to write everywhere $\mathscr{B}(K)$ instead of $\mathscr{C}(K)$. $\square$

Now we want to show that the equation (3.1) corresponds to the case described in Proposition 2.4. Denote

$$X = \mathscr{B}(K), \qquad X_0 = \mathscr{C}(K)$$

(then X is a Banach space, $X_0 \subset X$ a closed subspace). The function v defined in Section 1 (more precisely the restriction $v|_K$) will play the role of q in Proposition 2.4; further let $s \equiv 1$ (on K).

Define the projections P_n considered in Proposition 2.4 in the following way. For $n \in \mathbb{N}$ divide K into n disjoint arcs K_i^n. For example if $K = \psi(\langle a, b\rangle)$ then put for $i = 1, 2, \dots, n$

$$I_i^n = \left(a + (i-1)\frac{b-a}{n}, a + i\frac{b-a}{n}\right\rangle, \tag{3.3}$$

$$K_i^n = \psi(I_i^n) \tag{3.4}$$

and choose $z_i^n \in K_i^n$—for example

$$z_i^n = \psi\left(a + (i - \tfrac{1}{2})\frac{b-a}{n}\right). \tag{3.5}$$

Suppose that the diameters of the subarcs K_i^n satisfy

$$\max_{1\leqslant i\leqslant n}\left\{\operatorname{diam} K_i^n\right\} \to 0 \qquad \text{for } n \to \infty$$

(if we choose K_i^n in the way described above this condition is fulfilled, of course).

For $f \in \mathscr{B}(K)$ define $P_n f$ as a function which is constant on each arc K_i^n; for $z \in K_i^n$ put

$$P_n f(z) = f(z_i^n).$$

It is easily seen that $P_n s = s$, $\|P_n\| = 1$ and for $f \in X_0$ $[= \mathscr{C}(K)]$ we have

$$P_n f \to f \qquad \text{for } n \to \infty.$$

Now express the operator $\overline{\mathcal{H}}$ as a sum of two operators Q, C. Denote by $\mathscr{R}$ the set of all $r > 0$ for which there is a circumference S^r with radius r such that $\mathscr{H}_1(S^r \cap K) > 0$ (where $\mathscr{H}_1$ is the linear Hausdorff measure). It is shown in [9], lemma 1.10, that $\mathscr{R}$ is at most countable. It is shown also in [9] that for any $r > 0$, $r \notin \mathscr{R}$,

$$\sup_{\zeta \in K} \left(v_r^{\psi}(\zeta) + |\Delta(\zeta)| \right) = \sup_{\zeta \in K} v_r^{\psi}(\zeta).$$

As (1.29) is supposed to be fulfilled we can choose $r_0 > 0$, $r_0 \notin \mathscr{R}$, such that

$$\frac{1}{\pi} \sup_{\zeta \in K} \left(v_{r_0}^{\psi}(\zeta) + |\Delta(\zeta)| \right) < 1. \tag{3.6}$$

Now put for $\zeta \in K$, $f \in \mathscr{B}(K)$,

$$Qf(\zeta) = \frac{1}{\pi} \left(\sum_{I \in \gamma_{\zeta, r_0}} \int_I f(\psi(t)) \, \mathrm{d}\vartheta_\zeta(t) + \Delta(\zeta) f(\zeta) \right),$$

where γ_{ζ, r_0} has the meaning described before the formula (1.10) and further

$$\begin{aligned} Df(\zeta) &= \overline{W}_K f(\zeta) - Qf(\zeta), \\ Cf(\zeta) &= \overline{\mathcal{H}} f(\zeta) - Qf(\zeta) = \overline{W}_K f(\zeta) - Qf(\zeta) - \widehat{W}_K f(\zeta) \\ &= Df(\zeta) - \widehat{W}_K f(\zeta). \end{aligned}$$

It can be shown analogously to the proofs of Lemmas 1.11 and 1.12 in [9] that D maps the unit ball in $\mathscr{B}(K)$ onto a set of functions equicontinuous and uniformly bounded on K. Thus $D\colon X \to X_0$ and D is a compact operator on X and by Lemma 3.1

$$C\colon X \to X_0$$

and C is a compact operator on X. Further we see that

$$Q\colon X \to X, \qquad Q\colon X_0 \to X_0$$

(since $\overline{W}_K\colon X \to X$, $\overline{W}_K\colon X_0 \to X_0$) and $\|Q\| < 1$ due to (3.6).

Now we are in a position to apply Proposition 2.4. Under the given notation the equation (3.1) can be written in the form (2.27); the equation (2.28) has now the form

$$(I - \overline{\mathcal{H}} P_n) u_n + a_n v = g_0. \tag{3.7}$$

Fix $\widetilde{z} \in K$ and for $f \in X$ put

$$r(f) = f(\widetilde{z}).$$

Now the assumptions of Proposition 2.4 are fulfilled. Suppose, in addition, that $g_0 \in X_0$ [if g_0 is of the form (3.2) then it suffices to assume that $u_K \in \mathscr{C}(K)$]. First it follows from Proposition 2.4 that equations (3.1) and (3.7) (for all sufficiently large n) are solvable in X (even for $g_0 \in X$). If we add the condition $r(f_K) = 0$ to (3.1) and the condition $r(P_n u_n) = 0$ to (3.7) then the equations (3.1), (3.7) are uniquely solvable. If (f_K, a) is the solution of (3.1) and (u_n, a_n) is the solution of (3.7) (fulfilling the given conditions) then

$$u_n \to f_K, \qquad a_n \to a.$$

Now we describe the algorithm to which this method leads. Let I_i^n, K_i^n, z_i^n be of the form (3.3), (3.4), (3.5). Put further

$$\widetilde{z} = \psi(b) \quad \big(= \psi(a)\big);$$

then $\widetilde{z} \in K_n^n$ (for all $n \in \mathbb{N}$). Let χ_i^n denote the characteristic function of K_i^n and put

$$M_{ij}^n = \big[(I - \overline{\mathcal{H}})\chi_j^n\big](z_i^n) \tag{3.8}$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, (n-1)$ and

$$M_{in}^n = v(z_i^n) \tag{3.9}$$

for $i = 1, 2, \ldots, n$.

Put further

$$b_i^n = g_0(z_i^n) \tag{3.10}$$

for $i = 1, 2, \ldots, n$,

$$x_i^n = u_n(z_i^n) \tag{3.11}$$

for $i = 1, 2, \ldots, (n-1)$ and

$$x_n^n = a_n. \tag{3.12}$$

Let u_n be the solution of the equation (3.7) satisfying the condition $r(P_n u_n) = 0$, that is $u_n(z_n^n) = 0$ (as $\widetilde{z} \in K_n^n$). Then

$$P_n u_n = \sum_{j=1}^{n-1} u_n(z_j^n)\chi_j^n = \sum_{j=1}^{n-1} x_j^n \chi_j^n,$$
$$\overline{\mathcal{H}} P_n u_n = \overline{\mathcal{H}} \sum_{j=1}^{n-1} x_j^n \chi_j^n = \sum_{j=1}^{n-1} x_j^n \overline{\mathcal{H}} \chi_j^n$$

and the equality (3.7) can be written in the form

$$u_n - \sum_{j=1}^{n-1} x_j^n \overline{\mathcal{H}} \chi_j^n + a_n v = g_0.$$

In particular we get for $z = z_i^n$

$$u_n(z_i^n) - \sum_{j=1}^{n-1} x_j^n \big(\overline{\mathcal{H}} \chi_j^n\big)(z_i^n) + a_n v(z_i^n) = g_0(z_i^n).$$

Using the above notation we obtain that

$$\sum_{j=1}^{n} M_{ij}^n x_j^n = b_i^n \qquad (i = 1, 2, \ldots, n). \tag{3.13}$$

Note that now we see that the matrix $M = (M_{ij}^n)$ is regular (for all sufficiently large n) since (3.7) and consequently (3.13) are solvable for any right hand side.

Let us show how to evaluate easily the coefficients M_{ij}^n (for $j \neq n$). First consider the case $i \neq j$. Then $\chi_j^n(z_i^n) = 0$ and thus

$$M_{ij}^n = -\overline{\mathcal{H}} \chi_j^n(z_i^n) = W_K\big(\overline{g}(z_i^n), \chi_j^n\big) - W_K(z_i^n, \chi_j^n) = \frac{1}{\pi}\bigg(\int_{I_j^n} \mathrm{d}\vartheta_{\overline{g}(z_i^n)} - \int_{I_j^n} \mathrm{d}\vartheta_{z_i^n}\bigg).$$

The integrals on the right hand side are nothing else but the increments of the argument along the arc K_j^n with respect to the point $\overline{g}(z_i^n)$ (resp. z_i^n) which are easy to evaluate [they are equal to the angles “under which the endpoints of K_j^n are seen from the point $\overline{g}(z_i^n)$ or z_i^n”].

Now consider the case $j = i$. Then $z_i^n \in K_i^n$ and we get [using equalities (1.19), (1.16) and the fact that $\chi_i^n(z_i^n) = 1$]

$$\begin{aligned}
M_{ii}^n &= W_K\bigl(\overline{g}(z_i^n), \chi_i^n\bigr) + 1 - \overline{W}_K \chi_i^n(z_i^n) \\
&= W_K\bigl(\overline{g}(z_i^n), \chi_i^n\bigr) + 1 - W_K(z_i^n, \chi_i^n) - \frac{1}{\pi}\Delta(z_i^n) \\
&= W_K\bigl(\overline{g}(z_i^n), \chi_i^n\bigr) + 1 - W_K(z_i^n, \chi_i^n) - \frac{1}{\pi}\Bigl[\pi - \bigl(\alpha_-(z_i^n) - \alpha_+(z_i^n)\bigr)\Bigr] \\
&= W_K\bigl(\overline{g}(z_i^n), \chi_i^n\bigr) - \Bigl[W_K(z_i^n, \chi_i^n) - \frac{1}{\pi}\alpha_-(z_i^n) + \frac{1}{\pi}\alpha_+(z_i^n)\Bigr].
\end{aligned}$$

Here

$$W_K\bigl(\overline{g}(z_i^n), \chi_i^n\bigr) = \frac{1}{\pi}\int_{I_i^n} \mathrm{d}\vartheta_{\overline{g}(z_i^n)}.$$

For evaluating $W_K(z_i^n, \chi_i^n)$ one has to realize that $z_i^n \in K_i^n$. Denote for a while

$$\begin{aligned}
t_i &= a + (i - \tfrac{1}{2})\frac{b-a}{n}, \\
\mathbf{1}_{I_i^n} &= \left(a + (i-1)\frac{b-a}{n}, t_i\right), \\
\mathbf{2}_{I_i^n} &= \left(t_i, a + i\frac{b-a}{n}\right).
\end{aligned}$$

Recalling (1.15) we observe that

$$\lim_{t \to t_i+} \vartheta_{z_i^n}(t)$$

coincides with $\alpha_+(z_i^n)$ up to a multiple of 2π and analogously for

$$\lim_{t \to t_i-} \vartheta_{z_i^n}(t).$$

Suppose that $\vartheta_{z_i^n}$ has been chosen in such a way that

$$\lim_{t \to t_i+} \vartheta_{z_i^n}(t) = \alpha_+(z_i^n), \qquad \lim_{t \to t_i-} \vartheta_{z_i^n}(t) = \alpha_-(z_i^n).$$

Then we have

$$\begin{aligned}
&W_K(z_i^n,\chi_i^n)-\frac{1}{\pi}\alpha_-(z_i^n)+\frac{1}{\pi}\alpha_+(z_i^n)\\
&\quad=\frac{1}{\pi}\left\{\int_{\mathbf{1}_{I_i^n}}\mathrm{d}\vartheta_{z_i^n}+\int_{\mathbf{2}_{I_i^n}}\mathrm{d}\vartheta_{z_i^n}-\alpha_-(z_i^n)+\alpha_+(z_i^n)\right\}\\
&\quad=\frac{1}{\pi}\left\{\alpha_-(z_i^n)-\vartheta_{z_i^n}\left(a+(i-1)\frac{b-a}{n}\right)+\vartheta_{z_i^n}\left(a+i\frac{b-a}{n}\right)\right.\\
&\qquad\left.-\alpha_+(z_i^n)-\alpha_-(z_i^n)+\alpha_+(z_i^n)\right\}\\
&\quad=\vartheta_{z_i^n}\left(a+i\frac{b-a}{n}\right)-\vartheta_{z_i^n}\left(a+(i-1)\frac{b-a}{n}\right).
\end{aligned}$$

We see that also M_{ii}^n can be evaluated easily (analogously to M_{ij}^n for $i\neq j$). Thus the described numerical method is easy to implement.

R e m a r k . Consider the case $u_L\equiv 0$ on L (then $g_0=-u_K$), let n be sufficiently large and let $[x_1^n,x_2^n,\ldots,x_n^n]$ be the solution of (3.13),

$$f_K^n=\sum_{j=1}^{n-1}x_j\chi_j^n.$$

For $z\in S^+\cup L$ put

$$h(z)=\mathcal{H}_K f_K^n(z)+x_n^n v(z)=\sum_{j=1}^{n-1}x_j^n\mathcal{H}_K\chi_j^n(z)+x_n^n v(z)$$

[the evaluation of $\mathcal{H}_K\chi_j^n(z)$ is quite similar to the evaluation of M_{ij}^n for $i\neq j$]. Function h is an approximation of the solution of the Dirichlet problem on S^+ with the boundary condition u_K on K and $u_L\equiv 0$ on L. It is easy to see that h has the following properties:

(a) h is harmonic on S^+,
(b) h is continuos on $S^+\cup L$ and $h(z)=0$ for any $z\in L$,
(c) for each $\zeta\in K$ except the endpoints of K_i^n $(i=1,2,\ldots,n)$ the limit

$$\lim_{\substack{z\to\zeta\\ z\in S^+}}h(z)$$

exists and is finite,
(d) for $i=1,2,\ldots,n$

$$\lim_{\substack{z\to z_i^n\\ z\in S^+}}h(z)=u_K(z_i^n).$$

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